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VERTICAL ANGLES DEVIATIONS OF THE VERTICAL AND ADJUSTMENT

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CONTENTS

VERTICAL ANGLES, DEVIATIONS OF THE VERTICAL AND ADJUSTMENT

1 Introduction

The possibility of using vertical angle measurement as a means for determining deviations of the vertical has been investigated by several geodesists. In his papers [3], [4], [5] and [6], R. FINSTERWALDER did important pioneering work which was and is continued especially in Germany and Switzerland. F. KOBOLD gave in [10] a survey of the results that have been reached in this field, whereas W. HOFMANN in his comprehensive study [8] investigated the possibilities of determining accurate geoidal heights and deviations of the vertical by vertical angle measurements.

The present author hopes to make a contribution to the discussion and to the solution of some existing problems by examining the subject in the light of the calculus of observations.

One of the main characteristics of the mathematical model used by the authors mentioned is that the coefficient of refraction k is assumed to be constant over a whole net or at least in parts of the net. The use of a coefficient k usually implies the assumption of a circular path of the light-rays, and the assumption of its being constant has far-reaching consequences. Although it seems that in high mountainous areas k is more constant than in lower parts of the atmosphere, there is always a possibility that the accuracy of vertical angle measurements is considerably reduced by fluctuations of refraction.

The interesting article [14] by L. HRADILEK deals with the computation of a coefficient of refraction for each line of sight, on the basis of a formula given by A. A. IZOTOV and L. P. PELLINEN. This formula expresses k as a function of atmospheric pressure and temperature (measured at the station), in the temperature gradient (which can be introduced as an unknown) and in the so-called aequivalent height, which is a function of the height of the line of sight over the surface of the earth in the profile between the two stations concerned. In view of the difficulties which are peculiar to the mathematical model using a constant coefficient of refraction, this method seems to be a big step forward. It would have been interesting to connect the results of the present paper to those of [14], but when the author came in the possession of HRADILEK's paper, his own work was too far advanced to widen its scope to that extent.

Consequently, this paper is devoted mainly to the properties of the "classical" model of trigonometric levelling and to the examination of some approximation methods used in practice. The results of computations will, as a rule, not be interpreted physically; this means that no conclusion will be drawn as to, for example, what kind of heights are found by trigonometric levelling.

2 Notation and basic formulae

The following notation will be used.

P_i point on the surface of the earth, numbered i.

H_i length of the normal from P_i on the ellipsoid of reference.

H_{ij} $= \frac{H_i + H_j}{2}$

β_{ij} angle of elevation of P_j as measured in P_i, corrected for heights of instrument and target.

α_{ij} ellipsoidal angle of elevation corresponding to β_{ij}.

θ_i magnitude of the deviation of the vertical in P_i.

θ_{ij} component of θ_i in the direction from P_i to P_j.

ξ_i, η_i respectively North-South and East-West component of θ_i. The sign convention is such that if both ξ_i and η_i are positive, the astronomical zenith is situated North and East of the ellipsoidal zenith.

s_{ij} length of the ellipsoidal arc between the projections of P_i and P_j.

R_{ij} radius of curvature of the normal section of the ellipsoid from the projection of P_i to that of P_j.

k coefficient of refraction; $k = \frac{R_{ij}}{r_{ij}}$ if r_{ij} is the radius of curvature of the light-ray from P_i to P_j.

ψ_{ij} azimuth from P_i to P_j.

ρ^{cc} = 636620 centesimal seconds.

Stochastic quantities are indicated by an underscore e.g. $\underline{H}_i$, $\underline{\xi}_j$.

Figures 1 and 2 demonstrate the notation and conventions used. $Z_i^{\text{ell.}}$ and $Z_i^{\text{astr.}}$ are the ellipsoidal respectively astronomical zenith of P_i.

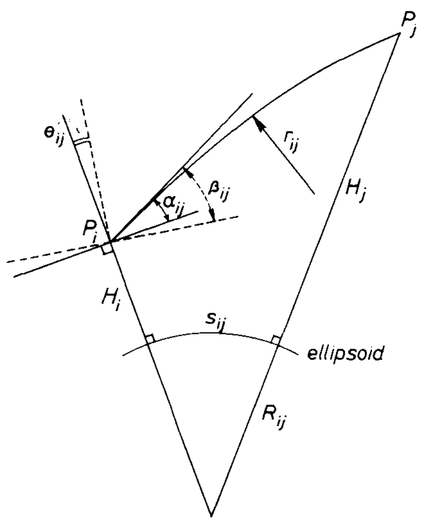

Figure 1

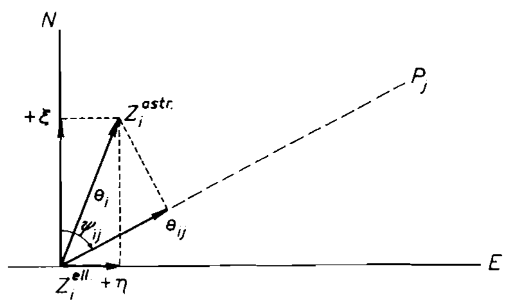

Figure 2

We shall base our investigation on W. JORDAN's formula, see [9], page 358:

$$\underline{H}_j - \underline{H}_i = \left(1 + \frac{H_{ij}}{R_{ij}}\right) s_{ij}\,\mathrm{tg}\,\underline{\alpha}_{ij} + \frac{s_{ij}^2(1-\underline{k})}{2R_{ij}\cos^2\underline{\alpha}_{ij}} \quad \ldots \quad (1)$$

Several other formulae have been developed. HOFMANN mentions in [8] the formulae by REICHENEDER and WILD-BAESCHLIN, and he develops still another formula. Their accuracy has been carefully investigated in HOFMANN's study. The differences prove to be small; JORDAN's formula is used here because it is generally known and the different parts are conveniently arranged. It is assumed that s_{ij} has been determined so accurately that it can be considered as a non-stochastic quantity in (1). The same applies to H_{ij}.

Not the ellipsoidal elevation angle $\underline{\alpha}_{ij}$ has been measured, but the angle $\underline{\beta}_{ij}$. Therefore we have to introduce:

$$\underline{\alpha}_{ij} = \underline{\beta}_{ij} - \underline{\theta}_{ij}$$

to be able to use the actually observed elevation angles in the ellipsoidal model. $\underline{\theta}_{ij}$ being a small quantity, $\mathrm{tg}(\underline{\beta}_{ij} - \underline{\theta}_{ij})$ can be expanded in Taylor's series. We will confine ourselves to the study of nets in which $|\mathrm{tg}\beta_{ij}| < 0.1$, $|\theta_{ij}| < 50^{cc}$ and s_{ij} has an order of magnitude of 10 km. In the second term on the right hand side of (1) we can then replace $\underline{\alpha}_{ij}$ by β_{ij} and consider β_{ij} as being non-stochastic. We then obtain from (1):

$$\underline{H}_j - \underline{H}_i = \boxed{\left(1 + \frac{H_{ij}}{R_{ij}}\right) s_{ij}\,\mathrm{tg}\,\underline{\beta}_{ij} + \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}} + \\ - \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}\,\underline{k} - \left(1 + \frac{H_{ij}}{R_{ij}}\right) \frac{s_{ij}}{\cos^2\beta_{ij}}\,\underline{\theta}_{ij} \quad \ldots \quad (2)$$

We call the framed part in equation (2): $\underline{h}_{ij}$ and introduce the following relation which is evident from Figure 2:

$$\theta_{ij} = \xi_i \cos \psi_{ij} + \eta_i \sin \psi_{ij} \quad \ldots \quad (3)$$

Furthermore we call the coefficient of θ_{ij} in (2) S_{ij}:

$$S_{ij} = \left(1 + \frac{H_{ij}}{R_{ij}}\right) s_{ij} \frac{1}{\cos^2\beta_{ij}} \quad \ldots \quad (4)$$

By expressing ξ_i and η_i in centesimal seconds we can now write (2) as follows:

$$\underline{H}_j - \underline{H}_i + \frac{S_{ij}}{\rho} \cos\psi_{ij}\,\underline{\xi}_i^{cc} + \frac{S_{ij}}{\rho} \sin\psi_{ij}\,\underline{\eta}_i^{cc} + \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}\,\underline{k} = \underline{h}_{ij}\ (+\underline{\varepsilon}_{ij}) \quad (5)$$

The addition $(+\underline{\varepsilon}_{ij})$ on the right hand side represents the correction to be given to $\underline{h}_{ij}$ if the number of observations in the net gives rise to a least squares adjustment.

3 The relation between computed deviations of the vertical and the coefficient of refraction

If in a triangulation net the relative horizontal positions of the stations are known with sufficient accuracy (see [8], page 25), one can measure vertical angles with the purpose of computing for each station P_i the quantities $\underline{H}_i$, $\underline{\xi}_i$ and $\underline{\eta}_i$. Each measured vertical angle furnishes an equation (5), and if there are more observations $\underline{h}_{ij}$ than unknowns $\underline{H}_i$, $\underline{\xi}_i$, $\underline{\eta}_i$ and $\underline{k}$ we get an adjustment problem.

Of course, vertical angle measurement cannot provide an absolute determination of H_i, ξ_i and η_i: in such a net one must know or assume one H, one ξ and one η. In practice one usually selects a certain point whose H is chosen about equal to its height above mean sea-level, and whose ξ and η are assumed to be zero. One then obtains H- respectively ξ- and η-differences with respect to this datum point, which we shall call P_0.

The coefficient of refraction k has in the above been introduced as an unknown because at first sight this seems natural. In reality, k will not have the same value for all observations; however in high mountain areas it usually proves to vary much less than in lower parts of the atmosphere, so that it seems reasonable to assume a mean value of k, which is indeed the simplest way in which this physical phenomenon can be introduced in the mathematical model. One may of course introduce different values of k for different parts of the net.

To simplify the argument we will now for a moment replace the reference ellipsoid by a sphere with an appropriate radius R – this is frequently practised in trigonometric levelling. If we consider an area where the deviations of the vertical increase proportionally to the distances and equally in all directions, we can conclude that the level surfaces are also spheres. One might now introduce, instead of ξ_i and η_i, an unknown $\Delta \underline{R}$ representing the "extra curvature" (in German: "zusätzliche Krümmung") of the level surfaces, respectively of the geoid. In order to linearize the formula one has then to introduce $\underline{k} = k_0 + \Delta \underline{k}$ in which k_0 is an approximate value. Instead of (5) we would then get:

$$\underline{H}_j - \underline{H}_i + \frac{s_{ij}^2}{2R\cos^2\beta_{ij}} \Delta \underline{k} + \frac{s_{ij}^2(1-k_0)}{2R^2\cos^2\beta_{ij}} \Delta \underline{R} = \underline{h}_{ij} - \frac{k_0}{2R\cos^2\beta_{ij}} \quad . \quad . \quad . \quad . \quad (6)$$

If we compare in (6) the coefficients of $\Delta \underline{k}$ and $\Delta \underline{R}$, it appears that the coefficient of $\Delta \underline{R}$ in each equation is a constant multiple of the coefficient of $\Delta \underline{k}$, the constant being $\frac{1-k_0}{R}$.

If we have m equations (6) with n unknowns ($n \leqslant m$), the rank of the matrix of the equations is not n but $n-1$: the column of coefficients of $\Delta \underline{R}$ is a multiple of the column of coefficients of $\Delta \underline{k}$, so that $\underline{k}$ and $\Delta \underline{R}$ cannot be determined separately. Increasing the number of equations (6) does not help: in an adjustment the matrix of the normal equations will be singular.

The reasoning followed above is generally known, see for example [4] and [7]. It will be clear that introduction of an ellipsoidal model, with different radii R_{ij} in different directions, does not remove the difficulty that makes it impossible to determine Δk and ΔR simultaneously. The variations of R_{ij} in different directions are so small that numerically the indicated relation remains valid.*) However, it is not self-evident that the same problem manifests itself in the equations (5), where we

*) It may be noted that in this case the assumption of a constant k implies that rays of light in different directions have slightly different radii of curvature.

have *irregular* deviations of the vertical and no constant extra curvature. We shall show that in this case, too, it is impossible to determine the coefficient of refraction and the deviations of the vertical simultaneously; the coefficient of k in each equation (5) is practically linearly dependent on the coefficients of the other unknowns. To prove this we write these coefficients under the unknowns to which they belong:

H_j	H_i	ξ_i	η_i	k
$+1$	-1	$+\frac{S_{ij}}{\rho}\cos\psi_{ij}$	$+\frac{S_{ij}}{\rho}\sin\psi_{ij}$	$+\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}$

If the statement is true, it must be possible to find coefficients α_j, α_i, β_i and γ_i, such that:

$$\alpha_j\cdot 1-\alpha_i\cdot 1+\beta_i\frac{S_{ij}}{\rho}\cos\psi_{ij}+\gamma_i\frac{S_{ij}}{\rho}\sin\psi_{ij}\approx\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} \quad \ldots\ldots (7)$$

The following set of values satisfies this condition:

$$\alpha_i=+\frac{s_{i0}^2}{2R\cos^2 B};\ \beta_i=+2\rho\frac{s_{i0}\cos\psi_{i0}}{2R\cos^2 B};\ \gamma_i=+2\rho\frac{s_{i0}\sin\psi_{i0}}{2R\cos^2 B};\quad ..\ (7a)$$

in which s_{i0} is the distance from P_i to the datum point P_0, ψ_{i0} is the azimuth P_i-P_0; R is the geometric mean of the radii in the meridian and prime vertical and $\frac{1}{\cos^2 B}$ is the mean value of $\frac{1}{\cos^2\beta_{ij}}$ in the net.

If we substitute these values in the left hand side of (7), we get:

$$-\frac{s_{i0}^2}{2R\cos^2 B}+\frac{s_{j0}^2}{2R\cos^2 B}+\frac{2s_{i0}S_{ij}\cos\psi_{ij}\cos\psi_{i0}}{2R\cos^2 B}+\frac{2s_{i0}S_{ij}\sin\psi_{ij}\sin\psi_{i0}}{2R\cos^2 B}\approx\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} \quad (7b)$$

Application of the law of cosines in $\triangle P_0P_iP_j$ in Figure 3 gives:

$$s_{j0}^2 = s_{i0}^2+s_{ij}^2-2s_{i0}s_{ij}\cos(\psi_{ij}-\psi_{i0})$$
$$s_{ij}^2 = -s_{i0}^2+s_{j0}^2+2s_{i0}s_{ij}\cos(\psi_{ij}-\psi_{i0})$$

If we neglect for a moment the differences between s_{ij} and S_{ij} and between $R\cos^2 B$ and $R_{ij}\cos^2\beta_{ij}$, it is readily seen that both sides of (7b) are equal. In Appendix I it is shown that the maximum difference of both sides is about 5 percent. This maximum value will seldom occur in trigonometric nets that are approximately flat; differences found in the Isartal net measured by Dr. HOFMANN are also given in Appendix I. The result is that from a numerical point of view, in equations (5) like in equations (6), the coefficients of k are linearly dependent on the coefficients of the other unknowns. If we have m observation equations with n unknowns ($m \geqslant n$), the matrix of coefficients of these equations will theoretically have the rank n, but numerically it will behave as if the rank were $n-1$.

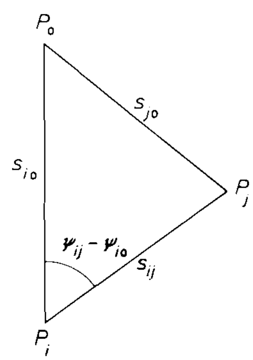

Figure 3

The matrix of coefficients of the normal equations will consequently be singular. The fact that equations (7b) are not exactly valid manifests itself even less in the normal equations than in the observation equations because the coefficient of $\underline{k}$ in each normal equation is a linear combination of its coefficients in the observation equations. The slight differences between right-hand and left-hand sides of (7b) are thereby cancelled out to a great extent.

In order to make the problem mathematically manageable we shall assume that the indicated linear dependency is exactly valid. The normal equations cannot be solved but we can only express $\underline{H}_i$, $\underline{\xi}_i$ and $\underline{\eta}_i$ as functions of $\underline{k}$.

We now generalize our notation and change over to the notation used in tensor calculus, for which reference is made to [12]. The unknowns are called $\underline{x}^\alpha$. The observation equations are indicated by the index i (which differs from the previously used index i which was a point number!) The coefficient of $\underline{x}^\alpha$ in the ith equation is called A^i_α. The column of coefficients of $\underline{k}$ can, because of the demonstrated linear dependency, be written as $-b^\alpha A^i_\alpha$, in which $-b^\alpha$ is the general notation of the coefficients α_i, β_i and γ_i in equation (7).

The observation $\underline{h}_{ij}$ is called $\underline{p}^i$; we assume that m vertical angles have been observed $(i = 1, \ldots, m)$.

In a net which includes n points beside the datum point we can write the observation equations as follows:

$$A^i_\alpha \underline{x}^\alpha - b^\alpha A^i_\alpha \underline{k} = \underline{p}^i + \underline{\varepsilon}^i \qquad (i, j, l = 1, \ldots, m;\ \alpha, \beta, \gamma = 1, \ldots, 3n;\ m \geqslant 3n) \quad (8)$$

$$A^i_\alpha (\underline{x}^\alpha - b^\alpha \underline{k}) = \underline{p}^i + \underline{\varepsilon}^i \quad \ldots \quad (9)$$

Let the matrix of weights have elements g_{ij}.

The normal equations are:

$$g_{ij} A^i_\alpha A^j_\beta (\underline{x}^\alpha - b^\alpha \underline{k}) = g_{ij} A^j_\beta \underline{p}^i$$

Putting:

$$g_{ij} A^i_\alpha A^j_\beta = g_{\alpha\beta}$$

and

$$g_{\alpha\beta} g^{\gamma\alpha} = \delta^\gamma_\beta = \begin{cases} 1 \text{ if } \gamma = \beta \\ 0 \text{ if } \gamma \neq \beta \end{cases}$$

the solution for the unknowns is:

$$\underline{x}^\alpha - b^\alpha \underline{k} = g^{\alpha\beta} g_{ij} A^j_\beta \underline{p}^i$$

$$\underline{x}^\alpha = b^\alpha \underline{k} + g^{\alpha\beta} g_{ij} A^j_\beta \underline{p}^i \quad \ldots \quad (10)$$

From (10) it is evident that the influence of $\underline{k}$ on the unknowns can be separated completely from the influence of the observations $\underline{p}^i$. From the values $(x^\alpha)_1$ corresponding to a certain value k_1 of the coefficient of refraction, one obtains very easily the values $(x^\alpha)_2$ corresponding to a value k_2, namely:

$$(x^\alpha)_2 = (x^\alpha)_1 + b^\alpha (k_2 - k_1) \quad \ldots \quad (11)$$

The corrections to the observations can be determined from (8):

$$\underline{\varepsilon}^i = A^i_\alpha \underline{x}^\alpha - b^\alpha A^i_\alpha \underline{k} - \underline{p}^i$$

or with (10):

$$\underline{\varepsilon}^i = A^i_\alpha\,(b^\alpha \underline{k} + g^{\alpha\beta} g_{jl} A^l_\alpha \underline{p}^j) - b^\alpha A^i_\alpha \underline{k} - \underline{p}^i$$

$$\underline{\varepsilon}^i = A^i_\alpha g^{\alpha\beta} g_{jl} A^l_\beta \underline{p}^j - \underline{p}^i \quad \ldots \quad (12)$$

This leads to the remarkable conclusion that the corrections are independent of $\underline{k}$. A further conclusion is that the quantity

$$\underline{E} = g_{ij} \underline{\varepsilon}^i \underline{\varepsilon}^j$$

by which the usual estimate for the variance factor σ^2 ("mean square error of an observation of unit weight ") is determined, is also independent of the value of $\underline{k}$. In practice one usually adopts a certain value of $\underline{k}$, found by means of astronomical observations, in order to get a determinate solution for the unknowns. If in an adjustment as discussed here a "wrong" value of $\underline{k}$ is used, this will influence the values one finds for the unknowns, but it does not influence the estimate one finds for the variance factor. Of course this does not mean that the coefficient of refraction does not have an influence on the accuracy of trigonometric levelling. If one finds a value E that is significantly too high one may conclude that there is a model error; we have found here that if this model error is caused by refraction, it is not the *value* of k that makes E too high, but the assumption that k is *constant* over the whole net.

Let us now consider a net of $n+1$ points $P_0, P_1, \ldots, P_n$. Not counting k, we have $3n$ unknowns, successively $H_1, \ldots, H_n$, $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$. The values of the coefficients b^α follow from (7a):

$$b^1 = -\frac{s_{10}^2}{2R\cos^2 B}, \ldots, \quad b^n = -\frac{s_{n0}^2}{2R\cos^2 B}$$

$$b^{n+1} = -\frac{2\rho s_{10}\cos\psi_{10}}{2R\cos^2 B}, \ldots, \quad b^{2n} = -\frac{2\rho s_{n0}\cos\psi_{n0}}{2R\cos^2 B}$$

$$b^{2n+1} = -\frac{2\rho s_{10}\sin\psi_{10}}{2R\cos^2 B}, \ldots, \quad b^{3n} = -\frac{2\rho s_{n0}\sin\psi_{n0}}{2R\cos^2 B}$$

Formula (10) was:

$$\underline{x}^\alpha = b^\alpha \underline{k} + g^{\alpha\beta} g_{ij} A^j_\alpha \underline{p}^i$$

We see that a change in the coefficient of refraction (k_2-k_1) has an effect on H_i which is proportional to the square of the distance s_{i0} between P_i and P_0, and proportional to (k_2-k_1). The effect on ξ_i and η_i is proportional to (k_2-k_1) and to the rectangular coordinate differences (X_i-X_0) respectively (Y_i-Y_0) in a system whose X-axis is directed to the North and whose Y-axis is directed to the East.

The components ξ_i and η_i form the vector θ_i; the changes in ξ_i and η_i can be composed to form the vector ζ_i, which according to the above goes through P_0 and whose length is proportional to s_{i0} and to (k_2-k_1). If we designate the values belonging to k_1 by $(\theta_i)_1$ and those belonging to k_2 by $(\theta_i)_2$, it is easily seen that one obtains $(\theta_i)_2$ by composing ζ_i and $(\theta_i)_1$ (see Figure 4).

One thus obtains a clear picture of the influence of a change in the coefficient of refraction on the computed deviations of the vertical. By varying k one can find a value for which the sum of the squares of the lengths of the θ_i vectors is a minimum. The system of deviations of the vertical thus found might be called a system in which there is "no extra curvature". It is, however, doubtful if this procedure has any physical meaning. An example is given in Paragraph 7.4.

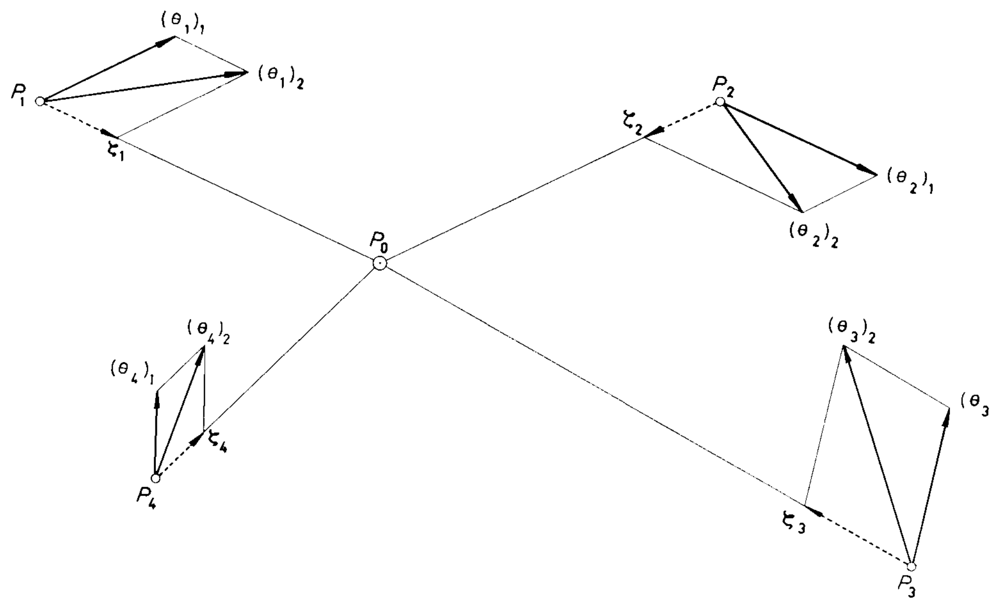

Figure 4

In the above we have found an indeterminate solution, in which one parameter is contained. If in the net beside the datum point, one ellipsoidal height H or one ξ or η is known, the system can be determined. If the coefficient of refraction cannot be determined from meteorological data or other considerations, the method of finding deviations of the vertical by vertical angle measurements will thus always have the character of an interpolation method. If more than one H, ξ or η in the net is known (ξ and η from gravimetrical or astronomical observations) one obtains an adjustment problem for the determination of k. An example is given in Paragraph 7.5.

4 The consequences of introducing k as an observation

In the line of thought followed in Section 3, k was left as a parameter in the solution, and after the indeterminate solution had been found, a value of k could be introduced to make the solution determinate. A somewhat different approach to the problem is to introduce the coefficient of refraction as an observation and transport it to the right-hand side of equations (5). This has been done, for example, by W. Hofmann in [8], page 60. A value of k can for instance be found by taking two points A and B in the same meridian and determining their difference in latitude (as accurate determinations of latitude are as a rule simpler to execute than observations for longitude or combinations of the two). If the distance AB is known and reciprocal vertical angles are measured, a value for k can then be computed, as follows easily from Figure 1. With the thus found value of k one can correct all measured angles for refraction and then carry out the adjustment. The procedure is approximately equivalent to the one treated in Section 3, with a subsequent choice of k in such a way that the "trigonometric difference in latitude" between A and B is equal to the astronomically observed difference. The method of Section 3 is more general because there one is not restricted to two points in the same meridian, even if only latitude observations are used.

A difficulty with the "separate" determination of k is that the vertical angles measured in A and B influence the accuracy directly. Usually, A and B will be stations of the net, and if the vertical angles used for the evaluation of k are the same that are used in the adjustment of the net, the computed coefficient of refraction is stochastically dependent on these observations, which complicates the adjustment problem. In the method of Section 3, all vertical angles are treated in the same way, and none are "favoured" by being used for a separate computation of k.

A point of principle, which to the author's knowledge has not received any attention in literature, is that vertical angles, corrected for refraction (making use of the same k) are not stochastically independent. The coefficient of refraction, whatever has been the way of determining it, cannot be considered as a quantity "without error". Let us consider again the equations (8). Correcting the observations for refraction means that the refraction term is transported to the right-hand side, so that we get "new" observations f^i:

$$\underline{f}^i = b^\alpha A^i_\alpha \underline{k} + \underline{p}^i$$

The general law of propagation of variances (or of weight-coefficients *)) as given by J. M. Tienstra, for which reference is made to [12], gives for the weight-coefficients, in Tienstra's notation:

$$\begin{aligned} \overline{f^i, f^j} &= \overline{(b^\alpha A^i_\alpha k + p^i),\ (b^\beta A^j_\beta k + p^j)} \\ &= b^\alpha A^i_\alpha b^\beta A^j_\beta \overline{k, k} + 2 b^\alpha A^i_\alpha \overline{k, p^j} + \overline{p^i, p^j} \end{aligned}$$

Now $\overline{k, p^j}$ is zero if the vertical angles used for determining k are not also used in the adjustment. In this case we have:

$$\overline{f^i, f^j} = b^\alpha A^i_\alpha b^\beta A^j_\beta \overline{k, k} + \overline{p^i, p^j} \qquad (13)$$

One can assume that the observed vertical angles are not correlated, so:

$$\overline{p^i, p^j} = 0 \qquad \text{for } i \neq j$$

But because of the first term on the right-hand side of (13), all observations corrected for refraction are correlated. Consequently the matrix of weight-coefficients $\overline{f^i, f^j}$ is not a diagonal matrix. The adjustment problem as indicated here takes the form of what Tienstra has called "Standard Problem IV"; it can be reduced to a „Standard Problem II": a problem in the form of observation equations with correlated observations. The computations of this problem can be carried out and one finds

*) The term cofactor as used by Tienstra is here replaced by: weight-coefficient, which agrees better with the terminology of mathematical statistics.

the corrections to be given to the "compound" observations $\underline{f}^i$. By the appropriate rules one could then compute how the correction given to $\underline{f}^i$ should be distributed to its composing parts, $b^\alpha A^i_\alpha \underline{k}$ and $\underline{p}^i$. However, we have seen in Section 3 that the quantity $\underline{E} = g_{ij}\underline{\varepsilon}^i\underline{\varepsilon}^j$ is independent of the value of k, no matter what weight matrix $||g_{ij}||$ we have. From this, one can conclude directly that the correction found for k will be zero: loosely speaking one might say that "variation of k cannot help minimize $\underline{E}$". But what will be the results of the adjustment for the unknowns $\underline{H}$, $\underline{\xi}$ and $\underline{\eta}$? In principle we have the same adjustment problem whether we introduce k as a parameter or as an observation; the latter is a special case of the former. At first sight the complicated computation of the adjustment of correlated observations seems bound to give values for the unknowns that are quite different from the values found from a straightforward, "Standard Problem II" with uncorrelated observations, as treated in Section 3, with a subsequent choice of k equal to the observed value. However, it can be proved that both methods give the same values for the unknowns. The proof is given in App. II; it is based on the fact that the matrix of weight-coefficients (13) is composed of a diagonal matrix $||\overline{p^i, p^j}||$ and a matrix $||b^\alpha A^i_\alpha b^\beta A^j_\beta||\overline{k, k}$, whose rank is one. It is to be noted that the variance of k whose effect we have just discussed is the variance caused by the determination of k, e.g. from astronomical measurements. It has been shown that this variance has no effect on the results of the computation (although it of course affects the "external" accuracy). This is plausible, for the same value of k is used to correct all observations and this procedure is equivalent to considering k as a constant. The whole situation bears a strong resemblance to the effect of a scale (systematic) error in distance measuring.

We have not discussed the random fluctuations of k within the net: the assumption was that they were negligible. If we do want to take account of these random fluctuations, we must increase the main diagonal of the matrix of weight-coefficients accordingly: the non-diagonal elements are zero, the diagonal ones follow from (13) for $i = j$, when for $\overline{k, k}$ the value $\overline{\varkappa, \varkappa}$ corresponding to the random fluctuations is taken, see Appendix III and, e.g., [9] page 396.

It may be that in reciprocal observations between two stations the random fluctuations are not independent; the same applies to all observations made on the same station within a certain time interval. One might therefore use a stochastic model in which non-zero correlation is assumed between reciprocal observations and between observations made from the same station. The difficulty is of course to assess numerical values. The influence of random fluctuations on the weight-coefficients is analysed in Appendix III.

5 **Approximation methods; R. Finsterwalder's method nr. 2**

Up till now we have only discussed a "rigorous" solution of the problem, in so far as we have adhered to the chosen mathematical model and have applied the method of least squares rigorously. In Section 3 we did assume a relation that was not quite exact, but this was admissible for the practical applications considered.

A rigorous least squares adjustment of vertical angles for obtaining deviations of the vertical has not often been carried out in practice. The practical geodesist usually shrinks back from the large amount of computational work that is involved and uses a less elaborate approximation method. The principle is of course very sound, as the "last millimetre" has no practical significance and even a very "rigorous" theory cannot describe reality completely.

But in a theoretical investigation, an approximation method should only be used after indicating sharply what approximations are made, and this can usually only be done by examining the relationship between the rigorous and the approximate method. An approximation method that has been found rather intuitively may be excellent, but when its connections with the rigorous method have not been investigated, it remains dangerous to draw theoretical conclusions from it.

R. FINSTERWALDER has in [6], page 130, given a survey of the different methods for computing trigonometric heights and deviations of the vertical from vertical angle measurements. These methods are:

1. Simultaneous computation of heights and deviations of the vertical in one adjustment.
2. The usual adjustment of trigonometric levelling ignoring deviations of the vertical; in the second step of the adjustment these deviations are computed.
3. To begin with, deviations of the vertical are computed from reciprocal observations. The influence of these deviations on the observed vertical angles is computed, and the angles are corrected for this influence, after which the adjustment of heights is effected.
4. The net is arranged in profiles running North-South and East-West, so that "vertical traverses" are formed, which at each end are closed on astronomical observations.

Method nr. 1 has been treated in Section 3; method nr. 2 will be discussed in this section and method nr. 3 in Section 6. Method nr. 4 will not be discussed because it is only a special case of method nr. 1 as treated in Section 3.

Method nr. 2 has been described in detail in [3] and [5]. As remarked before, one starts in this method adjusting the net ignoring the deviations of the vertical. The coefficient of refraction is supposed to be known. One obtains a number of observation equations of the form:

$$\underline{H}_j - \underline{H}_i = \underline{h}_{ij} + \underline{\varepsilon}_{ij} \quad \ldots \quad (21)$$

in which $\underline{h}_{ij}$ has been corrected for refraction. The indices i and j are here again point numbers. From this first height adjustment one finds the corrections $\underline{\varepsilon}_{ij}$. These corrections are considered as observations of the influence of $\underline{\theta}_{ij}$ on the observed height difference $\underline{h}_{ij}$.

If we call the corresponding correction to the vertical angle concerned $\underline{\varepsilon}'_{ij}$, we have:

$$\underline{\varepsilon}_{ij} = \frac{S_{ij}\underline{\varepsilon}'_{ij}}{\rho} \quad \ldots \quad (21a)$$

where S_{ij} is defined by (4).

One consequently puts:

$$\underline{\xi}_i \cos \psi_{ij} + \underline{\eta}_i \sin \psi_{ij} = \underline{\varepsilon}'_{ij} \quad \ldots \quad (22)$$

Every observed vertical angle furnishes an equation (22). If there are a sufficient number of equations, one can compute all quantities ξ_i and η_i from them. If there are more equations than unknowns, the observations $\underline{\varepsilon}'_{ij}$ will again receive corrections; one then obtains the second step of the adjustment, expressed by the observation equations:

$$\xi_i \cos \psi_{ij} + \eta_i \sin \psi_{ij} = \underline{\varepsilon}'_{ij} + \underline{v}'_{ij} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

in which $\underline{v}'_{ij}$ are the corrections to be given in this second step. This adjustment gives values for ξ_i and η_i by means of which the originally observed vertical angles can be corrected for deviations of the vertical. The thus corrected angles are again used in a second adjustment of the heights only, and these final heights can be considered as the ellipsoidal heights.

In practice, the heights from the second height adjustment proved to differ only very slightly from these found in the first height adjustment. This was explained by saying that in a flat and symmetrical net the influences of different deviations of the vertical would cancel each other.

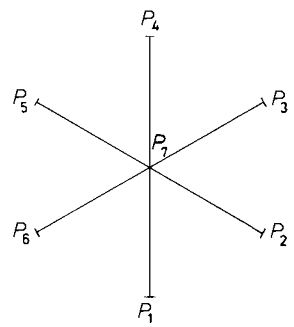

Figure 5

If in Fig. 5 vertical angles are measured in P_7 towards $P_1, \ldots, P_6$, the effects of a deviation of the vertical in P_7 on the observations for P_1 and P_4 will be equal but of opposite sign. The same applies to the angles towards P_2 and P_5 etc. However, this reasoning can never explain the indicated phenomenon because the deviations of the vertical and the observations in the points $P_1, \ldots, P_6$ have not been taken into account. Nevertheless it was found that the heights resulting from the second adjustment of heights were always practically the same as those from the first, even in irregularly shaped nets.

The real cause is that ξ_i and η_i are determined from (23). By means of the method of least squares one makes the corrections $\underline{v}'_{ij}$ "as small as possible", in practice they turn out to have the order of magnitude of the standard deviation of an angular observation. By correcting the original angles for deviations of the vertical, one obtains the following corrected observations for the height-differences:

$$(\underline{h}_{ij})_{\text{corrected}} = \underline{h}_{ij} - (\xi_i \cos \psi_{ij} + \eta_i \sin \psi_{ij})\, S_{ij}$$

or with (23):

$$(\underline{h}_{ij})_{\text{corrected}} = \underline{h}_{ij} - S_{ij}\, (\underline{\varepsilon}'_{ij} + \underline{v}'_{ij})$$

or putting $S_{ij} \underline{v}'_{ij} = \underline{v}_{ij}$:

$$(\underline{h}_{ij})_{\text{corrected}} = \underline{h}_{ij} - \underline{\varepsilon}_{ij} - \underline{v}_{ij}$$

In the second height adjustment we get the following observation equations

$$\underline{H}_j - \underline{H}_i = (\underline{h}_{ij} - \underline{\varepsilon}_{ij} - \underline{v}_{ij}) + \underline{e}_{ij} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

$\underline{e}_{ij}$ being the least squares-corrections resulting from this second adjustment.

We now use our general notation again, and write (24) as the l-th equation of a system of observation equations:

$$A^l_\alpha \underline{x}^\alpha = (\underline{h}^l - \underline{\varepsilon}^l - \underline{v}^l) + \underline{e}^l$$

Let the matrix of weights have elements g_{lm}; the normal equations are:

$$g_{lm}A^l_\alpha A^m_\beta \underline{x}^\alpha = g_{lm}A^m_\beta \,(\underline{h}^l - \underline{\varepsilon}^l - \underline{v}^l)$$
$$= g_{lm}A^m_\beta \,(\underline{h}^l - \underline{v}^l) - g_{lm}A^m_\beta \,\underline{\varepsilon}^l$$

Because of (21), $\underline{\varepsilon}^l$ is the correction that was given to $\underline{h}^l$ as a result of the first height adjustment. The second height adjustment is done with exactly the same matrices $||A^l_\alpha||$ and $||g_{lm}||$. Consequently, according to a well-known formula from the theory of least squares (see e.g. [12], page 145) we have:

$$g_{lm}\, A^m_\beta \underline{\varepsilon}^l = 0$$

The normal equations are then:

$$g_{lm}A^l_\alpha A^m_\beta \underline{x}^\alpha = g_{lm}A^m_\beta \,(\underline{h}^l - \underline{v}^l)$$

In this formula, $\underline{v}^l$ has the order of magnitude of the standard deviation of $\underline{h}^l$, see [5], page 26; consequently the second height adjustment will give the same results x^α as the first height adjustment, within the limits of the accuracy of the observations.

Method nr. 2 gives results that agree rather well with a rigorous least squares solution, see [5] and Section 7 of this paper. However, it is very difficult to deduct this method from the rigorous one: indeed it has not been created on the base of the rigorous method by modifications on sharply indicated points, but by heuristic reasoning.

Starting from the rigorous method one can obtain an adjustment problem which only contains unknowns H_i: for this it suffices to eliminate all ξ_i and η_i from the observation equations. But by doing this one can never retain the same number of observation equations, because the number of equations decreases as the number of unknowns. The remaining equations are linear combinations of the original ones, and the right-hand sides will consist of linear combinations of the observations, so that one will get a so-called "Standard Problem IV": condition equations containing unknowns. This adjustment problem differs greatly from the one expressed by equation (21).

Another simplification is that in the right-hand sides of the equations (22), the quantities $\underline{\varepsilon}'_{ij}$ are by no means non-correlated. By equations (21a) they are connected to the corrections resulting from the adjustment of equations (21). An adjustment of equations (22) according to the method of observation equations using a diagonal weight matrix is theoretically wrong.

Summing up, one can say that method 2 is based on weak theoretical foundations which make it dangerous to draw far-reaching conclusions from it. The practical results however seem to be quite satisfactory.

6 The method of reciprocal observations

The method nr. 3 (according to R. Finsterwalder's classification) starts by computing the deviations of the vertical from reciprocal observations. The principle follows from Figure 6, where for simplicity refraction has been ignored.

The following equations can be established:

$$\underline{A} = \underline{B} + \gamma_{ij}$$

$$\underline{\theta}_{ij} + \frac{\pi}{2} - \underline{\beta}_{ij} = \pi - \left(\frac{\pi}{2} - \underline{\beta}_{ji} + \underline{\theta}_{ji}\right) + \gamma_{ij}$$

$$\underline{\theta}_{ij} + \underline{\theta}_{ji} = \underline{\beta}_{ij} + \underline{\beta}_{ji} + \gamma_{ij}$$

$$\xi_i \cos \psi_{ij} + \eta_i \sin \psi_{ij} + \xi_j \cos \psi_{ji} + \eta_j \sin \psi_{ji} = \underline{\beta}_{ij} + \underline{\beta}_{ji} + \gamma_{ij}$$

Introducing refraction and writing $\psi_{ji} = \psi_{ij} + \pi$ we obtain:

$$\xi_i \cos \psi_{ij} - \xi_j \cos \psi_{ij} + \eta_i \sin \psi_{ij} - \eta_j \sin \psi_{ij} = \underline{\beta}_{ij} + \underline{\beta}_{ji} + \left(1 - \frac{k}{\cos \beta_{ij}}\right) \gamma_{ij} \quad . \quad (25)$$

(see e.g. [8], page 58). It is clear that, here also, we have to have a datum point to which the ξ- and η-differences can be referred, as equations (25) can only furnish differences of deviations of the vertical.

In literature it is usually stated that this method can indeed furnish the above mentioned differences ξ_i and η_i, if only one has a sufficiently large number of equations (25). If in a net consisting of four points all possible vertical angles from and

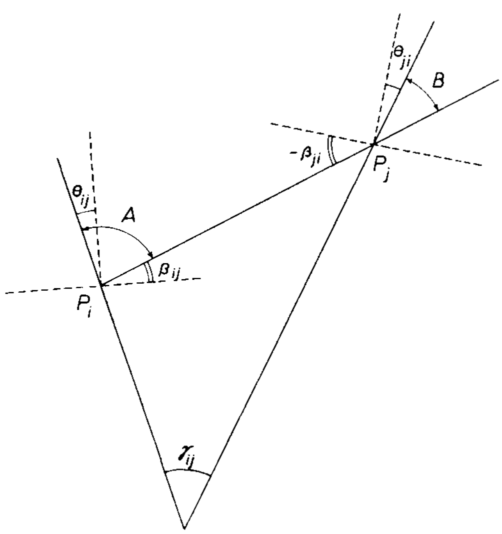

Figure 6

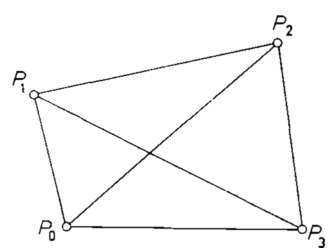

Figure 7

towards all points have been measured, one can establish 6 equations (25). One point is the datum point, the other ones have together $3 \times 2 = 6$ unknown components ξ_i and η_i, so that one obtains 6 equations in 6 unknowns.

If there are more than four points, there are also more unknowns, but if all possible vertical angles are measured the number of equations (25) increases more than the number of unknowns. One could then from an adjustment of this over-determined system find values for ξ_i and η_i.

Let us now consider a net of four points P_0, P_1, P_2 and P_3 (see Figure 7). The matrix of coefficients of equations (25) is in this case:

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	$-\cos\psi_{01}$			$-\sin\psi_{01}$		
2		$-\cos\psi_{02}$			$-\sin\psi_{02}$	
3			$-\cos\psi_{03}$			$-\sin\psi_{03}$
4	$\cos\psi_{12}$	$-\cos\psi_{12}$		$\sin\psi_{12}$	$-\sin\psi_{12}$	
5	$\cos\psi_{13}$		$-\cos\psi_{13}$	$\sin\psi_{13}$		$-\sin\psi_{13}$
6		$\cos\psi_{23}$	$-\cos\psi_{23}$		$\sin\psi_{23}$	$-\sin\psi_{23}$

We will examine the rank of this matrix. The rank is not altered if we multiply each row by the length of the corresponding side, and change the signs of the first three rows:

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	$s_{01}\cos\psi_{01}$			$s_{01}\sin\psi_{01}$		
2		$s_{02}\cos\psi_{02}$			$s_{02}\sin\psi_{02}$	
3			$s_{03}\cos\psi_{03}$			$s_{03}\sin\psi_{03}$
4	$s_{12}\cos\psi_{12}$	$-s_{12}\cos\psi_{12}$		$s_{12}\sin\psi_{12}$	$-s_{12}\sin\psi_{12}$	
5	$s_{13}\cos\psi_{13}$		$-s_{13}\cos\psi_{13}$	$s_{13}\sin\psi_{13}$		$-s_{13}\sin\psi_{13}$
6		$s_{23}\cos\psi_{23}$	$-s_{23}\cos\psi_{23}$		$s_{23}\sin\psi_{23}$	$-s_{23}\sin\psi_{23}$

If we now consider the net as being flat, and imagine a rectangular coordinate system whose origin is P_0, whose positive X-axis is directed northward and whose positive Y-axis is directed eastward, the elements of the thus obtained matrix can be interpreted as coordinate differences in this system and we can write the matrix as follows:

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4	x_2-x_1	$-(x_2-x_1)$		y_2-y_1	$-(y_2-y_1)$	
5	x_3-x_1		$-(x_3-x_1)$	y_3-y_1		$-(y_3-y_1)$
6		x_3-x_2	$-(x_3-x_2)$		y_3-y_2	$-(y_3-y_2)$

The matrix has now been reduced to a simple form and we can apply the Gauss algorithm to investigate its rank. This is done in Appendix IV: the rank turns out to be 5. This means that the matrix is singular and that the system represents only five independent equations with six unknowns. (In view of the fact that the right-hand sides of equations (25) have an arbitrary value, the system will as a rule be inconsistent.) Consequently, the statement that in a net of four points a determination of relative deviations of the vertical is possible by vertical angle measurement only, is false.

We can now continue by examining a net of five points (see Figure 8). If all possible vertical angles have been observed, there will be 10 equations.

If we indicate the equations by the numbers 0–1, 0–2, 0–3, . . ., 3–4, we have seen above that the matrix of equations 0–1, 0–2, 0–3, 1–2, 1–3 and 2–3 has the rank 5. By adding a point P_4 we get two additional unknowns ξ_4 and η_4, and four additional equations. To determine the unknowns we need 8 independent equations, but we have only 7 of them: according to the above the coefficients of equation 0–4, for instance, are linearly dependent on those of equations 0–1, 0–3, 1–3, 1–4 and 3–4 (in quadrangle $P_0P_1P_4P_3$) and the same applies to 3–4 with respect to 1–2, 1–3, 2–3, 1–4 and 2–4 (in quadrangle $P_0P_2P_4P_1$) so that both 0–4 and 3–4 are dependent on 1–4, 2–4 and the equations of quadrangle $P_0P_1P_2P_3$.

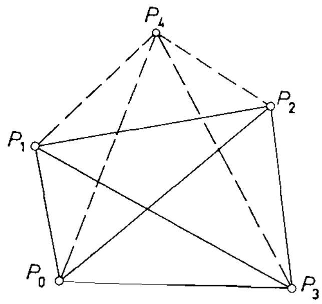

Figure 8

The extra observations do not make it possible to remove the singularity. The reasoning can easily be extended for nets with more points: equations (25) will always have a matrix whose rank is one less than the number of unknowns. Consequently the matrix of the normal equations will also be singular: its deficiency is one, and the unknowns cannot be uniquely determined. The method can only be used as an interpolation method. As will be seen in the following, we cannot introduce the coefficient of refraction as an unknown for the same reason as indicated in Section 3, so for application of this method it is necessary to know two components ξ or η relative to the datum point of the net, of which one might be said to serve for determining the coefficient of refraction and the other for reducing the number of unknowns.

The deduction of the method according to Figure 6 is as rigorous as required for numerical applications; nevertheless it is an approximation method. This is evident if one takes equations (5) for the observation from P_i to P_k and for the observation from P_k to P_i, and adds both equations. The unknowns H_i and H_k are then eliminated. Because in the cases considered here one can put $S_{ik} = S_{ki}$, the thus obtained equation must, after division by S_{ik}, be identical with (25). In the two equations that were added, the coefficient of k was linearly dependent on the other coefficients, consequently in the resulting equation this must also be the case, which proves the abovementioned impossibility of determining k as an unknown.

Every equation (25) is equivalent to the sum of two corresponding equations (5), so equations (25) contain only part of the information contained in the original equations (5). If refraction is known and if one component ξ or η in the net is known, the system (25) can furnish values for ξ_i and η_i, but the solution will not be complete; it can be considered as a first step.

In [8], a solution has been found from the normal equations resulting from equations (25), in spite of the singularity proved here. This was possible by putting one of the unknowns equal to zero (see [8], page 65) which was in this case approximately right. The observations were then corrected for the computed deviations of the vertical, after which the adjustment of heights was performed. The fact that this second adjustment was necessary was of course caused by the incompleteness of the first adjustment.

But exactly this incompleteness of the first step means that the values of adjusted observations and unknowns resulting from it are not the final values. In the second step of the adjustment they must again receive corrections. If the second step is to be correct one must consequently:

1. Compute corrections for $\underline{\beta}_{ij}$ from the corrections to $(\underline{\beta}_{ij}+\underline{\beta}_{ji})$ found in the first adjustment.
2. Compute the weight-coefficients of the corrected $\underline{\beta}_{ij}$ as well as those of the computed $\underline{\xi}_i$ and $\underline{\eta}_i$.

The observations corrected for deviations of the vertical are by no means uncorrelated, which makes the second step of the adjustment very complicated; it is a so-called "Standard Problem IV". Finally, if one realizes what is the number of conditions that are fulfilled in each step, it is easily seen that in the second step of the adjustment not all observation equations must be used but only half of them, namely one of each two from which an equation (25) is composed. However, in a rigorous adjustment this makes no difference.

Method nr. 3 is consequently an approximation method, in which the deviations of the vertical are found from an adjustment on too few conditions; in [8] the ellipsoidal heights are then found by an adjustment with a wrong matrix of weights.

7 Numerical applications

The theoretical considerations given in the previous sections have been established in connection with numerical computations having regard to the trigonometric levelling net measured by Dr. W. Hofmann around the valley of the river Isar. The observations have been published in [8], and the present author very gratefully acknowledges Dr. Hofmann's kind cooperation in furnishing additional data and information. Only height differences that had been observed reciprocally were used, and the following computations were carried out:

1. Adjustment according to Section 3, using the same weights as used in [8].
2. Adjustment considering the observations as being correlated by the correction for refraction.
3. Adjustment according to R. Finsterwalder's method nr. 2.
4. Adjustment according to Section 3, using weights computed from (13) for $i = k$ to allow for random fluctuations of refraction, and using different numerical values of the coefficient of refraction.
5. "Rigorous" adjustment of the results of item 4 to fit three astronomically determined deviations of the vertical in latitude.

7.1 In the first adjustment, the coefficient of refraction was treated as an unknown to begin with, so that the observation equations had exactly the form of (5). The normal equations were established, using the same weights as given in [8], page 55; for the observations $\underline{h}_{ij}$ the values computed by Dr. Hofmann were taken.

The unknown k got the last number, so that in solving the normal equations it would be the first to be computed. By reducing the normal equations according to the Gauss algorithm, it was seen that after eliminating all other unknowns the coefficient of k in the resulting equation was zero: the computation was done with five decimal places and the numerical value of the diagonal element in question was $3 \cdot 10^{-5}$, a deviation from zero that can be ascribed entirely to rounding off. The matrix of coefficients thus turned out to be singular.

After this, the term containing k in the equations (5) was transported to the right-hand side and so the observed height differences were corrected for refraction. For k, the value 0.2012 was taken, in accordance with the value given by Dr. HOFMANN. The observations corrected for refraction were considered as being uncorrelated; the weights assigned to them were again those of [8], page 55, so that the computations already executed could be used throughout. The approach of the problem thus was indentical to that of Section 3, because the value $k = 0.2012$ was just a value of the parameter, not an observation. The matrix of the normal equations was inverted and the unknowns computed, they are given in Table I in column 1. The estimate for the variance factor was:

$$\hat{\sigma}^2 = \frac{g_{ij}\varepsilon^i\varepsilon^j}{m-n} = 0.006969$$

In the chosen system of weights an observation on a side of 8.5 km had unit weight. The estimate of the standard deviation of the mean of a measured angle as computed from the angle observations was 1.6^{cc}, which in view of the large number of 18 observations for each angle can be considered a good approximation of the standard deviation. The observation of the height difference of unit weight thus has a variance

$$\frac{(8.5)^2 \cdot 10^6}{(\rho^{cc})^2}(1.6)^2 = 0.000456 \text{ m}^2$$

There were $28-15 = 13$ supernumerous observations, consequently $\frac{\hat{\sigma}^2}{\sigma^2}$ has an $\underline{F}_{13,\infty}$ probability distribution (see [11]). With the aid of this distribution we can test statistically the hypothesis that no model error is present in our setup of the adjustment problem. The F-value found here is $0.006969/0.000456 = 15.3$, whereas a one-sided test on the 5% level of significance has a critical value

$$F_{0.95\,;\,13,\infty} \approx 1.72$$

and on the 5‰ level:

$$F_{0.995\,;\,13,\infty} \approx 2.30$$

so that the hypothesis must be rejected because the F-value found is very significantly too high, even on the 5‰ level. It is highly probable that the assumption of a constant k is the model error in question, as there does not seem to be any other possible influence that could explain the high F-value found.

7.2 The problem was also treated according to Section 4, taking into account the correlation between the observations corrected for refraction. The weight-coefficients were computed according to (13). The standard deviation of k was rather arbitrarily assumed to be 10% of the numerical value of k, so the values used were:

$$k = 0.2012; \quad \sigma_k = 0.02; \quad \sigma_k{}^2 = 0.0004$$

The standard deviation of an observed angle was according to [8] 1.6^{cc}. The weight-coefficients were expressed in dm^2, which made it convenient to take unity as variance factor. For solving the adjustment problem, the thus obtained matrix of weight-coefficients had to be inverted to give the matrix of weights. This inversion of a 28×28 matrix was simplified by a transformation of the observation equations: the two equations pertaining to the same height difference were subtracted by which the resulting right-hand part did not contain the coefficient of refraction any more. The thus obtained 14 equations had non-correlated right-hand members. Half of the original equations were again included in the system, which as a consequence was equivalent to the original system. The matrix of weight-coefficients of the new system consisted of four sub-matrices of which only the lower right-hand one was not a diagonal matrix. Therefore the relationship of Frobenius–Schur (see e.g. [1], page 20) could be used in the inversion.

The matrix of weights thus being known, the normal equations were established and their matrix inverted. The result of the inversion proved to be insufficiently accurate: the product of the matrix and its computed reciprocal was not equal to unity with sufficient approximation. Therefore an iteration process was used for improving the reciprocal matrix (see e.g. [2], page 120). This numerical difficulty is explained by the fact that the matrix of weight-coefficients consisted of the sum of a matrix with relatively small elements (from the angle observations) and a matrix with relatively large elements (from refraction).

The latter matrix was singular (its rank being one), and by its preponderance the determinant of the matrix of weight-coefficients was, popularly speaking, small compared to the value of its elements. This cause of instability was also present in the matrix of weights. The inversion of the matrix of weight-coefficients was not completely checked by "back-multiplication" to give the unit matrix or by re-inversion, so that the instability did not manifest itself until the inversion of the matrix of normal equations.

The values found for the unknowns are given in Table I, column 2. According to Section 4, solutions 1 and 2 should be the same; the differences are due to the fact that in solution 1 the weights were determined with fewer decimals than in solution 2, and to the instability of the just mentioned inversion. Solution 2 gave as estimate for the variance factor

$$\hat{\sigma}^2 = 15.5$$

whereas $\sigma^2 = 1$, so that this solution generates practically the same $\underline{F}_{13,\infty}$ value as solution 1.

7.3 Heights and deviations of the vertical were computed by R. FINSTERWALDER's method nr. 2, in which for the second step of the adjustment (equations (23)) equal weights were used. No "ellipsoidal" heights were computed. The results are given in Table I, column 3. The coefficient of refraction used was $k = 0.2012$.

7.4 Weight-coefficients were computed allowing for random fluctuations of the coefficient of refraction. The formula used was formula (3) of Appendix III, which corresponds to formula (13) for $i = j$, if $\overline{\varkappa, \varkappa}$ is substituted for $\overline{k, k}$. The standard deviations assumed were:

$$\sigma_\beta = 1.6^{cc}; \quad \sigma_\varkappa = 0.014$$

The value $\sigma_\varkappa = 0.014$ is in accordance with values found in literature. The weight-coefficients were expressed in dm^2; the variance factor was taken to be $\sigma^2 = 1$. The weights were the reciprocals of the weight-coefficients; they ranged from 0.35 to 20.53.

The solution was identical to solution 1, except for the weights, but the different weights proved, as usual, to make little difference in the value found for the unknowns. The stochastic model turned out to be much more satisfactory than the one used in solution 1, where all variance was assumed to be caused by the angle measurements proper. The estimate for the variance factor was

$$\hat{\sigma}^2 = 1.43$$

thus generating an $\underline{F}_{13,\infty}$ value that was not significantly too high.

The unknowns were computed both for $k = 0.2012$ and for $k = 0.13$; both solutions gave the same $\hat{\sigma}^2$ and the same corrections to the observations. The unknowns for $k = 0.2012$ are given in Table I, together with their standard deviations (column 4 and 6). Starting from this solution we can compute the value of k for which the sum of squares of the deviations of the vertical is a minimum. If we put

$$k = 0.2012 + \Delta k$$

and call the solution sought: y^a, and the solution for $k = 0.2012$: x^a, we must have, according to (11):

$$\begin{aligned} y^a &= x^a + b^a \Delta k \\ \Sigma(y^a)^2 &= \Sigma(x^a + b^a \Delta k)^2 \\ &= \Sigma(x^a)^2 + 2\Sigma b^a x^a \Delta k + \Sigma(b^a)^2(\Delta k)^2 \end{aligned}$$

It is evident that the extreme value of this function is a minimum, which occurs when:

$$\Sigma(b^a)^2 \Delta k + \Sigma b^a x^a = 0$$

$$\Delta k = -\frac{\Sigma b^a x^a}{\Sigma(b^a)^2}$$

In this case Δk proved to be -0.0229, so that the value of k which corresponds to "no extra curvature" is $0.2012 - 0.0229 = 0.1783$. The values of the unknowns for $k = 0.1783$ are given in Tabel I, column 5. The solutions for $k = 0.2012$ and $k = 0.1783$ are pictured in Figure 9, together with the vector construction which illustrates the connection between the two solutions.

7.5 In the net under consideration, the differences in latitude between the datum point and three other points had been determined astronomically. This provided a possibility to find k from a rigorous adjustment. The stations in question were 1,2 and 5 (see Figure 9).

If we denote the values of the components as determined by adjustment nr. 4 by $\underline{\xi}_1^t$, $\underline{\xi}_2^t$ and $\underline{\xi}_5^t$, the corresponding astronomically determined components by $\underline{\xi}_1^a$, $\underline{\xi}_2^a$ and $\underline{\xi}_5^a$, and the relevant coefficients by b_6, b_7 and b_{10} (in accordance with the numbering system given in Section 3), we can establish the following condition equations containing the unknown $\Delta\underline{k}$:

$$(\underline{\xi}_1^t+\underline{\varepsilon}_1^t)+b_6\,\Delta\underline{k} = \underline{\xi}_1^a+\underline{\varepsilon}_1^a$$

$$(\underline{\xi}_2^t+\underline{\varepsilon}_2^t)+b_7\,\Delta\underline{k} = \underline{\xi}_2^a+\underline{\varepsilon}_2^a$$

$$(\underline{\xi}_5^t+\underline{\varepsilon}_5^t)+b_{10}\Delta\underline{k} = \underline{\xi}_5^a+\underline{\varepsilon}_5^a$$

in which the ε's represent least-squares corrections. By introducing "compound" observations $\underline{t}_i$, the adjustment problem was reduced to a case of observation equations with correlated observations. By applying the law of propagation of weight-coefficients to:

$$\underline{t}_i = \underline{\xi}_i^a-\underline{\xi}_i^t$$

the weight-coefficients of $\underline{t}_i$ were found; for the trigonometrically determined components the weight-coefficients resulting from adjustment nr. 4 were used, for the astronomically determined components a standard deviation of 1^{cc} was introduced (see [8], page 65). The matrix of weights was found by inverting the matrix of weight-coefficients, whereafter the one normal equation was easily established and solved. The quantities $\underline{\xi}_i^t$ had been referred to $k = 0.2012$, and the value of Δk was found to be $\Delta k = 0.0004$, so that the result was $k' = 0.2016$; its standard deviation as computed from the adjustment was $\sigma_k = 0.0062$. The estimate $\hat{\sigma}^2$ of the variance factor gave rise to an $\underline{F}_{2,\infty}$ value of 1.59, which is not significantly too high, so that no model errors are manifest. As a result of this adjustment, all values H, ξ and η computed received corrections which were computed by the appropriate rules of least squares theory; the final values are given in Table I, column 7. They contain the influence of Δk, which explains why some of their standard deviations (column 8) are larger than the corresponding ones in column 6. It appears that the standard deviations of the unknowns $\underline{\xi}_i$ and $\underline{\eta}_i$ are very large compared to their size, so that the computed values have only a limited significance.

For comparison, the values of the unknowns found by Dr. Hofmann are given in Table I, column 9.

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TABLE 1

	Solution 1	Solution 2	Finster-walder	Solution 4			Final	Standard	Hofmann
	k=0.2012	k=0.2012	k=0.2012	k=0.2012	k=0.1783	Stand. dev.	solution 5	deviations	k=0.2012
	1	2	3	4	5	6	7	8	9
H_1-H_0	+204.011	+204.017	+204.029	+204.022	+203.072	2.9 cm	+204.047	2.1 cm	+204.022
H_2-H_0	−139.486	−139.475	−139.438	−139.493	−139.410	3.5	−139.454	2.8	−139.425
H_3-H_0	+ 90.874	+ 90.893	+ 91.043	+ 90.869	+ 90.998	4.5	+ 90.896	5.7	+ 90.923
H_4-H_0	−110.802	−110.775	−110.553	−110.794	−110.607	5.4	−110.816	7.9	−110.723
H_5-H_0	−186.641	−186.631	−186.598	−186.633	−186.578	2.9	−186.655	3.1	−186.636
ξ_0^{cc}	0	0	0	0	0	0	0	0	0
ξ_1	− 3.5	− 1.9	− 6.4	− 1.3	+ 10.8	4.8cc	+ 2.7	0.98cc	+ 0.4
ξ_2	+ 10.9	+ 11.6	+ 7.6	+ 10.3	+ 22.4	5.3	+ 15.8	0.98	+ 14.5
ξ_3	+ 1.4	+ 2.6	+ 1.3	+ 0.1	+ 6.5	5.2	+ 6.3	3.9	+ 2.5
ξ_4	− 4.7	− 5.8	− 4.1	− 8.2	− 13.0	6.7	− 1.5	5.8	− 0.1
ξ_5	− 16.4	− 17.2	− 9.2	− 19.4	− 22.8	5.3	− 10.2	0.98	− 9.9
η_0	0	0	0	0	0	0	0	0	0
η_1	− 3.7	− 3.7	− 1,9	− 3.0	− 3.2	4.9	− 4.8	4.8	0.0
η_2	+ 21.2	+ 19.6	+ 22.1	+ 21.3	+ 11.5	4.6	+ 19.7	5.4	+ 19.8
η_3	+ 19.0	+ 16.6	+ 12.8	+ 17.2	− 1.1	5.1	+ 15.8	6.3	+ 17.3
η_4	+ 36.6	+ 33.8	+ 26.5	+ 39.8	+ 17.0	5.4	+ 39.1	8.9	+ 19.7
η_5	+ 22.3	+ 20.3	+ 15.7	+ 20.8	+ 8.7	4.4	+ 20.0	6.0	+ 13.9

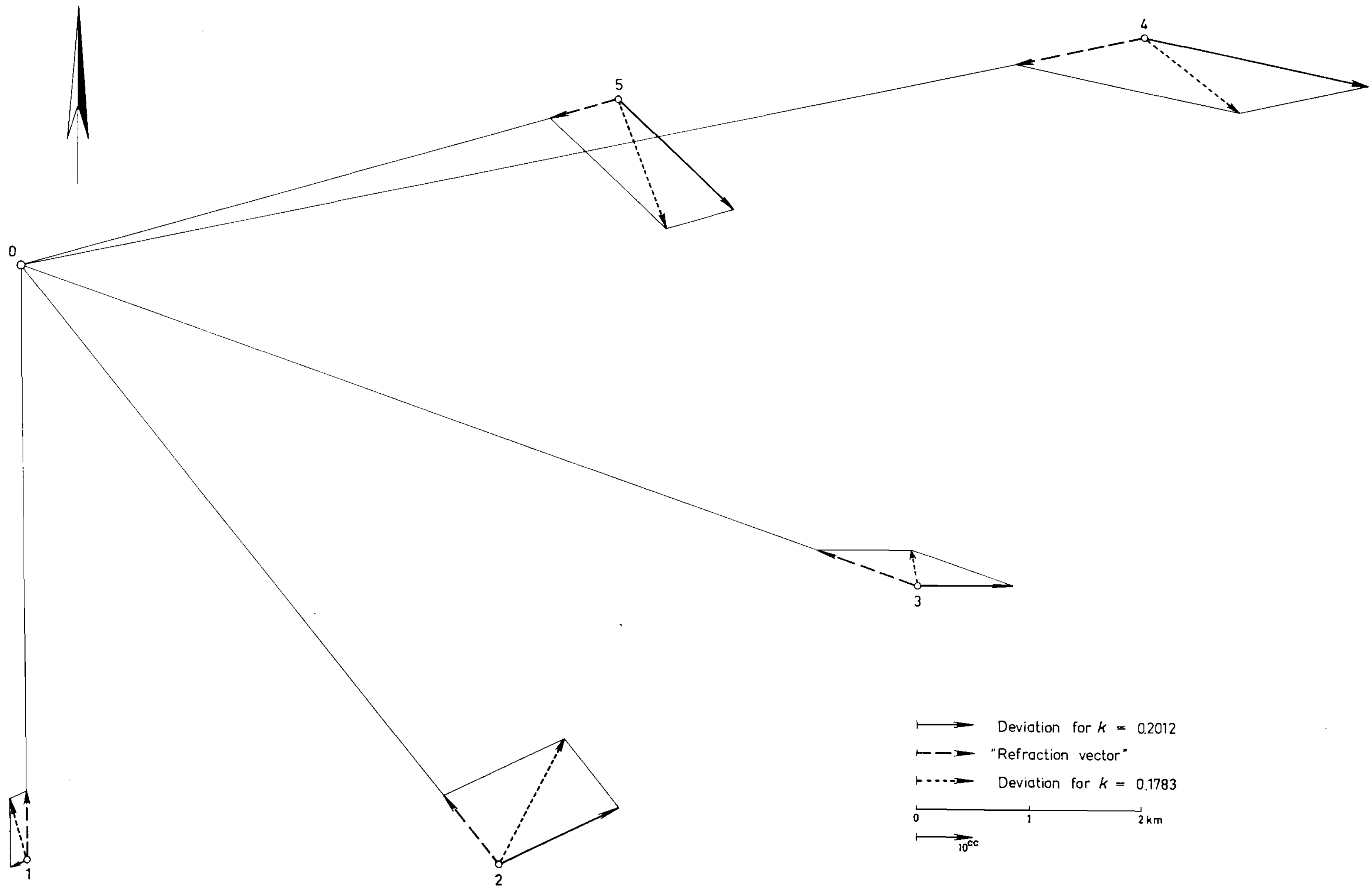
0
1
2
3
4
5
Deviation for k = 0.2012
"Refraction vector"
Deviation for k = 0.1783
0
1
2 km
10cc

Figure 9

APPENDIX I

The equation (7b) to be investigated can be written

$$\frac{-s_{i0}^2+s_{j0}^2+2s_{i0}S_{ij}\cos(\psi_{ij}-\psi_{i0})}{R\cos^2 B}=\frac{s_{ij}^2}{R_{ij}\cos^2\beta_{ij}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7c)$$

The numerator on the left-hand side can be transformed as follows:

$$-s_{i0}^2+s_{j0}^2+2s_{i0}s_{ij}\cos(\psi_{ij}-\psi_{i0})+2(S_{ij}-s_{ij})s_{i0}\cos(\psi_{ij}-\psi_{i0}) =$$
$$= +s_{ij}^2+2(S_{ij}-s_{ij})s_{i0}\cos(\psi_{ij}-\psi_{i0})$$

We now assume that $|\mathrm{tg}\beta_{ij}|_{\max}=0.1$, and consequently $\left(\frac{1}{\cos^2\beta_{ij}}\right)_{\max}=1.01$;

$$(S_{ij}-s_{ij})_{\max}=0.01\,s_{ij}$$

For nets used in practice one can assume $(s_{i0})_{\max}=2s_{ij}$. In these circumstances we can choose $\frac{1}{\cos^2 B}=1.005$, and the boundaries of the left-hand member of (7c) can be indicated as follows:

$$\frac{1}{R}\cdot 1.005\,(s_{ij}^2+0.04\,s_{ij}^2)=\frac{1}{R}\cdot 1.045\,s_{ij}^2$$
$$\frac{1}{R}\cdot 1.005\,(s_{ij}^2-0.04\,s_{ij}^2)=\frac{1}{R}\cdot 0.965\,s_{ij}^2$$

Because at 50° latitude the radii of curvature of the ellipsoid in meridian and prime vertical differ about 20 km, we can put:

$$(R_{ij})_{\max}=R+10\text{ km}$$
$$(R_{ij})_{\min}=R-10\text{ km}$$

So $$\frac{R}{R_{ij}}=\frac{R}{R\pm 10}=1\pm\frac{10}{R}\approx 1\pm 0.002$$

We can thus put the maximum and minimum values of the right-hand member of (7c) as follows:

Maximum: $\frac{1.01\,s_{ij}^2}{R_{ij}}$ or $\frac{1.012\,s_{ij}^2}{R}$

Minimum: $\frac{s_{ij}^2}{R_{ij}}$ or $\frac{0.998}{R}\,s_{ij}^2$

The maximum difference between left- and right-hand members is thus seen to be $\pm\frac{0.047}{R}\,s_{ij}^2$ or about 5% of the value of the right-hand member. This value 5% will seldom occur in practice; as an example the following table gives the results for the net measured by Dr. Hofmann; the numbers under the heading $b^{\alpha}A^{i}_{\alpha}$ represent the left-hand sides of equations (7), those under the heading $\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}$ the right-hand sides.

	$b^a A_a^i$	$\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}$		$b^a A_a^i$	$\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}$		$b^a A_a^i$	$\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}}$
1	2.1957	2.1990	11	1.3967	1.3820	21	11.9553	11.9241
2	3.6423	3.6395	12	3.6471	3.6395	22	8.1694	8.1405
3	5.6313	5.6167	13	3.7008	3.6997	23	1.7492	1.7403
4	8.1640	8.1408	14	1.5703	1.5723	24	1.7426	1.7404
5	2.3842	2.3802	15	1.5810	1.5723	25	2.0080	2.0163
6	2.2011	2.1988	16	5.3884	5.3712	26	3.6999	3.6987
7	5.7521	5.7450	17	5.6365	5.6166	27	5.7530	5.7447
8	11.9452	11.9249	18	2.0300	2.0163	28	2.3909	2.3803
9	5.3861	5.3715	19	2.1807	2.1827			
10	1.3764	1.3820	20	2.1883	2.1826			

APPENDIX II

In this appendix, matrix notation will be used. Matrices and vectors are represented, respectively, by capitals and lower case in bold type, scalars by Greek letters. The transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^{\mathsf{T}}$; stochastic vectors are underscored.

Suppose we have a number of correlated observations; their matrix of weight-coefficients $\mathbf{C}$ can be written as the sum of a symmetric matrix $\mathbf{Q}$ and a symmetric matrix of rank one. The matrix of rank one can be written as a vector postmultiplied by its transpose:

$$\mathbf{C} = \mathbf{Q}+\mathbf{q}\mathbf{q}^{\mathsf{T}} \quad \ldots \quad (1)$$

There is a theorem on the reciprocal of the sum of two matrices one of which has the rank one, which theorem is proved e.g. in [1]. In our case we have, according to this theorem:

$$\mathbf{C}^{-1} = \mathbf{Q}^{-1}-\gamma\, \mathbf{Q}^{-1}\mathbf{q}\mathbf{q}^{\mathsf{T}}\mathbf{Q}^{-1}$$

in which

$$\gamma = \frac{1}{1+\alpha} \text{ and } \alpha = \mathbf{q}^{\mathsf{T}}\mathbf{Q}^{-1}\mathbf{q} \quad \ldots \quad (2)$$

$\mathbf{C}^{-1}$ is the matrix of weights. If we call $\mathbf{Q}^{-1}$: $\mathbf{G}$, we have:

$$\mathbf{C}^{-1} = \mathbf{G}-\gamma\mathbf{G}\mathbf{q}\mathbf{q}^{\mathsf{T}}\mathbf{G}$$

By putting:

$$\sqrt{\gamma}\mathbf{G}\mathbf{q} = \mathbf{r} \quad \ldots \quad (3)$$

we see that the matrix of weights can be written in essentially the same form as the matrix of weight-coefficients:

$$\mathbf{C}^{-1} = \mathbf{G}-\mathbf{r}\mathbf{r}^{\mathsf{T}} \quad \ldots \quad (4)$$

Let us now consider an adjustment problem, in which the observations have a matrix of weight-coefficients $\mathbf{C}$ as meant in (1). We write the problem in the form of observation equations:

$$\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{f}}+\underline{\mathbf{v}} \qquad (\underline{\mathbf{v}} = \text{correction})$$

Because of (4), the normal equations are:

$$\mathbf{A}^{\mathsf{T}}(\mathbf{G}-\mathbf{r}\mathbf{r}^{\mathsf{T}})\mathbf{A}\underline{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}(\mathbf{G}-\mathbf{r}\mathbf{r}^{\mathsf{T}})\underline{\mathbf{f}}$$

$$(\mathbf{A}^{\mathsf{T}}\mathbf{G}\mathbf{A}-\mathbf{A}^{\mathsf{T}}\mathbf{r}\mathbf{r}^{\mathsf{T}}\mathbf{A})\underline{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}\mathbf{G}\underline{\mathbf{f}}-\mathbf{A}^{\mathsf{T}}\mathbf{r}\mathbf{r}^{\mathsf{T}}\underline{\mathbf{f}} \quad \ldots \quad (5)$$

If we put

$$\mathbf{A}^{\mathsf{T}}\mathbf{r} = \mathbf{l} \quad \ldots \quad (6)$$

we can write (5) as:

$$(\mathbf{A}^{\mathsf{T}}\mathbf{G}\mathbf{A}-\mathbf{l}\mathbf{l}^{\mathsf{T}})\underline{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}\mathbf{G}\underline{\mathbf{f}}-\mathbf{A}^{\mathsf{T}}\mathbf{r}\mathbf{r}^{\mathsf{T}}\underline{\mathbf{f}}$$

In general, det $(\mathbf{A}^{\mathsf{T}}\mathbf{G}\mathbf{A}-\mathbf{l}\mathbf{l}^{\mathsf{T}}) \neq 0$, so:

$$\underline{\mathbf{x}} = (\mathbf{A}^{\mathsf{T}}\mathbf{G}\mathbf{A}-\mathbf{l}\mathbf{l}^{\mathsf{T}})^{-1}(\mathbf{A}^{\mathsf{T}}\mathbf{G}\underline{\mathbf{f}}-\mathbf{A}^{\mathsf{T}}\mathbf{r}\mathbf{r}^{\mathsf{T}}\underline{\mathbf{f}}) \quad \ldots \quad (7)$$

We now use the above-mentioned theorem again. If we put

$$\beta = -\mathbf{l}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{l} \quad \ldots \quad (8)$$

and

$$\delta = \frac{1}{1+\beta} \quad \ldots \quad (9)$$

(7) becomes:

$$\underline{\mathbf{x}} = \{(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1} + \delta(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{l}\mathbf{l}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\}\ (\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} - \mathbf{A}^{\mathrm{T}}\mathbf{r}\mathbf{r}^{\mathrm{T}}\underline{\mathbf{f}}) \quad \ldots \quad (10)$$

Now it is essential that in the special case of Section 4 we could write:

$$\mathbf{q} = \mathbf{A}\mathbf{b} \quad \ldots \quad (11)$$

Hence, with (3) and (6):

$$\left.\begin{aligned} \mathbf{r} &= \sqrt{\gamma}\mathbf{G}\mathbf{q} = \sqrt{\gamma}\mathbf{G}\mathbf{A}\mathbf{b} \\ \mathbf{l} &= \mathbf{A}^{\mathrm{T}}\mathbf{r} = \sqrt{\gamma}\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b} \\ \mathbf{l}\mathbf{l}^{\mathrm{T}} &= \gamma\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A} \end{aligned}\right\} \quad \ldots \quad (12)$$

Now (10) becomes:

$$\underline{\mathbf{x}} = \{(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1} + \delta\gamma\mathbf{b}\mathbf{b}^{\mathrm{T}}\}\ (\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} - \gamma\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}})$$

$$\underline{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} - \gamma\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} + \delta\gamma\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} - \delta\gamma^2\mathbf{b}[\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b}]\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}}$$

The part in square brackets is, according to (2):

$$\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b} = \mathbf{q}^{\mathrm{T}}\mathbf{G}\mathbf{q} = \alpha$$

Consequently:

$$\underline{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} - \gamma(1-\delta+\alpha\gamma\delta)\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}} \quad \ldots \quad (13)$$

(8) gives with (2), (11) and (12):

$$\beta = -\mathbf{l}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{l} = -\gamma\mathbf{b}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A}\mathbf{b}$$

$$\beta = -\gamma\mathbf{q}^{\mathrm{T}}\mathbf{G}\mathbf{q} = -\gamma\alpha$$

In (13) we have then, using (9):

$$1-\delta+\alpha\gamma\delta = 1-\delta-\beta\delta = 1 - \frac{1}{1+\beta} - \frac{\beta}{1+\beta} = 0$$

So that

$$\underline{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{G}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{G}\underline{\mathbf{f}}$$

This is just the solution which would have been found if a matrix of weights **G** had been used. Q.e.d.

APPENDIX III

If the coefficient of refraction is introduced as an observation, equations (5) of Section 2 can be written as follows:

$$\underline{H}_j - \underline{H}_i + \frac{S_{ij}}{\rho} \cos \psi_{ij} \underline{\xi}_i^{cc} + \frac{S_{ij}}{\rho} \sin \psi_{ij} \underline{\eta}_i^{cc} = \underline{h}_{ij} - \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} \underline{k} \; (+\underline{\varepsilon}_{ij}) \quad (1)$$

We now suppose that random fluctuations of refraction occur, and introduce a corresponding stochastic quantity $\underline{\varkappa}_{ij}$ (whose mean is zero). If we then replace the total "observational" part of (1) by $\underline{f}_{ij}$, we have:

$$\underline{f}_{ij} = \underline{h}_{ij} - \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} \underline{k} - \frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} \underline{\varkappa}_{ij} \quad \ldots\ldots\ldots\ldots \quad (2)$$

For brevity we write:

$$-\frac{s_{ij}^2}{2R_{ij}\cos^2\beta_{ij}} = c_{ij}$$

Further we replace the indexing system by a system in which the observations are successively numbered from 1 to m. We retain the indexes i and j, but, like in Section 3, i and j do not represent station numbers any more! In tabular form we can then write equations (2) as follows:

	h_1	h_2	.	.	.	h_m	k	$\varkappa_1$	$\varkappa_2$	.	.	.	$\varkappa_m$
$f_1 =$	1						c_1	c_1					
$f_2 =$		1					c_2		c_2				
.			.				.			.			
.				.			.				.		
.					.		.					.	
$f_m =$						1	c_m						c_m

We can introduce the following system of weight-coefficients in which some plausible assumptions of zero correlation are made:

	h_1	h_2	...	h_m	k	$\varkappa_1$	$\varkappa_2$	...	$\varkappa_m$
h_1	$\overline{h_1, h_1}$	0	...	0	0	0	0	...	0
h_2	0	$\overline{h_2, h_2}$	...	0	0	0	0	...	0
.	.	.	...	.	.	.	.	...	.
.	.	.	...	.	.	.	.	...	.
.	.	.	...	.	.	.	.	...	.
h_m	0	0	...	$\overline{h_m, h_m}$	0	0	0	...	0
k	0	0	...	0	$\overline{k, k}$	0	0	...	0
$\varkappa_1$	0	0	...	0	0	$\overline{\varkappa_1, \varkappa_1}$	$\overline{\varkappa_1, \varkappa_2}$	...	$\overline{\varkappa_1, \varkappa_m}$
$\varkappa_2$	0	0	...	0	0	$\overline{\varkappa_1, \varkappa_2}$	$\overline{\varkappa_2, \varkappa_2}$	...	$\overline{\varkappa_2, \varkappa_m}$
.	.	.	...	.	.	.	.	...	.
.	.	.	...	.	.	.	.	...	.
.	.	.	...	.	.	.	.	...	.
$\varkappa_m$	0	0	...	0	0	$\overline{\varkappa_1, \varkappa_m}$	$\overline{\varkappa_2, \varkappa_m}$	...	$\overline{\varkappa_m, \varkappa_m}$

It is easy now to apply the law of propagation of weight-coefficients (which corresponds to the execution of two matrix multiplications). We see (in ordinary notation):

$$\overline{f^i, f^j} = \overline{h^i, h^j} + c_i \cdot c_j \cdot \overline{k, k} + c_i \cdot c_j \cdot \overline{\varkappa_i, \varkappa_j}$$

It has been shown in Section 4 and Appendix II that the term containing $\overline{k, k}$ can be left out without affecting the results. If we assume that $\overline{\varkappa_i, \varkappa_j} = 0$ for $i \neq j$ and $\overline{\varkappa_i, \varkappa_i} = \overline{\varkappa_j, \varkappa_j} = \overline{\varkappa, \varkappa}$ we get:

$$\overline{f_i, f_j} = 0 \quad \text{when } i \neq j$$

$$\overline{f_i, f_i} = \overline{h_i, h_i} + (c_i)^2 \, \overline{\varkappa, \varkappa} \qquad (3)$$

which means that the effect of the random fluctuations is an increase in the main diagonal of the matrix of weight-coefficients.

APPENDIX IV

Investigation of the rank of the matrix (see e.g. [13], page 86):

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4	x_2-x_1	$-(x_2-x_1)$		y_2-y_1	$-(y_2-y_1)$	
5	x_3-x_1		$-(x_3-x_1)$	y_3-y_1		$-(y_3-y_1)$
6		x_3-x_2	$-(x_3-x_2)$		y_3-y_2	$-(y_3-y_2)$

Add (row 1 + row 2) to row 4, (row 1 + row 3) to row 5 and (row 2 + row 3) to row 6:

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4	x_2	x_1		y_2	y_1	
5	x_3		x_1	y_3		y_1
6		x_3	x_2		y_3	y_2

Now apply the Gauss algorithm:

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4		x_1		$y_2 - \frac{x_2}{x_1} y_1$	y_1	
5			x_1	$y_3 - \frac{x_3}{x_1} y_1$		y_1
6		x_3	x_2		y_3	y_2

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4				$y_2 - \frac{x_2}{x_1} y_1$	$y_1 - \frac{x_1}{x_2} y_2$	
5			x_1	$y_3 - \frac{x_3}{x_1} y_1$		y_1
6			x_2		$y_3 - \frac{x_3}{x_2} y_2$	y_2

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4				$y_2 - \frac{x_2}{x_1} y_1$	$y_1 - \frac{x_1}{x_2} y_2$	
5				$y_3 - \frac{x_3}{x_1} y_1$		$y_1 - \frac{x_1}{x_3} y_3$
6					$y_3 - \frac{x_3}{x_2} y_2$	$y_2 - \frac{x_2}{x_3} y_3$

	ξ_1	ξ_2	ξ_3	η_1	η_2	η_3
1	x_1			y_1		
2		x_2			y_2	
3			x_3			y_3
4				$y_2 - \frac{x_2}{x_1} y_1$	$y_1 - \frac{x_1}{x_2} y_2$	
5					$\frac{x_1 y_3 - x_3 y_1}{x_2}$	$y_1 - \frac{x_1}{x_3} y_3$
6					$y_3 - \frac{x_3}{x_2} y_2$	$y_2 - \frac{x_2}{x_3} y_3$

By the next step the element in the lower right-hand corner becomes:

$$\frac{y_2 x_3 - x_2 y_3}{x_3} - \frac{y_3 x_2 - x_3 y_2}{x_1 y_3 - x_3 y_1} \cdot \frac{y_1 x_3 - x_1 y_3}{x_3} = \frac{y_2 x_3 - x_2 y_3}{x_3} + \frac{y_3 x_2 - x_3 y_2}{x_3} = 0$$

The matrix is singular; its rank is 5.

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