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THEORY AND PRACTICE

OF

PENDULUM OBSERVATIONS AT SEA

BY

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TECHNISCHE BOEKHANDEL EN DRUKKERIJ J. WALTMAN JR. - DELFT
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PREFACE.

The difficulties experienced during the pendulum observations in the west and north part of Holland, where the ground is so subject to microseisms that no satisfactory results can be obtained, gave rise to the elaboration of a method which would make it possible to eliminate these disturbances. The method is described in: „Observations de pendule dans les Pays Bas, 1913-1921", par F. A. VENING MEINESZ; J Waltman Jr., Delft; in its simplest form it consists in swinging two pendulums in the same plane on the same support with equal amplitudes and in opposite phase. The mean of the two periods is free from the effect of any small movement of the apparatus. The old four pendulum Sterneck apparatus of the Stückrath type could be used for carrying out this method.

The good results obtained, led to a study of the possibility of determining gravity in some analogous way on board a moving ship. Of course it would be necessary to eliminate the effect of much larger movements but it was possible to change the method in such a way that this could be done, provided at least that the ship's movements are small. This method is described in the following publication.

Small ship's movements can be obtained by carrying out the observations on board of a submerged submarine; the writer is indebted to Prof. Dr. F. K. TH. VAN ITERSON for first mentioning this idea to him. Besides reducing the rolling and pitching of the vessel, a submarine has the advantage that during submergence the ship is not subject to vibrations, a circumstance which is of vital importance for the possibility of swinging pendulums.

For carrying out the method it was necessary to provide the Stückrath apparatus with a photographic recording apparatus. This was constructed at the Meteorological Institute at de Bilt (Holland) by its chief mechanic L. M. VAN REST in the summer of 1923.

In the autumn of 1923 the Dutch Geodetic Commission authorised the writer to make a voyage in a submarine of the Royal Dutch Navy, the K.II, from Holland to Java by way of the Suez canal, in order to put the method to a test. The voyage gave satisfactory results; it appeared possible to get an accuracy corresponding to a mean error of 1 : 200.000—1 : 300.000 in the gravity.

The results of this voyage which were published in: "Observations de pendule sur la mer, 1923, publication provisoire", gave rise to the devising of a new pendulum apparatus more specially adapted to the method, which was again constructed by Mr. VAN REST.

The apparatus was put to the test in the autumn of 1925 during a voyage on board the Dutch submarine H^r. M^s. K. XI from Holland to Port-Said. The apparatus reduces the time required to compute an observation from several days to a few hours.

During the spring of 1926 a further improvement was made by suspending the apparatus in gimbals and from May till Dec. 1926 a third voyage was carried out, this time on board of H^r. M^s. K. XIII, going from Holland to Java by way of Panama („Détermination de la pesanteur en mer, publication provisoire"). The final publications of the results of this and the former voyage will soon appear.

In 1928 a last alteration was made by the construction at de Bilt of a new recording apparatus, which brings the apparatus to its final shape as described in this publication. Arrangements have been made with the NEDERLANDSCHE SEINTOESTELLEN FABRIEK at Hilversum (Holland) for undertaking the construction of further copies of this apparatus.

Thanks are due to the Netherlands Navy for making these extensive investigations at sea possible, which led to the elaboration of method and apparatus.

Thanks are likewise due to Prof. VAN EVERDINGEN, Director of the Meteorological Institute at de Bilt for allowing the construction of the apparatus at the Institute, to Dr. C. SCHOUTE, Adj. Director of the Institute for his valuable advice during the construction and to Mr. L. M. VAN REST, who carried it out and gave his time, his care and his ability to the subject.

CONTENTS.

Figures.

ERRATA.

page 22 put A_2 near right upper angle of figure,

page 42 change φ in ψ in the figure,

page 52 insert at the end of line 18, after „put down": parallel to the knife-edges of the other pendulums and

page 60 insert at the end of line 5, after „planes": and parallel to each other

page 61 insert a new line at the end of § 2, B:
If the adjustment which can thus be obtained is not sufficient, we can make use of the possibility of turning the normal position of the knife-edge about a vertical axis by means of the adjusting screws of the slow lifting levers.

Summary of the contents.

To facilitate the use of this publication, a short summary will be given of the contents.

The first chapter contains the theory of the method for determining gravity at sea by means of pendulum observations. The results of these theoretical considerations, i. e. the formulae for the corrections and reductions which have to be applied to the observed pendulum-period, are put together in § 8. § 9 indicates the principal arrangement of the new apparatus and the next paragraphs give some considerations about the most practical initial conditions for the pendulum amplitudes and phases in case this apparatus is used; sea-observations and land-observations have to be treated separately. The conclusions are repeated in Chapter II, which treats of the execution of the observations.

The first chapter is followed by several appendices: Appendix II considers the case of observations carried out with a four-pendulum Sterneck apparatus. As the observations and computations are troublesome with this apparatus and take much more time, this case is not treated in detail.

The other appendices contain further theoretical researches, which treat of problems related to the subject: The first shows how the fundamental equations may be solved more elegantly by means of imaginary quantities; the third gives some details about negligible effects of the ship's movements and the last treats the problem of three pendulums swinging together on the apparatus while mounted on a fixed support, so that only the sway of the apparatus interferes with the pendulum movement.

Chapter II gives a description of the apparatus, of the way to adjust it, of the practical execution of the observations and a short note on the determination of the sway.

Chapter III contains the methods of computation. The first five paragraphs treat of the determination of the observed pendulum period and of the reductions and corrections which have to be applied; § 5 gives an example of the complete reduction.

§ 6 is an appendix which contains a short summary of the methods of computation, which are necessary if the observations are made with the four-pendulum Sterneck apparatus.

The last paragraphs of this chapter treat of the determination of the result for gravity and of the reductions which have to be applied to this result.

According to the above summary he who wishes only to study the practical execution of the observations and computations may confine himself to § 1, the first page of § 2, § 8 and § 9 of the first chapter, to chapter II and to chapter III with the exception of § 6 The theory of the method is given by chapter I, while the appendices I, III and IV of this chapter contain investigations on some related problems. Appendix II of chapter I and § 6 of chapter III refer to the use of a four-pendulum Sterneck apparatus.

CHAPTER I.

Theoretical Considerations and Formulae.

§ 1. *General Remarks.*

The principal difficulty, which has to be met in making pendulum observations at sea, is of course the difficulty, caused by the disturbances to the pendulum movement by the movements of the ship. It is true that the angular movements caused by the rolling or the pitching, may be lessened by suspending the pendulum apparatus in gimbals. as is done for the apparatus, which will be described in this publication, but they cannot be removed completely and the translations of the apparatus cannot be taken away at all. In order to be able to solve the problem, it is necessary to assume that the ship's movements are so small and gradual, that the connection between the knife-edge of the pendulum and the agate plane on which it swings is not interrupted during the observation.

An important point in this regard is, that the experience during a great many observations and experiments has shown, that there need be no fear for microscopic shifting of the knife-edge, which might go on imperceptibly and which would make the following method illusory. There has never been found a trace of such a movement. If the movement of the apparatus becomes too strong, the knife-edge slips suddenly an appreciable amount, which is shown at once by the photographic record. There have however been made only a few experiments on board of a ship, which was subject to vibrations caused by the engines. The results gave the impression that this increases the tendency of the knife-edge to slip; they were too few, however, to prove that the above statement about microscopic shifts is also true in this case, although there was no evidence of them.

The disturbances, caused by the ship's movements may be grouped under three heads:

1. The effect of the horizontal accelerations of the knife-edge.
2. The effect of the vertical accelerations of the knife-edge.
3. The effect of the rotational movements of the apparatus.

They will be examined successively in the following paragraphs.

§ 2. *Effect of the horizontal accelerations of the knife-edge.*

When the rotational movements are kept below a certain limit by an adequate suspension of the apparatus in gimbals, this effect is by far the most important disturbance of the pendulum movement; it is in fact the only cause of its great irregularity, as all the other disturbances are comparatively small.

It can be shown, that this principal disturbance may be eliminated completely by swinging two pendulums simultaneously on the same support in the same vertical plane, provided we may assume that the components of the horizontal accelerations of both knife-edges in the plane of oscillation are equal. This is the case if the knife-edges do not slip, so that their distance remains constant, and if we may neglect two effects. First the small variation of the horizontal component of this distance by the rotation of the apparatus round a horizontal axis, which will be considered in § 4; and second the elastical variation of the distance by the stresses, exerted by the two pendulums on the support. As the two agate planes are firmly connected, a rough computation shows, that this effect can be neglected.

Leaving out all other disturbance terms besides the horizontal accelerations, we get for the equations of motion of the two pendulums

$$\begin{aligned} \ddot{\theta}_1 + \frac{g}{l_1}\,\theta_1 + \frac{\ddot{y}}{l_1} &= 0 \\ \ddot{\theta}_2 + \frac{g}{l_2}\,\theta_2 + \frac{\ddot{y}}{l_2} &= 0 \end{aligned} \qquad \ldots\ldots\ldots \quad (1\,A)$$

in which θ_1 and θ_2 are the angles of elongation, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ their second differential coefficients according to the time, g the acceleration of the gravity, l_1 and l_2 the mathematical lengths of the pendulums and $\ddot{y}$ the component in the plane of oscillation of the horizontal accelerations of the knife-edges.

It is clear that $\ddot{y}$ can be eliminated from these two equations, and this is the fundamental principle of the method. If the pendulums are isochronous, so that $l_1 = l_2$, the result of this elimination is very simple: the difference of the equations gives

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + \frac{g}{l}\,(\theta_1 - \theta_2) = 0 \quad \ldots\ldots \quad (1\,B)$$

which has the same shape as the equation of motion of an undisturbed pendulum of the same mathematical length l and an angle of elongation $(\theta_1 - \theta_2)$. We reach in this way the important conclusion that the difference of the angles of elongation may be considered as the angle of elongation of a fictitious pendulum, which is not disturbed by the horizontal accelerations of the apparatus, and which is isochronous with the original pendulums.

In order to deduce the formulae for the case the original pendulums are not isochronous, we must go somewhat deeper into the theory of the pendulum motion. On putting

$$\frac{g}{l} = n^2$$

we obtain for the general equation of motion of a pendulum

$$\ddot{\theta} + n^2\,\theta + S = 0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

in which S means some arbitrary disturbance term, which we need not yet define.

The solution is put into useful shape by introducing, instead of θ, two new variables a and φ, which are defined by the equations

$$\theta = a \cos \varphi \quad \ldots\ldots\ldots\ldots \quad (3\,A)$$

$$\dot{\theta} = -\, a\, n \sin \varphi \quad \ldots\ldots\ldots \quad (3\,B)$$

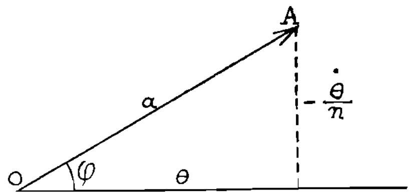
We will call a the amplitude of the pendulum movement, φ the phase, and the vector OA, of which the elongation θ is the horizontal projection, the pendulum-vector.

Differentiation of $(3\,A)$ combined with $(3\,B)$ gives

$$(\dot{\varphi} - n)\, a \sin \varphi - \dot{a} \cos \varphi = 0$$

and substitution of $(3\,A)$ and $(3\,B)$ in (2)

$$(\dot{\varphi} - n)\, a \cos \varphi + \dot{a} \sin \varphi - \frac{S}{n} = 0.$$



The combination of both equations gives

$$(\dot{\varphi} - n)\, a = \frac{S}{n} \cos \varphi \quad \ldots\ldots\ldots \quad (4\,A)$$

$$\dot{a} = \frac{S}{n} \sin \varphi \quad \ldots\ldots\ldots \quad (4\,B)$$

If there is no disturbance, that is to say if S is zero, this becomes

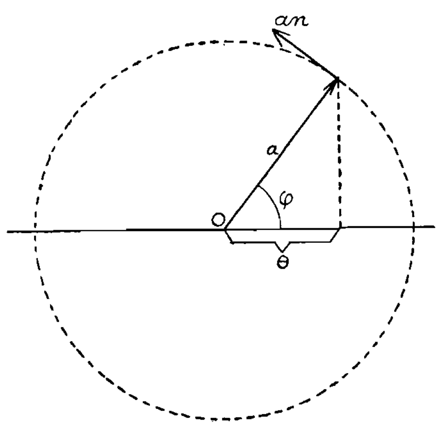
$$\dot{\varphi} = n$$

$$\dot{a} = 0$$

In this case the amplitude is constant and the phase is increasing in a constant ratio n, that is to say the pendulum-vector has a constant length and is revolving with a constant angular velocity n. If we define the period of the pendulum T as half of the time in which it describes a complete revolution, we have

$$T = \frac{\pi}{n} = \pi \sqrt{\frac{l}{g}} \quad \ldots\ldots\ldots \quad (5)$$

We get in this way the well-known solution for the undisturbed pendulum and the pendulum-vector, revolving with constant angular velocity, gives the ordinary way of representing the pendulum movement as the projection of a circular movement.



The formulae (4) provide the means to deduce the disturbances in this movement, caused by the term S in the equation (2). We can for instance deduce from $(4\,A)$ a formula for expressing the disturbance δT of the period of the pendulum. Defining the period τ of the disturbed pendulum in the same way as has been done for the period T of the undisturbed pendulum, as half of the time in which the pendulum-vector describes a complete revolution, we find by dividing ($4A$) by a and by integrating over a duration 2τ for the left member

$$2\pi - 2n\tau = 2n(T-\tau) = -2n\,\delta T$$

and so we get

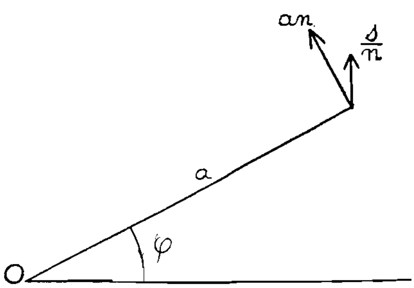
$$\delta T = -\frac{1}{2n^2}\int_0^{2\tau}\frac{S}{a}\cos\varphi\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (6A)$$

The mean variation of the amplitude per unity of time during the same period is

$$\dot{a} = \frac{1}{2n\tau}\int_0^{2\tau} S\sin\varphi\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6B)$$

which formula gives a better insight in the variation of the amplitude for a longer duration than the momentary value of $\dot{a}$.

We may easily derive from (4) the effect of S on the movement of the pendulum-vector OA. As $a\dot{\varphi}$ represents the component of the velocity of A perpendicular to OA and $\dot{a}$ the component of this velocity in the direction of OA, we can express the formulae (4) for the disturbed pendulum in this way, that the velocity of A, which for the undisturbed pendulum is equal to an in a direction perpendicular to OA, is increased with a vertical component $\frac{S}{n}$.



Returning now to the original problem of eliminating the effect of the horizontal acceleration in case the original pendulums are not isochronous, we find by introducing

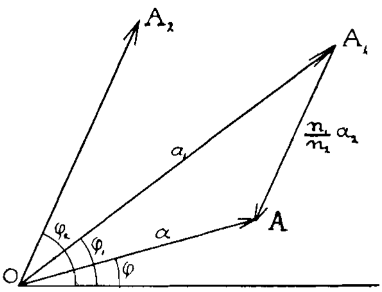
$$n_1^2 = \frac{g}{l_1}\cdot \quad n_2^2 = \frac{g}{l_2}$$

in the equations ($1A$), for the values of the disturbance terms S_1 and S_2 caused by the horizontal acceleration

$$S_1 = \frac{n_1^2}{g}\ddot{y} \quad S_2 = \frac{n_2^2}{g}\ddot{y}.$$

The velocity of the extremity A_1 of the first pendulum is therefore increased with a vertical disturbance $\frac{n_1}{g}\ddot{y}$ and that of the extremity A_2 of the second pendulum-vector with $\frac{n_2}{g}\ddot{y}$. It is clear that these disturbances are eliminated for a vector OA, which is obtained by taking the vectorial difference of OA_1 and $\frac{n_1}{n_2}OA_2$, because the disturbances of both parts neutralize each other (see fig. next page). The extremity A of the vector OA has therefore a velocity, which only results from the normal angular velocities n_1 of OA_1 and n_2 of A_1A. The horizontal projection of this vector is the difference of the horizontal projections of OA_1 and AA_1, that is to say it is

$$\theta_1 - \frac{n_1}{n_2}\theta_2 .$$



The velocity of A may be expressed in several ways; in order to get a formula, adapted to the apparatus, which will be described afterwards, the following way is the most useful.

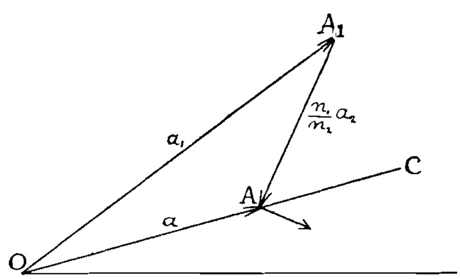
We separate the velocity of A into two parts:

1. The velocity resulting from the rotation of the triangle $O\,A_1\,A$ round O with an angular velocity n_1. This keeps the length a constant, and gives $O\,A$ a constant angular velocity n_1. If this were the only movement of A, $O\,A$ could be considered as the pendulum-vector of a fictitious pendulum, which is analogous to the fictitious pendulum, introduced in the case of absolute isochronism. This fictitious pendulum has the same property of being undisturbed by the horizontal acceleration; it is isochronous with the first pendulum, so that its period is T_1 and its angle of elongation, which we will represent by θ, is given by the horizontal projection of $O\,A$

$$\theta = \theta_1 - \frac{n_1}{n_2}\theta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,A)$$

This formula allows the determination of its position from the positions of the original pendulums.

2. An additional velocity, resulting from a rotation of $A\,A_1$ around A_1 with an angular velocity $n_2 - n_1$. This causes a deviation of the movement of the fictitious pendulumvector from the ordinary circular movement.



As $\angle\, A_1\, A\, C = \varphi_2 - \varphi$, we get for the velocity of A in the direction of $O\,A$

$$\dot{a} = (n_2 - n_1)\,\frac{n_1}{n_2}\,a_2\,sin\,(\varphi_2 - \varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,B)$$

and in the direction perpendicular to $O\,A$

$$a\,(\dot{\varphi} - n_1) = -\,(n_2 - n_1)\,\frac{n_1}{n_2}\,a_2\,cos\,(\varphi_2 - \varphi) \quad . \quad . \quad . \quad . \quad (7\,C)$$

from which formula we may deduce in the same way as for formula $(6\,A)$ a formula for the deviation $\delta\,T$ of the period of the fictitious pendulum, which corresponds to this deviation of the phase-velocity. We find, by dividing $(7\,C)$ by a and integrating over the double period $2\,T$, for the left member

$$(2\,\pi - 2\,n_1\,T) = 2\,n_1\,(T_1 - T) = -\,2\,n_1\,\delta\,T$$

and therefore $$\delta T = + \frac{n_2 - n_1}{2\, n_2} \int_0^{2T} \frac{a_2}{a} \cos(\varphi_2 - \varphi)\, dt. \quad . \quad . \quad . \quad (7\,D)$$

with $$T = T_1 + \delta T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,E)$$

In case there are other disturbances of the pendulum movement, represented by the terms S_1 and S_2 in the equations of motion of the two original pendulums, there is of course a corresponding disturbance of the fictitious pendulum. S_1 gives a vertical velocity $\frac{S_1}{n_1}$ to A_1 and S_2 a vertical velocity $\frac{S_2}{n_2}$ to A_2; the velocity of A brought about by these two effects is therefore $\frac{S_1}{n_1} - \frac{n_1}{n_2} \frac{S_2}{n_2} = \frac{1}{n_1} \left(S_1 - \frac{n_1^2}{n_2^2} S_2\right)$ in a vertical direction, which corresponds to a disturbance

$$S = S_1 - \frac{n_1^2}{n_2^2} S_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,F)$$

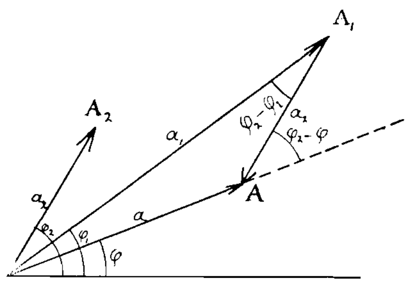
in the equation of motion of the fictitious pendulum

$$\ddot{\theta} + n_1^2\, \theta + S = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,G)$$

Formulae $(6\,A)$ and $(6\,B)$ give the resulting disturbances of the period and the amplitude, by substituting in them the value of S from formula $(7\,F)$ and n_1 for n.

The formulae (7) give the solution for an arbitrary value of the difference between the periods T_1 and T_2 of the original pendulums. We will now apply them to the actual case, that $T_2 - T_1$ is small with regard to T, so that the square of the ratio $\frac{T_2 - T_1}{T}$ may be neglected. In this case we may drop the factor $\frac{n_1}{n_2}$ in the formulae and further we may put

$$n_2 - n_1 = - \frac{n^2}{\pi} (T_2 - T_1) = - \frac{\pi}{T^2} (T_2 - T_1)$$



derived from $T_1 = \frac{\pi}{n_1}$ and $T_2 = \frac{\pi}{n_2}$.

In the integral $(7\,D)$ the quantities a_2, a, and $\cos(\varphi_2 - \varphi)$ are practically constant. We get in this case

$$\theta = \theta_1 - \theta_2 \quad . \quad . \quad . \quad . \quad . \quad (8\,A)$$

$$T = T_1 + \delta T \quad . \quad . \quad . \quad . \quad (8\,B)$$

$$\delta T = - (T_2 - T_1) \frac{a_2}{a} \cos(\varphi_2 - \varphi) \quad (8\,C)$$

$$a = - \frac{\pi}{T^2} (T_2 - T_1)\, a_2 \sin(\varphi_2 - \varphi) \quad . \quad . \quad . \quad . \quad : \quad (8\,D)$$

in which $(8\,A)$ gives the means to determine the position of the fictitious pendulum, while $(8\,B)$ and $(8\,C)$ allow the computation of T_1, if T has been determined.

In case there are other disturbances S_1 and S_2 for the original pendulums, we have for the fictitious pendulum

$$S = S_1 - S_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\,E)$$

It may be noticed that $a_2 \cos (\varphi_2 - \varphi)$ is the component of the pendulum-vector of the second pendulum in the direction of the fictitious pendulum-vector, while $a_2 \sin (\varphi_2 - \varphi)$ is the component perpendicular to this direction.

To apply the described method of eliminating the effect of the horizontal accelerations, we may record the two original pendulums and deduce from these records the positions of the fictitious pendulum, so that we may determine its period T, but this is an elaborate proceeding, which is still more painstaking as the records of the original pendulums are very irregular because of the disturbances. The new apparatus for maritime gravity survey, which will be described in chapter II, simplifies the computations a great deal by recording at once the angle of elongation of the fictitious pendulum $\theta = \theta_1 - \theta_2$, so that we can determine directly its period. As this curve is nearly regular because of the removal of the principal disturbance, this determination does not take longer, than it would take for an ordinary pendulum observation on land. For computing the correction δT it is however necessary to make also a record of one of the original pendulums. This curve is of course disturbed, but as it is wanted only for the computation of some small reductions — it will also be needed for the determination of the reduction to zero amplitude — this does not mean a serious increase of labour. This record must be continuous during the whole observation, because it is too much disturbed to allow interpolation between separate momentary values. When however the apparatus is used on land, the record is so regular, that a few intermediate records are sufficient.

If the above mentioned method of recording only the two original pendulums is applied, it is of course more useful to express δT in the data of these pendulums instead of using formula (8 C), which expresses it in the data of the fictitious pendulum and of one of the original ones. We may refer for this question to Appendix II of this chapter.

§ 3. *Effect of the vertical acceleration of the knife-edge.*

The vertical acceleration of the knife-edge has a much smaller effect than the horizontal. The corresponding term S in the equation of motion of the pendulum is

$$S = \frac{\ddot{x}}{l}\,\theta$$

in which $\ddot{x}$ is the vertical acceleration of the knife-edge. Comparing this with the formula of S for the horizontal acceleration

$$S = \frac{\ddot{y}}{l}$$

we see that in the first case S contains a factor θ, which is of the order of one percent.

It is not possible to follow the same method of § 2 for eliminating the effect of the vertical acceleration. We may of course eliminate $\ddot{x}$ from the equations of motion of two pendulums, which we suppose subject to the same vertical acceleration $\ddot{x}$, but we obtain in this way an equation, which only contains the amplitude and not the period of the pendulums or g. This result may be expected. The vertical acceleration $\ddot{x}$ is inseparable from the constant vertical acceleration g and we eliminate g at the same time that we eliminate $\ddot{x}$.

The effect of $\ddot{x}$ can easily be expressed by substituting the value of S

$$S = \frac{\ddot{x}}{l}\,\theta = \frac{n^2}{g}\,a\cos\varphi\,\ddot{x}$$

in the equations ($6A$) and ($6B$). ($6A$) gives

$$\delta T = -\frac{1}{2g}\int_0^{2\tau} \ddot{x}\cos^2\varphi\,dt = -\frac{1}{4g}\int_0^{2\tau}\ddot{x}\,(1 + \cos 2\varphi)dt$$

so that

$$\delta T = -\frac{\tau}{2g}\,[\ddot{x}] - \frac{1}{4g}\int_0^{2\tau}\ddot{x}\cos 2\varphi\,dt \quad . \quad . \quad . \quad . \quad (9A)$$

in which $[\ddot{x}]$ indicates the mean value of $\ddot{x}$ during the time 2τ.

($6B$) gives

$$\dot{a} = \frac{\pi a}{4g\tau^2}\int_0^{2\tau}\ddot{x}\sin 2\varphi\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9B)$$

We may apply these formulae also to the fictitious pendulum, if we suppose $\ddot{x}$ to be the same for both original pendulums, because the term S for the fictitious pendulum is according to ($8E$)

$$S = S_1 - S_2 = \frac{\ddot{x}}{l}\,(\theta_1 - \theta_2) = \frac{\ddot{x}}{l}\,\theta .$$

so that for this pendulum S is the same function of $\ddot{x}$ and θ as for the real pendulums.

If we neglect quantities of higher order, we see that the first term of ($9A$) is the variation of T corresponding to a variation of g with $\ddot{x}$ in the formula $T^2 = \pi^2\,\frac{l}{g}$ (5), because differentiation of this formula gives

$$dT = -\frac{T\,dg}{2g}.$$

This disturbance has therefore the meaning that, if we deduce the gravity from the observed period of the pendulum by applying the ordinary formula (5), we will not find g but $g + [\ddot{x}]$, that is to say $g +$ the mean value of the vertical acceleration during the time of the observation.

If we suppose that $\ddot{x}$ is not strongly variable during the period 2τ, the second term of ($9A$) is smaller than the first, because the factor $\cos 2\varphi$ takes positive and negative values during a complete period 2τ. If $\ddot{x}$ is constant during that time, we find zero. The supposition that $\ddot{x}$ is slowly variable, compared with the pendulum movement, is realized on board of all ships; the period of their movement is always several times greater than the pendulum period. An apparatus with much longer pendulums might in fact give serious complications on board ship.

The results of all the experiments and observations have shown, that for the actual apparatus this last disturbance-term is negligible. This can easily be controlled by observing the amplitude. We may remark that, if $\ddot{x}$ is an irregularly varying quantity without terms of the same period as the pendulum, the integrals in the formulae (9 A) and (9 B) will have the same order of magnitude, because there is no reason why the multiplication with *sin* 2 φ or with *cos* 2 φ will make any difference in this regard. We may even say that the mean value of these integrals for a great many periods 2 τ will be about the same. Indicating the mean value of a quantity by [ ] we get therefore for the mean disturbances, caused by these terms

$$[\delta T] = \frac{\tau^2}{\pi a} [\dot{a}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We may conclude that the disturbance of the period is negligible if there is no appreciable variation of the amplitude.

The only effect of the vertical acceleration is therefore the effect, represented by the first term of (9 A), i. e. a disturbance δT corresponding to $[\ddot{x}]$ in the result for g; the amplitude of the fictitious pendulum remains constant. As $\ddot{x}$ is a more or less periodically fluctuating quantity, the disturbance δT has also that character. The photographic records of the fictitious pendulum of which fig. 7, 8 and 9 gives an example, show this character very clearly; while the amplitude remains rigorously constant, the white time-marks, caused by the chronometers, give a fluctuating curve, revealing the fluctuating value of the pendulum-period.

The disturbance of the result for the gravity g of an observation is the mean value of $\ddot{x}$ for the whole duration t of the observation. This is given by

$$\frac{1}{t} \int_0^t \ddot{x}\, dt = \frac{1}{t} (\dot{x}_t - \dot{x}_0)$$

that is to say it is the difference of the vertical velocities at the beginning and at the end of the observation, divided by the duration t. In two ways we can make this as small as we wish to make it, by increasing t and by taking at the beginning and at the end the mean of the positions of the fictitious pendulum for a great many seconds, so that we may suppose, that the mean $\dot{x}_0$ and $\dot{x}_t$ are small.

In this way we can easily make this disturbance negligible. Observations of half an hour are more than sufficient for the purpose.

§ 4. *The effect of the angular movements of the apparatus.*

The angular movements of the apparatus have the following effects:

1. Relative movements of the knife-edges of the two pendulums, whose combination gives the fictitious pendulum.

Because of the angular movements of the line, which joins the two knife-edges, these knife-edges get relative horizontal and vertical accelerations, while we supposed in § 2, that their movements were exactly the same. The movement of the fictitious pendulum is therefore disturbed. We assume that the line through the knife-edges is horizontal in its normal position, as is the case for the pendulum-apparatus described in the next chapter.

By substituting the corresponding S terms of the equation of motion of the fictitious pendulum in the formulae ($6A$) and ($6B$) for the disturbances $\delta\tau$ and $\dot{a}$, we find that for each term the result is similar to that of page 11: the disturbances are irregularly varying quantities, whose mean values for a long duration are related in the way, indicated by a formula which is analogous to (10) (See App. III). We reach therefore the same conclusion, that, if the variation of the amplitude is negligible, the disturbance of the period is also negligible.

2. The rotation of each pendulum system.

Each pendulum system being free to follow the rotation round an axis, which coincides with the knife-edge, the movement is only affected by rotations round axes in the swinging-plane, that is to say by a rotation round a vertical axis and by a rotation round a horizontal axis in the swinging-plane.

The effect of the rotation of a system is generally threefold:

I. The variation of the direction of the external forces in the rotating system. In this case this is gravity, which is not affected by a rotation of the system about a vertical axis, so that there remains only the variation caused by a rotation about a horizontal axis in the swinging plane.

II. The acceleration, which each point of the system gets by the rotation. As to this effect, the rotation about both axes has to be taken into account.

III. The acceleration of Coriolis. This acceleration is perpendicular to the velocity and to the axis of rotation. Both being parallel to the swinging-plane, the acceleration of Coriolis is perpendicular to that plane so that it has no effect on the movement of the pendulum.

We will successively examine the first two effects and we will assume that the pendulums are nearly isochronous, so that the effect of the difference of the periods can be neglected for this research.

I. The effect of the variation of the direction of gravity with regard to the pendulum system, caused by the rotation of this system round a horizontal axis contained in the swinging-plane.

If β is the angle of deviation of the swinging-plane with regard to the vertical position, the component of the gravity in this plane is $g \cos \beta$. This is the active part of the gravity as far as the pendulum movement is concerned, so that the gravity g is replaced by $g \cos \beta$. This cause has therefore the same effect as a fluctuation of gravity, so that we can apply at once the results of § 3. The variation of gravity, which we have called $\ddot{x}$ in § 3 has here the value

$$-g(1 - \cos \beta) = -\frac{1}{2} g \beta^2$$

if we suppose β to be small; so that we get, by substituting $\ddot{x} = -\frac{1}{2} g \beta^2$ in the first term of (9 A) and by indicating the mean value of β^2 by $[\beta^2]$

$$\delta T = \frac{1}{4} T [\beta^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As to the second term of (9 A) we can again follow the same reasoning as has been held there, and we may say, that its mean value during a long time is related to the mean value of the variation of the amplitude in the way, given by formula (10), so that it is negligible if the variation of the amplitude is negligible. The condition, which has to be fulfilled if we wish this to be the case, is that β does not contain any term of the same period as the pendulum.

We find thus that if the amplitude remains undisturbed, the disturbance by this cause is restricted to δT as given by formula (11). δT has however another character, as that which is given by the first term of (9 A), because its mean value does not tend to zero for an infinite duration; β^2 has always the same positive sign, so that its mean value is a positive quantity, which in general is not negligible. We will have to determine it by making a continuous record of β, so that the mean value of β^2 during the whole observation may be computed. It is therefore a necessary condition for an apparatus for maritime gravity survey, that the deviation β of the swinging-plane can be recorded.

However, even if this angle is recorded, we will have to take care, that it does not exceed a rather narrow limit, e.g. 1^0, because the fact that the second power of β enters in the formula implies that greater deviations would necessitate a more accurate recording and an extremely laborious computation for the determination of $[\beta^2]$. This has been the principal motive for suspending the apparatus in gimbals.

If the swinging-planes of the two pendulums, from which the fictitious pendulum is derived, coincide exactly, the disturbance of the period of the fictitious pendulum is also given by formula (11), but if these planes enclose a small angle, the deviations β_1 and β_2 of both planes are not equal and we get a supplementary term for the disturbance of the fictitious pendulum. As

$$\delta T_1 = \frac{1}{4} T_1 [\beta_1^2] \qquad \delta T_2 = \frac{1}{4} T_2 [\beta_2^2]$$

and as we derive from formulae (8 B) and (8 C)

$$\delta T = \delta T_1 - (\delta T_2 - \delta T_1) \frac{a_2}{a} \cos (\varphi_2 - \varphi)$$

we find when we neglect the difference of T_1 and T_2

$$\delta T = \frac{1}{4} T_1 [\beta_1^2] + \frac{1}{4} T_1 [(\beta_1^2 - \beta_2^2) \frac{a_2}{a} \cos (\varphi_2 - \varphi)] \ . \quad (12\ A)$$

while formula (8 D) gives

$$\dot{a} = + \frac{\pi}{4 T_1} [(\beta_1^2 - \beta_2^2) a_2 \sin (\varphi_2 - \varphi)] \quad . \quad . \quad . \quad . \quad (12\ B)$$

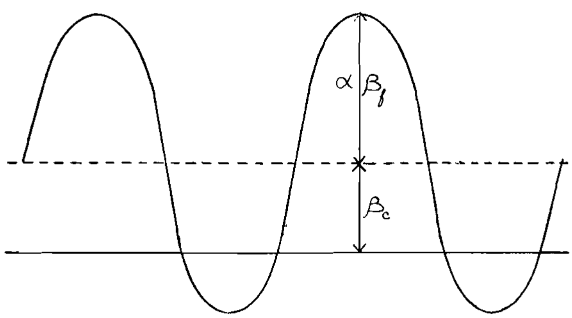
$(\beta_1^2 - \beta_2^2)$ is a fluctuating quantity; if we call the constant difference $(\beta_1 - \beta_2)$, Δ, it is $2\,\Delta\,(\beta_1 - \frac{1}{2}\,\Delta)$, and β_1 fluctuates.

We see that we may apply the same reasoning as has already been presented. If the amplitude does not show any trace of fluctuations, and provided the angle $(\varphi_2 - \varphi)$ is not by chance very small during the whole observation, the second term of $(12\,A)$ is negligible.

The apparatus ought not to give rise to this question by having exactly parallel planes, so that the knife-edges, resting on these planes, do not enclose a vertical angle. Still, although all precautions are taken by grinding the planes together after their being fixed in the apparatus, it might be that afterwards deformations occur, and so it is well to check this point from time to time by levelling all the planes. As long as Δ is only a few minutes of arc, this term is negligible, provided the mean β_1 likewise does not exceed a few minutes of arc; this is easily realized if the gimbals are working correctly.

The disturbance of the period of the fictitious pendulum is therefore generally given by the first term of $(12\,A)$. For applying this formula we need $[\beta_1^2]$. We will deal in chapter III with the determination of this quantity from the photographic record of β_1; we will give here a general formula for this computation.

We may separate β_1 in a constant or nearly constant part β_c and a fluctuating part β_f.



The first part represents the error in the mean position of the apparatus, caused by a defect in the adjustment of its equilibrium-position in the gimbal-suspension. The second part is caused by the swinging of the apparatus in the gimbals, so that it is a periodical quantity, which we will assume to be purely periodical. If the period of the gimbal-swinging is T_f (i. e. the complete period $2\,T_f$), and its amplitude α, we have

$$\beta_1 = \beta_c + \beta_f = \beta_c + \alpha \cos \pi \frac{t}{T_f} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the mean value of β_1^2 is

$$[\beta_1^2] = \beta_c^2 + \frac{1}{2}\,\alpha^2,$$

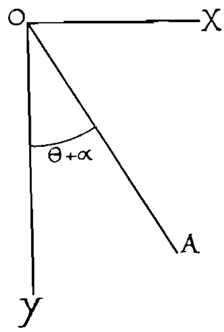
so that the disturbance of the fictitious pendulum is

$$\delta\,T = \frac{T}{4}\left(\beta_c^2 + \frac{1}{2}\,\alpha^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

For β_c^2 and α^2 we will have to substitute their mean values during the whole observation.

II. The effect of the acceleration, which each point of the pendulum system gets by the rotation round a horizontal axis in the swinging-plane and by the rotation round a vertical axis.

For deducing these effects, we will determine the corresponding terms in the equation of motion of the pendulum. This equation is found by dividing the equation of the momentum of the forces by the moment of inertia, both with regard to the origin of the system of coordinates, which is chosen as usual in the knife-edge.



In a point A with the polar coordinates ρ and $\theta + \alpha$ (θ is again the angle of elongation of the pendulum) the rotation about the horizontal axis OX gives a vertical acceleration

$$\rho\,(\dot{\beta})^2$$

in which $\dot{\beta}$ is the angular velocity. The rotation about OY gives in A a horizontal acceleration

$$\rho\,(\dot{\gamma})^2\,(\theta + \alpha)$$

in which $\dot{\gamma}$ represents the angular velocity round the vertical axis. The accelerations, caused by both rotations in a direction perpendicular to the swinging-plane, do not affect the movement, so that we need not examine them.

The above-mentioned accelerations in the swinging-plane give together a contribution to the momentum about O

$$\{(\dot{\beta})^2 - (\dot{\gamma})^2\}\,(\theta + \alpha)\,\rho^2\,dm$$

in which dm is the element of mass of the pendulum in A. Terms with higher powers of θ are as usual neglected, which is of course admissible for these small disturbances.

Integrating over the whole pendulum and dividing by the moment of inertia, we find for the disturbance term S in the equation of motion

$$S = \{(\dot{\beta})^2 - (\dot{\gamma})^2\}\,\theta\,.$$

By comparing this term with the term, caused by the fluctuation of the gravity with the acceleration $\ddot{x}$, which is considered on page 10, we notice that it is the same function of θ, so that we may at once write down the corresponding disturbances by substituting in (9 A)

$$\ddot{x} = l\,\{(\dot{\beta})^2 - (\dot{\gamma})^2\} = g\,\frac{T^2}{\pi^2}\,\{(\dot{\beta})^2 - (\dot{\gamma})^2\}\,.$$

Supposing again that the amplitude does not show any disturbance, we may neglect the second term of (9 A) and we get for the mean disturbance of the period during the whole observation

$$\delta T = \frac{T^3}{2\,\pi^2}\,\{-[(\dot{\beta})^2] + [(\dot{\gamma})^2]\} \quad \ldots\ldots \quad (15)$$

in which [ ] again indicates the mean value for this duration. Because $\dot{\beta}$ and $\dot{\gamma}$ are the same for all the pendulums, we may apply this formula also to the fictitious pendulum by substituting T_1 for T.

We will now examine the two terms of (15) separately. For $\dot{\beta}$ can be substituted the value, obtained by differentiating (13)

$$\dot{\beta} = - \frac{\pi}{T_f} \alpha \sin \pi \frac{t}{T_f}$$

so that we get

$$[(\dot{\beta})^2] = \frac{1}{2} \frac{\pi^2}{T_f^2} \alpha^2$$

and for the corresponding δT

$$\delta T = - \frac{T_1^3}{4 T_f^2} \alpha^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

Of course we have to realize that the assumption made in formula (13), that β is a purely periodical quantity, is an approximation so that deviations will occur. They will generally have the effect of increasing $[(\dot{\beta})^2]$.

About the second term of (15) we may in the same way get an estimate by assuming that $\dot{\gamma}$ is also a purely periodical quantity. This is a still more rough approximation, as $\dot{\gamma}$ represents the movement of the ship. Supposing the ship to be steered along a straight course, so that there is no systematic part of γ, and supposing ω to be the amplitude and $2\ T_s$ the period of this ship's movement, we find

$$[(\dot{\gamma})^2] = \frac{1}{2} \frac{\pi^2}{T_s^2} \omega^2$$

and the corresponding δT

$$\delta T = \frac{1}{4} \frac{T_1^3}{T_s^2} \omega^2.$$

We see easily that this disturbance may be neglected. Substituting for ω the improbably big value of $30' = 0.009$, and for T_s the rather small value of 5 seconds, we get: $\delta T = 1\ 10^{-7}$ sec.

If however the ship is not steering along a straight course, this disturbance may get a value, which is not negligible. Suppose for instance that the ship describes a circle, so that it makes a complete turn of 360° in 5 minutes, we find $\dot{\gamma} = 0.021$ and $\delta T = 28\ 10^{-7}$ sec.

Assuming that the course is always straight during an observation, this term can be neglected. The total effect of the angular movements of the apparatus is then found by combining formulae (14) and (16) and we get

$$\delta T = \frac{1}{4} T_1 [\beta_c^2 + C\alpha^2] \quad . \; . \; . \; . \; . \; . \; . \quad (17A)$$

in which

$$C = \frac{1}{2}\left(1 - 2 \frac{T_1^2}{T_f^2}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (17B)$$

This is the total disturbance by the angular movements, which we have to take into account, provided the amplitude does not show fluctuations.

From the deductions of this and the foregoing paragraphs we conclude, that it is important to examine the amplitude of the fictitious pendulum. If there is any trace of another variation than that, which is due to the damping by the friction of the air or to the deviation of isochronism of the two original pendulums as given by formula ($8\,D$), we will have to study the question and try to remove the cause. Generally a superficial survey of the amplitudes on the photographic record will be sufficient. If these curves show no fluctuations of the amplitudes, the question will probably be settled, as all the disturbances, which have been considered, have a fluctuating character.

A cause for irregularities may for instance be the presence of some friction in the swinging mechanism of the gimbals. This prevents the apparatus for moments of slow angular motion of the ship to get its normal position, and then suddenly the friction may be overcome and the equilibrium recovered, so that the apparatus gets a sudden impulse. This point has to be set right, because the effect cannot be determined and it may even have disastrous results, if it gives rise to a slipping of the knife-edges. During the first experiments with a provisional gimbal arrangement, this effect occurred and the corresponding irregularities of the records could be stated.

Another point, which has to be taken care of, is that the apparatus does not get movements of the same period as the pendulums, according to the assumptions, made in these paragraphs. This implies the necessity of bringing the plane of the knife-edges as well as possible in the same plane with the knife-edges of the gimbals, for otherwise the horizontal forces, exerted by the pendulums on the support, would give the apparatus a movement, which would have the same period of the pendulums. This adjustment may be made with the foot-screws of the apparatus, with which it rests on the cradle which is fixed to the gimbals. A further consideration of this point shows, that this adjustment need not be very accurate; deviations of a few millimeters do not matter.

In the following paragraphs we will deduce the formulae for the other disturbances and corrections, which we have to take into account for reducing the period of the fictitious pendulum.

§ 5. *The corrections for the effects of the temperature and of the air.*

Each pendulum-period is affected in the ordinary way by the temperature and the air. The first effect is proportional to the temperature and as the observations are made at atmospheric pressure, the second effect is practically proportional to the density of the air; for this pressure range the deviation of proportionality may be neglected.

Let t and D be the temperature and the density of the surrounding air; let c_1, c_2 be the temperature constants and d_1, d_2 the density constants of the two original pendulums, so that the corrections for these pendulums are

$$\delta T_1 = c_1 t + d_1 D$$
$$\delta T_2 = c_2 t + d_2 D.$$

We find then, by applying formulae (8 B) and (8 C), for the period of the fictitious pendulum

$$\delta T = \delta T_1 - (\delta T_2 - \delta T_1) \frac{a_2}{a} \cos(\varphi_2 - \varphi)$$

$$= c_1 t + d_1 D - [(c_2 - c_1) t + (d_2 - d_1) D] \frac{a_2}{a} \cos(\varphi_2 - \varphi) \quad . \quad (18\,A)$$

and for the amplitude we get the term, given by (8 D).

Besides the effect on the periods, the air has also an effect on the amplitudes of the original pendulums. We suppose that this damping is the same for both pendulums. In that case the sides a_1 and a_2 of the triangle $O A_1 A$ of page 19 decrease in the same ratio, so that the angles of $O A_1 A$ remain unchanged, that is to say, the period of the fictitious pendulum is not affected by the damping, while the amplitude a decreases also in the same ratio.

Formula (18 A) represents therefore the complete correction of the period, while the variation of the amplitude of the fictitious pendulum is given by the combination of the damping and the term given by (8 D).

$$\dot{a} = -ka - \frac{\pi}{T^2} [(c_2 - c_1) t + (d_2 - d_1) D] \, a_2 \sin(\varphi_2 - \varphi) \qquad (18\,B)$$

in which k is the damping-coefficient, which is dependent on the density of the air.

We will combine the last terms of (18 A) and (18 B) with the corrections for the deviation of isochronism, given by the formulae (8 C) and (8 D), so that we get the formulae

$$\delta T = - U_{21} \frac{a_2}{a} \cos(\varphi_2 - \varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\,A)$$

$$\dot{a} = - \frac{\pi}{T_1^2} U_{21} a_2 \sin(\varphi_2 - \varphi) \quad . \quad . \quad . \quad . \quad . \quad (19\,B)$$

with

$$U_{21} = (T_2 - T_1) + (c_2 - c_1) t + (d_2 - d_1) D \quad . \quad (19\,C)$$

U_{21} simply represents the difference of the periods of the original pendulums for the temperature and the density of the air during the observation.

The remaining parts of the formulae (18 A) and (18 B) are identical with the corrections of a single pendulum

$$\delta T = c_1 t + d_1 D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,A)$$

$$\dot{a} = -ka \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,B)$$

We could at once have written down the formulae (19) and (20) by substituting in the formulae (8) the periods of the pendulums for the temperature and the density of the air during the observation, instead of the reduced periods.

The value of U_{21} can be tabulated in order to facilitate the application of the formulae (19). If the difference of the density constants is not abnormally large, we may replace the last term by a constant term, found by multiplying $d_2 - d_1$ with the mean atmospheric density, so that the only variable of the table for U_{21} is the temperature t.

§ 6. *The reduction to infinitely small amplitude.*

For the two pendulums this reduction has the ordinary shape

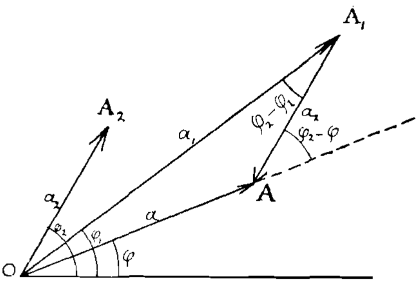
$$\delta T_1 = \frac{1}{16} T_1 a_1^2$$

$$\delta T_2 = \frac{1}{16} T_2 a_2^2$$

so that we find for the fictitious pendulum, in the same way as in the former paragraph by applying ($8B$) and ($8C$)

$$\delta T = \frac{1}{16} T_1 \left[a_1^2 - (a_2^2 - a_1^2) \frac{a_2}{a} \cos (\varphi_2 - \varphi) \right]$$

In order to express δT solely in the quantities, which are recorded, that is to say in the quantities related to the fictitious pendulum and to pendulum Nr. 2, we will eliminate a_1 by substituting



$$a_1^2 = a^2 + a_2^2 + 2\, a a_2 \cos (\varphi_2 - \varphi)$$

and we get

$$\delta T = \frac{1}{16} T_1 \left[a^2 + a_2^2 + 3\, a a_2 \cos (\varphi_2 - \varphi) + 2\, a_2^2 \cos^2 (\varphi_2 - \varphi) \right] \quad (21\,A)$$

or

$$\delta T = \frac{1}{16} T_1 \left[\left\{ a + 1\frac{1}{2} a_2 \cos (\varphi_2 - \varphi) \right\}^2 + a_2^2 - \frac{1}{4} a_2^2 \cos^2 (\varphi_2 - \varphi) \right] \quad (21\,B)$$

For the amplitude of the fictitious pendulum we find by applying ($8D$) and by eliminating a_1

$$\dot{a} = \frac{\pi}{16\, T_1} a a_2 \sin (\varphi_2 - \varphi) \left[a + 2\, a_2 \cos (\varphi_2 - \varphi) \right] \quad . \quad . \quad (21\,C)$$

If we use an apparatus, which does not record the fictitious pendulum, but only the two original ones, we have to deduce another expression for δT and $\dot{a}$. This has been done in Appendix II to this chapter.

§ 7. *The correction for the rate of the chronometer.*

This affects only the unity of time without having anything to do with the pendulum movement itself. The corresponding correction for the period of the fictitious pendulum is therefore found in the same way as for any other space of time

$$\delta T = - \frac{r}{86400} T = - 115{,}7\, r\, T\, 10^{-7} \text{ sec.} \quad . \quad . \quad . \quad (22)$$

in which r is the daily rate in seconds.

§ 8. *Synopsis of the formulae for the reduction of the period of the fictitious pendulum.*

Putting together the results of the preceding paragraphs, we get the following list of reductions for the period of the fictitious pendulum; they represent the values, which we have to subtract from the observed period to obtain the reduced period. This reduced period is the same as the period T_1 of the first original pendulum

$$T = T_1$$

Reductions.

Temperature: $\delta T = c_1 t$ . . . . (20 *A*)

Air: $\delta T = d_1 D$ . . . . (20 *A*)

Amplitude:

$$\delta T = \frac{1}{16} T_1 \left[a^2 + a_2^2 + 3 a a_2 \cos(\varphi_2 - \varphi) + 2 a_2^2 \cos^2(\varphi_2 - \varphi)\right] \quad (21\,A)$$

or

$$= \frac{1}{16} T_1 \left[\left\{a + 1\tfrac{1}{2} a_2 \cos(\varphi_2 - \varphi)\right\}^2 + a_2^2 - \tfrac{1}{4} a_2^2 \cos^2(\varphi_2 - \varphi)\right] \quad (21\,B)$$

Deviation of isochronism:

$$\delta T = - U_{21} \frac{a_2}{a} \cos(\varphi_2 - \varphi) \quad . \; . \; . \; . \; . \; . \quad (19\,A)$$

$$U_{21} = (T_2 - T_1) + (c_2 - c_1)\, t + (d_2 - d_1)\, D \quad . \; . \; . \quad (19\,C)$$

Tilt of swinging-plane:

$$\delta T = \tfrac{1}{4} T_1 \left[\beta_c^2 + C \alpha^2\right] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (17\,A)$$

$$C = \tfrac{1}{2} \left(1 - 2 \frac{T_1^2}{T_f^2}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (17\,B)$$

Rate of the chronometer:

$$\delta T = - 115.7\, r\, T\, 10^{-7} \text{ sec.} \quad . \; . \; . \; . \; . \; . \; . \quad (22)$$

in which

a, a_2 = amplitudes of the fictitious and of the second pendulum,
φ, φ_2 = phases „ „ „ „ „ „ „ „
T_1, T_2 = periods of the two pendulums,
c_1, c_2 = temperature constants of the two pendulums,
d_1, d_2 = air-density constants of the two pendulums,
t = temperature of the pendulums,
D = density of the surrounding air,
β_c = constant part of the tilt of the swinging-plane,
α = amplitude of the periodical part of this angle of tilt.
T_f = period of this last movement.

The amplitude ought not to show any evidence of other effects than those which are mentioned hereafter. If there is any other disturbance, the cause must be traced and removed. For this control it will generally be sufficient to see that there is no fluctuation of the amplitude having a period of a few seconds or less, because the disturbances, which may be feared, are all more or less fluctuating with short periods.

If a further control is desired, the total variation of the amplitude of the fictitious pendulum during the whole observation may be computed. We ought to find the combination of the following terms, which do not contain short-periodical parts

Damping:

$$\dot{a} = -ka \quad \ldots\ldots\ldots\ldots \quad (20B)$$

Deviation of isochronism:

$$\dot{a} = -\frac{\pi}{T^2} U_{21}\, a_2 \sin(\varphi_2 - \varphi) \quad \ldots\ldots \quad (19B)$$

Amplitude:

$$\dot{a} = \frac{\pi}{16\,T}\, a a_2 \sin(\varphi_2 - \varphi)\, [a + 2a_2 \cos(\varphi_2 - \varphi)] \quad \ldots \quad (21C)$$

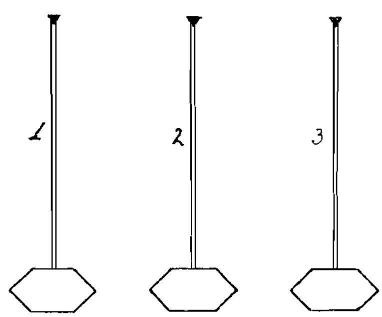
If there is any irregular variation of the amplitude, we may also expect a disturbance of the period. To get an estimate of its magnitude, we can use the relation of the mean values of both quantities for a great many observations, which is given by the formula

$$[\delta T] = \frac{T^2}{\pi a}\, [\dot{a}]\,. \quad \ldots\ldots\ldots\ldots \quad (10)$$

in which [ ] indicates mean values.

§ 9. *Principal arrangement of the apparatus.*

The apparatus for maritime gravity determinations, which will be described in detail in the next chapter, contains three pendulums, swinging all in the same plane, which are as nearly as possible isochronous; we will distinguish them by the suffixes 1, 2 and 3. The three knife-edges are at equal distances of about 13 cm. in the same horizontal plane; this plane may be brought to coincidence with the plane of the knife-edges of the gimbals by using the footscrews with which the apparatus rests in the cradle.



The pendulums are combined in two pairs: 1 and 2, respectively 2 and 3, that is to say that, if θ_1, θ_2 and θ_3 are the three angles of elongation, the angles $\theta_1 - \theta_2$ and $\theta_3 - \theta_2$ are recorded. These angles may be considered as the angles of elongation of two fictitious pendulums, which we will indicate by the suffixes 12 and 32. It might be thought, that the control on the stability of the pendulums, obtained by the comparison of the results of these two fictitious pendulums, is not absolute, because pendulum No. 2 forms part of both pairs, but we will show in the next paragraph, that this objection can be met by an adequate choice of the amplitudes and the phase-differences of the original pendulums.

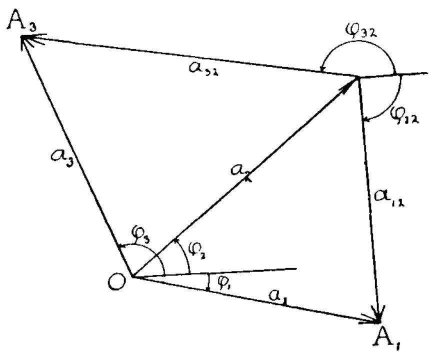
In order to be able to compute the reduction to infinitely small amplitude and the reduction for the deviation of isochronism, the apparatus also records pendulum 2; in this way we get for each pair besides the principal records a record of one of the original pendulums, which is necessary for these computations. This middle pendulum is recorded with regard to the position of an auxiliary pendulum, which moves in a plane parallel to the swinging-plane of the principal pendulums, and which is so strongly damped, that its own oscillations are negligible.

A similar damped pendulum, which can move in a plane perpendicular to the swinging-plane, serves to record the deviation β of this plane about a horizontal axis in the plane; the apparatus makes a record of the position of this auxiliary pendulum with regard to a fixed line of the apparatus.

Further details of the apparatus will be described in the next chapter.

§ 10. *Graphical representation of the pendulum-vectors.*

We may get a better insight than that given by the formulae, in the relations of the three original pendulums and the two fictitious pendulums, by a graphical representation of their pendulum-vectors. If the pendulum-vectors of the original pendulums are constructed from the origin O, the vectors of the fictitious pendulums are given by their differences $A_2 A_1$ and $A_2 A_3$.



As the disturbances of the fictitious pendulums are small in comparison with those of the original pendulums, the shape of the triangle $A_1 A_2 A_3$ varies only slightly, while the position of O is strongly variable, so that O describes a very irregular orbit. The whole triangle $A_1 A_2 A_3$ has of course a rotation, corresponding to the periods of the pendulums, or to be more accurate, the vectors OA_1, OA_2 and OA_3 rotate with angular velocities $\frac{\pi}{T_1}$, $\frac{\pi}{T_2}$ and $\frac{\pi}{T_3}$, besides the effect of the disturbances, to which the pendulums are subject.

§ 11. *Choice of the amplitudes and the phase-differences; control of the pendulums.*

As long as the difference of the reduced periods of the two fictitious pendulums is constant during the voyage, we have a great probability, that neither of the original pendulums has undergone a change. If however there is a discrepancy, it requires some consideration to decide, which pendulum is probably responsible for it. We have to realise the effect of the change of the pendulums on the difference of the periods of the fictitious pendulums.

Suppose the changes of the periods of the three pendulums are Δ_1, Δ_2 and Δ_3. We find then the changes of the periods of the fictitious pendulums by applying the formulae, 8 B and 8 C.

$$\Delta_{12} = \Delta_1 - (\Delta_2 - \Delta_1)\,\frac{a_2}{a_{12}}\,\cos(\varphi_2 - \varphi_{12})$$

$$\Delta_{32} = \Delta_3 - (\Delta_2 - \Delta_1)\,\frac{a_2}{a_{32}}\,\cos(\varphi_2 - \varphi_{32}) \quad . \quad . \quad . \quad . \qquad (23)$$

The change of the difference of the periods of the fictitious pendulums is therefore

$$\Delta = \Delta_{12} - \Delta_{32} = \Delta_1 - \Delta_3 - (\Delta_2 - \Delta_1)\,\frac{a_2}{a_{12}}\,\cos(\varphi_2 - \varphi_{12}) +$$
$$(\Delta_2 - \Delta_3)\,\frac{a_2}{a_{32}}\,\cos(\varphi_2 - \varphi_{32}) \qquad (24A)$$

and the change of the mean of their periods

$$\delta = \frac{1}{2}(\Delta_{12} + \Delta_{32}) = \frac{1}{2}(\Delta_1 + \Delta_3) - \frac{1}{2}(\Delta_2 - \Delta_1)\,\frac{a_2}{a_{12}}\,\cos(\varphi_2 - \varphi_{12}) -$$
$$- \frac{1}{2}(\Delta_2 - \Delta_3)\,\frac{a_2}{a_{32}}\,\cos(\varphi_2 - \varphi_{32}) \qquad (24B)$$

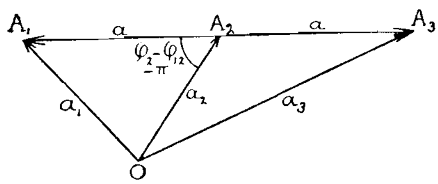
If we choose the initial amplitudes of the original pendulums and their phase-differences in such a way, that for the fictitious pendulums the amplitudes are equal

$$a_{12} = a_{32} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25A)$$

and the phases opposite

$$\varphi_{12} = \varphi_{32} + \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25B)$$

so that the graphical representation gives



in which the position of O is arbitrary, we find for Δ and δ

$$\Delta = (\Delta_1 - \Delta_3) - 2\left\{\Delta_2 - \frac{1}{2}(\Delta_1 + \Delta_3)\right\}\frac{a_2}{a}\,\cos(\varphi_2 - \varphi_{12}). \quad (26A)$$

$$\delta = \frac{1}{2}(\Delta_1 + \Delta_3) + \frac{1}{2}(\Delta_1 - \Delta_3)\,\frac{a_2}{a}\,\cos(\varphi_2 - \varphi_{12}) \quad . \quad . \quad (26B)$$

This situation has the advantage, that the mean of the periods of the fictitious pendulums depends only on the periods of the outer pendulums, Nos. 1 and 3; a change of the middle pendulum does not affect it.

A change of one of the outer pendulums is revealed at once by a change Δ of the difference $T_{12} - T_{32}$, but we cannot reverse this statement; if there is a change Δ, this can also be caused by a change Δ_2 of the second pendulum. A decision can easily be procured by making more observations in the same conditions for the fictitious pendulums, but with varying values of $a_2 \cos(\varphi_2 - \varphi_{12})$, which in the graphical representation means different positions of O. The quantities $\Delta_1 - \Delta_3$ and $\Delta_2 - \frac{1}{2}(\Delta_1 + \Delta_3)$ can then be computed with formula (26 A) from the values of Δ which have been found. It may be remarked that those varying values of a_2 ($\cos(\varphi_2 - \varphi_{12})$ are obtained automatically, even if the initial circumstances are the same, because they depend on the middle pendulum, which is irregularly disturbed, so that these values will also vary irregularly.

It is not difficult to determine in this way the eventual change of $T_1 - T_3$ so that the control of the constancy of T_1 and T_3 can easily be obtained.

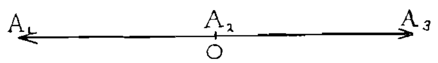
Other initial conditions may give less favourable circumstances. If for instance the amplitudes of the fictitious pendulums are equal and their phases the same, we find by applying the formulae (24), that the difference of their periods depends only on pendulums Nos. 1 and 3, so that a change of pendulum No. 2 would not be revealed, while the mean of the periods is not independent of this change. We might therefore incur the risk, that the difference of the periods of the fictitious pendulums is constant, so that there is no reason to suspect trouble, while the result for the gravity is vitiated by a change of the middle pendulum. This initial situation has therefore to be avoided.

The fact that in the first-mentioned case, the mean of the fictitious pendulums depends only on the two outer pendulums may procure the means of continuing a voyage of observation, even if one of the pendulums is suspected of irregular variations. We must then put this pendulum in the middle place in the apparatus and carefully attend to the conditions given by formulae (25).

The easiest way to realise these initial conditions is to begin by freeing the middle pendulum without giving it amplitude and by giving the outer pendulums equal amplitudes in opposite phases

$$a_2 = 0 \qquad a_1 = a_3 \qquad \varphi_3 = \varphi_1 + \pi. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In the graphical representation this means, that in the beginning the origin O coincides with A_2; during the course of the observation it will of course leave A_2 and describe an irregular curve, determined by the disturbance by the horizontal acceleration.



While the conditions (25) for the fictitious pendulums practically remain true during the whole observation, because the disturbances are small, the conditions (27) will soon be destroyed.

The apparatus has an arrangement, which makes it possible to give the pendulums amplitude at exactly the same moment in the same or in opposite phases, so that the realisation of the conditions (27), as far as the phase is concerned, is easy. It is however somewhat difficult to get exactly equal amplitudes for the outer pendulums. We will consider this question in detail in Chapter II (see page 63).

The way of realising the conditions (25) by the initial conditions (27) has some further advantages. Firstly the amplitude a_2 being zero at the beginning, it will remain small if the horizontal accelerations are not too large. In this case the correction terms with a_2 in the formulae (19 A) and (21 A or B) are small, so that the computation of the terms, containing $a_2 \cos(\varphi_2 - \varphi)$, requires less accuracy, and this is still more so for the terms containing a_2^2. As those computations are rather tedious because they have to be extended over the whole duration of the observation and because the curve of the second pendulum, from which they have to be deduced, is irregular, this is a decided practical advantage. If the disturbances are strong, so that a_2 gets a great value during the observations, the advantage is of course lost.

A second advantage ensues, because of the fact, that the conditions (25) $a_{12} = a_{32}$ and $\varphi_{12} = \varphi_{32} + \pi$, make the third term of the reduction to infinitely small amplitude, given by formula (21 A): $\frac{3}{16} T_1 \, a a_2 \cos(\varphi_2 - \varphi)$ equal with opposite sign for the two fictitious pendulums, so that an error in this term does not affect the mean of the two periods. If a_2 is small, this term is the principal one of the terms, which depend on the second pendulum, because the other terms contain the factor a_2^2. As the computation of these terms is troublesome for the reasons mentioned above, the fact that the principal one is eliminated from the mean result is a valuable point; it constitutes a second reason for allowing a less accurate computation of these terms. We will further enlarge on the methods of making these computations in chapter III.

Lastly we may notice that, because of the initial conditions (27), the horizontal stresses, exerted by the pendulums on the support, are in balance at the beginning of the observation, so that at least for the first part of the observation there is no sway of the apparatus, caused by the pendulums themselves. This is of course no important advantage, as the apparatus eliminates the effect of the much larger horizontal movements, caused by the ship's motion. Still we will see in the next paragraph, that for the observations on land, this advantage is valuable, and it is of course well to execute the observations, which have to be made before and after the voyages in the central gravity base station, as nearly as possible in the same conditions as the observations during the voyages.

§ 12. *Choice of the amplitudes and the phase-differences for land observations.*

If the apparatus is used on land, which will for instance occur for every determination at the gravity base station and for control determinations at intermediate ports, other considerations prevail as to the most suitable initial amplitudes and phase-differences. The investigation of all the possibilities has not yet been completed at the moment of writing this paper, but it has been pursued far enough to be sure, that the methods indicated in this paragraph, are satisfactory.

Another point, which has not been investigated, is the possibility of using the apparatus on land for a quick gravity survey, for which not the utmost accuracy is required, by applying the idea, which is roughly sketched on page 65 of the next chapter. The suggestions, made in this paragraph, do not refer to that possibility, for which probably the same arrangements as those, indicated in the preceding paragraph for the maritime gravity survey, will be useful. We will suppose, that a greater accuracy is required and that in addition precise data are wanted on the differences of the pendulum-periods, in order to be able to check the constancy of the pendulums.

If we wish to obtain a greater accuracy, it will be desirable to diminish the error resulting from the fluctuation of the rate of the chronometer, which is present in the maritime observations because the observations take only half an hour while the rate is deduced from two time-signals which are a great many hours apart. We will therefore have to extend the land observations over the complete period from one time-signal to the next one.

We can do this by making a few half-hour observations at equal intervals during this period, but it is safer to fill up the whole period with pendulum observations, so that the fluctuations of the rate may be completely eliminated from the results. This may of course be done in the usual way by taking the mean after giving each result a weight, proportional to the duration t of the observation:

$$T = \frac{t' T' + t'' T'' + \ldots + t^{(n)} T^{(n)}}{t' + t'' + \ldots + t^{(n)}} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

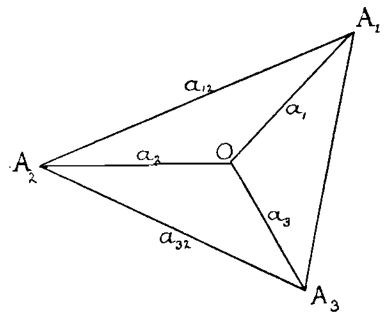
This continuous series of observations makes it desirable, in order to save work, to make the observations as long as possible and to try to avoid the necessity of recording the middle pendulum during the whole duration of the observation. This is generally possible if we dispose of such a firm foundation for the apparatus, that we may neglect the outside disturbances of the pendulum motion, so that the movement of the knife-edges is only caused by the sway of the support, brought about by the pendulums themselves.

It can in fact be shown, that we may give such phase-differences and amplitudes to the three pendulums, that these phase-differences as well as the ratio of the amplitudes are constant during an indefinite time, so that each pendulum swings with a constant period and a constant damping. In that case no intermediate records of the second pendulum are necessary as the quantities, which are required, may be computed from their initial and final values. There are three different solutions of the problem.

We will not go into these theoretical considerations and intricate computations as there is a simple way in our case to find a sufficient approximation to the required solution. The exact solutions would not be of any more use, first because it is not practically possible to realise exactly the necessary amplitudes and phase-differences and second because the solutions, which are mentioned, are only rigorously true for infinitely small amplitudes, so that for the usual amplitudes deviations will occur anyway.

The approximate solution, which will be applied, is the choice of such amplitudes and phase-differences, that the stresses, exerted by the three pendulums are in balance, so that there is no sway of the apparatus. As these stresses are proportional to the angles of elongation of the pendulums, we have to make the sum of these angles zero and because the angles are the projections of the rotating pendulum-vectors $O A_1$, $O A_2$ and $O A_3$ on the X-axis, this is realised when the resultant of these vectors is zero. We may express this result in this way, that the sway of the apparatus is zero, if the origin O coincides with the centre of gravity of the triangle $A_1 A_2 A_3$.

If this situation is realised, the only cause of a change of the regular movement of the pendulums is the small deviation of isochronism of the pendulums; this will bring about a slow deformation of the triangle $A_1 A_2 A_3$, so that gradually O will not longer coincide with the centre of gravity and sway will begin to take place. In this way the regularity of the movement will be destroyed in the long run. It depends principally on the value of the differences of the pendulum periods how long the observations can be pursued before the amplitude of the second pendulum shows such irregularities, that continuous recording becomes necessary. If these differences do not exceed some 50 10^{-7} sec., observations of some two or three hours will be possible without this complication. *)



In no case longer observations will be advisable because of the damping of the pendulums, which can not be diminished as the apparatus can not be pumped out.

The approximate solution leaves us a much greater liberty of choice for the initial circumstances than the three exact solutions, which have been mentioned: there is yet an infinite number of possibilities.

We have already noticed in the preceding paragraph, that the conditions (27), chosen for the maritime observations

$$a_2 = 0 \qquad a_1 = a_3$$
$$\varphi_1 = \varphi_3 + \pi$$

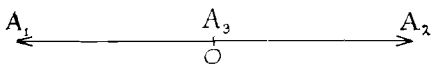
make the sway zero; O is indeed coinciding with the centre of gravity of $A_1 A_2 A_3$. They constitute therefore a possible solution, which moreover turns out to be an appropriate one. Besides giving the advantage that the observations on land are made under the same conditions as the maritime observations, it just procures the two principal results, which are desired: As a_2 is zero or at least very small, we see (form 8 B and C) that the two fictitious pendulums have the periods T_1 and T_3, and we have seen in the previous paragraph that for the maritime observations the mean result of the two fictitious pendulums depends on these same two periods. We might therefore adequately choose these conditions for the whole series of observations between two time-signals.

It will however be desirable to get also data for the middle pendulum, partly because we wish to have this pendulum as a reserve during the voyage in case one of the outer pendulums goes wrong and partly because we wish to use the control of the maritime observations, given by the difference of

*) A more thorough investigation of the problem is given in the Appendix IV to this chapter. It is shown there, that if the approximate solution is not completely realised, small irregularities will occur with a period $\frac{T^2}{3u}$ (u being the sway correction for a single pendulum, swinging alone); we may take them into account by making two short intermediate records. See also Chapter III.

the two fictitious pendulum periods, which depends also on the middle pendulum. We require therefore data concerning this pendulum. The easiest way to procure these data is to determine the difference between the period of this pendulum and of one of the outer ones, as this can be done with a few observations without requiring a complete new set between two time-signals.

We can for instance make a few additional observations under the conditions

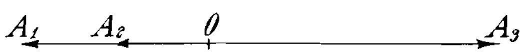


$$a_1 = a_2 \qquad a_3 = 0$$
$$\varphi_1 = \varphi_2 + \pi$$

which obviously also answer the purpose of making the sway zero. We deduce at once from formulae (8 B) and (8 C) that the periods of the fictitious pendulums have the value T_2 and $\frac{1}{2}(T_1 + T_2)$ so that the difference gives $T_1 - T_2$ independently of the rate of the chronometer during the observation.

If we do not wish to make any additional observation besides the principal sets, we may also obtain the required data on the middle pendulum by making one or two observations in each set under the conditions

$$a_1 = 2\,a_2 \qquad a_3 = 3\,a_2 \qquad \varphi_1 = \varphi_2 = \varphi_3 + \pi \quad . \; . \; . \quad (29\,A)$$



in stead of under the conditions (27). O coincides with the centre of gravity of A_1 A_2 A_3, so that the sway is again zero.

The two fictitious pendulums have the periods (see 8 B and 8 C)

$$T_{12} = 2\,T_1 - T_2 \text{ and } T_{32} = \frac{1}{4}\,T_2 + \frac{3}{4}\,T_3 \quad . \; . \; . \quad (29\,B)$$

so that the difference gives

$$2\,T_1 - \frac{3}{4}\,T_3 - \frac{5}{4}\,T_2$$

from which we may obtain the difference between T_2 and T_1 or T_3. The result is again independent of the rate of the chronometer during the observation.

Eliminating T_2 from (29 B) we get

$$\frac{1}{5}\,T_{12} + \frac{4}{5}\,T_{32} = \frac{2}{5}\,T_1 + \frac{3}{5}\,T_3$$

and by adding $^1/_{10}$ of the difference $T_1 - T_3$, which can be determined from the other observations of the set, we obtain

$$\frac{1}{2}(T_1 + T_3)$$

This result for the mean of the periods of the fictitious pendulums may be used in the ordinary way with the results for this quantity, given by the other observations of the set, to determine the value of $\frac{1}{2}(T_1 + T_3)$ for the whole set, according to formula (28).

Appendix I to Chapter I.

The solution of the equation of motion (2) (page 4) may be expressed in a more elegant way by the introduction of complex quantities

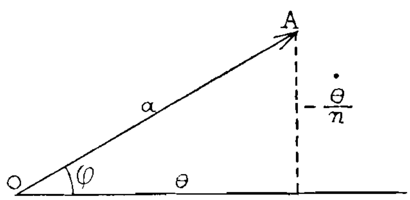
Substitute in stead of the variable θ the complex variable q, given by

$$q = \theta - \frac{i}{n}\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

in which

$$i = \sqrt{-1}.$$

According to the generally accepted way of representing a complex quantity by a vector, q may be represented by the vector OA, which we have called the pendulum-vector. Where no misunderstanding is possible, q may be identified with the pendulum-vector in order to avoid unnecessary phraseology.



The introduction of q in the equation of motion gives

$$\dot{q} - inq - i\frac{S}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

of which the solution is

$$q = (q_0 + \delta q^t)\, e^{int} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32\,A)$$

q_0 being the value of q for $t = 0$ and

$$\delta q^t = \frac{i}{n}\int_0^t S e^{-int}\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32\,B)$$

If there is no disturbance-term S, q is a vector of constant length, rotating with a constant angular velocity n, and θ is the projection of q on the horizontal axis.

If S is not zero, we may deduce at once the equations ($4\,A$) and ($4\,B$) for the disturbances of the period and of the amplitude by dividing (31) by q and separating the real and the imaginary parts. As further $\dot{q}$ represents the velocity of the extremity of the pendulum-vector, the formula (31) gives the disturbance of this velocity, caused by the term S, namely, a vertical component $= \frac{S}{n}$.

By dividing ($32\,A$) by $e^{i(nt+\varphi_0)}$ and by separating the real and the imaginary parts, we find the following slightly different formulae for the disturbances of the period and of the amplitude, which in several cases may be useful, e. g. when the amplitude is small

$$\sin \frac{t}{T}\, n\,\delta T = -\frac{1}{n a}\int_0^t S \cos(nt + \varphi_0)\, dt \quad . \quad . \quad . \quad (33\,A)$$

$$a \cos \frac{t}{T}\, n\,\delta T - a_0 = \frac{1}{n}\int_0^t S \sin(nt + \varphi_0)\, dt \quad . \quad . \quad . \quad (33\,B)$$

in which δT is the mean disturbance of the period during some arbitrary time t.

If the angle $\frac{t}{T}\, n\,\delta T$ is small, we find the approximate formulae

$$\delta T = -\frac{T}{n^2 a t}\int_0^t S \cos(nt + \varphi_0)\, dt \quad . \quad . \quad . \quad . \quad . \quad (34\,A)$$

$$\dot{a} = \frac{1}{n t}\int_0^t S \sin(nt + \varphi_0)\, dt \quad . \quad . \quad . \quad . \quad . \quad (34\,B)$$

which are similar to formulae ($6\,A$) and ($6\,B$) if we subtitute $t = 2\tau$.

For further details see: „Observations de pendule dans les Pays-Bas", page 9 a. f.

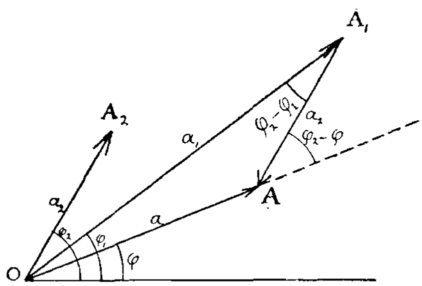
Appendix II to Chapter I.

Deduction of other formulae for the correction for deviation of isochronism and for the reduction to infinitely small amplitude of the fictitious pendulum.

The following formulae may be useful in case the two original pendulums are recorded without recording at the same time the fictitious pendulum itself. We have then to express the formulae, instead of in the quantities

$$a,\ a_2,\ (\varphi_2 - \varphi)$$

in three other quantities, sufficient to define triangle $O A_1 A$ and which are directly recorded; namely in



$$a_1,\ a_2,\ (\varphi_2 - \varphi_1)$$

or we may keep a, as a is regular and only slowly variable, so that it is known throughout the whole observation, if it has been computed at the beginning and at the end. We can then express the formulae solely in the amplitudes a_1, a_2 and a. As a_1 and a_2 may more easily be determined than the phase-difference ($\varphi_2 - \varphi_1$), because the measurement of the amplitudes of the disturbed curves is less difficult than the measurements of the phases, we will choose this last way.

From the triangle $O\,A_1\,A$ we may deduce the following formula for eliminating $(\varphi_2 - \varphi)$ and expressing in a_1, a_2 and a

$$cos\,(\varphi_2 - \varphi) = \frac{a_1^2 - a_2^2 - a^2}{2\,a\,a_2}\,.$$

Substituted in the formulae $(19\,A)$ and $(21\,A)$ we get for the period of the fictitious pendulum

$$T = \frac{1}{2}\,(T_1 + T_2) + U_{21}\,\frac{(a_2^2 - a_1^2)}{2\,a^2} \quad \ldots\ldots \quad (35)$$

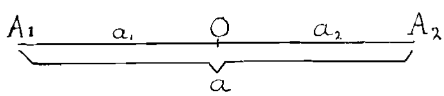
and for the reduction to infinitely small amplitude

$$\delta\,T = \frac{1}{32}\,T\left[a_1^2 + a_2^2 + \left(\frac{a_2^2 - a_1^2}{a}\right)^2\right] \quad \ldots\ldots \quad (36)$$

In case the initial conditions of the observation have been

$$a_1 = a_2 \qquad \varphi_1 = \varphi_2 + \pi \quad \ldots\ldots \quad (37)$$

as has for instance been the case during the voyage, made in 1923 from Holland to Java via the Suez-canal during which the observations were made with the ordinary four-pendulum Sterneck apparatus, the sum $a_1 + a_2$ does not deviate much from a, if the ship's movements are not very strong. This is easily verified by examining the graphical representation of the pendulum-vectors: the disturbances by the ship's movements give a variation of the position of O but do not affect the relative positions of A_1 and A_2. In this case we may profitably introduce the quantity ρ, given by



$$\rho = a_1 + a_2 - a$$

which is generally so small, that in the following formulae the terms with ρ may be neglected. We get then

$$T = \frac{1}{2}\left(T_1 + T_2\right) + U_{21}\,\frac{(a_2 - a_1)}{2\,a}\left(1 + \frac{\rho}{a}\right) \quad \ldots \quad (38\,A)$$

$$\delta\,T = \frac{1}{64}\,T\left[a^2 + 3\,(a_2 - a_1)^2\right]\left(1 + 2\,\frac{\rho}{a} + \ldots\right) \quad \ldots \quad (38\,B)$$

The initial conditions (37) are the most suitable for all instruments, which have not more than two pendulums swinging in the same plane.

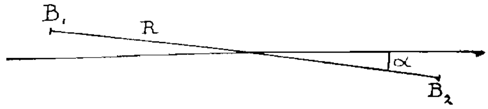
Appendix III to Chapter I.

Deduction of the formulae for the disturbances of the fictitious pendulum, caused by the fact, that the knife-edges of the original pendulums have relative movements, if the line connecting these knife-edges deviates from its normal horizontal position.

Supposing B_1 and B_2 are the knife-edges of the original pendulums, $2R$ their distance and α the deviation from the horizontal, the coordinates of B_1 are

$$x_1 = R \sin \alpha = R\alpha$$

$$y_1 = R \cos \alpha - R = -\frac{1}{2} R\alpha^2$$



and of B_2 the same values with contrary sign.

We will examine separately the effect on the fictitious pendulum of the horizontal and vertical accelerations.

I. Horizontal accelerations.

The disturbance terms S_1 and S_2 of the two original pendulums are

$$S_1 = \frac{\ddot{y}_1}{l} = -\frac{R}{2l}\frac{d^2}{dt^2}(\alpha^2) \qquad S_2 = \frac{\ddot{y}_2}{l} = +\frac{R}{2l}\frac{d^2}{dt^2}(\alpha^2)$$

so that we get for the fictitious pendulum

$$S = S_1 - S_2 = -\frac{R}{l}\frac{d^2}{dt^2}(\alpha^2)$$

Introduced in formulae ($6A$) and ($6B$), we find

$$\delta T = \frac{R}{2ga}\int_0^{2T}\frac{d^2}{dt^2}(\alpha^2)\cos\varphi\, dt \quad \ldots\ldots \quad (39A)$$

$$\dot{a} = -\frac{R}{2\pi l}\int_0^{2T}\frac{d^2}{dt^2}(\alpha^2)\sin\varphi\, dt \quad \ldots\ldots \quad (39B)$$

As it may be assumed that $\frac{d^2}{dt^2}(\alpha^2)$ is an irregularly variable quantity, without terms having the same period as the pendulums, the reasoning of page 11 a.f. may be applied and we may conclude that δT is negligible, if the amplitude remains stable.

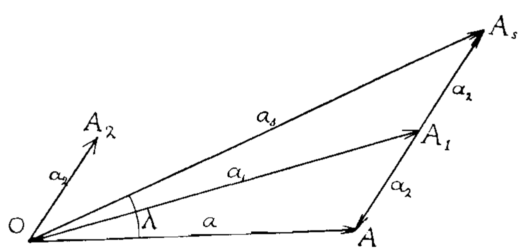
II. Vertical accelerations.

The disturbance terms S_1 and S_2 are

$$S_1 = \frac{\ddot{x}_1}{l}\theta_1 = \frac{R}{l}\ddot{\alpha}\theta_1 \qquad S_2 = \frac{\ddot{x}_2}{l}\theta_2 = -\frac{R}{l}\ddot{\alpha}\theta_2$$

so that for the fictitious pendulum S is

$$S = S_1 - S_2 = \frac{R}{l}\ddot{\alpha}(\theta_1 + \theta_2)$$



$\theta_1 + \theta_2$ may be considered to be the projection of the resultant vector OA_s of the two pendulum vectors OA_1 and OA_2. Representing its length by a_s, its phase by φ_s, and the phase-difference with the fictitious pendulum, $\angle A_s OA$, by λ; we have

$$S = \frac{R}{l} \ddot{\alpha} \, a_s \cos \varphi_s = \frac{R}{l} \ddot{\alpha} \, a_s \cos (\varphi + \lambda) \quad . \quad . \quad . \quad . \qquad (40)$$

In case the ship's movements are not very strong, the shape of the triangle $O \, A_s \, A$ does not vary much, so that the same may be said of a_s and also of λ, if a_s is not small. This last case which occurs for instance when the initial conditions (37) are realised, may be dismissed at once, as S contains a_s as factor, so that in that case S is likewise small.

Introducing (40) in formulae ($6A$) and ($6B$) we get for the disturbances

$$\delta \, T = - \frac{R}{4 \, g \, a} \int_0^{2T} \ddot{\alpha} \, a_s \cos \lambda \, d t - \frac{R}{4 \, g \, a} \int_0^{2T} \ddot{\alpha} \, a_s \cos (2 \varphi + \lambda) \, d t. \qquad (41 \, A)$$

$$\dot{a} = - \frac{R}{4 \pi l} \int_0^{2T} \ddot{\alpha} \, a_s \sin \lambda \, d t + \frac{R}{4 \pi l} \int_0^{2T} \ddot{\alpha} \, a_s \sin (2 \varphi + \lambda) \, d t. \qquad (41 \, B)$$

With regard to the second terms the usual reasoning may be applied. If $\ddot{\alpha}$ is an irregularly varying quantity without terms having the double period of the pendulum, $\delta \, T$ is negligible if the amplitude does not show fluctuations.

For getting an approximative estimate of the first terms we may consider a_s and λ as constant during the time of the integration, so that we get

$$\delta \, T = - \frac{T}{2 \, g} R \, [\ddot{\alpha}] \frac{a_s}{a} \cos \lambda \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42 \, A)$$

$$\dot{a} = - \frac{T}{2 \, \pi \, l} R \, [\ddot{\alpha}] \frac{a_s}{a} \sin \lambda \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42 \, B)$$

in which $[\ddot{\alpha}]$ represents the mean of $\ddot{\alpha}$ during the time, which we consider.

If λ has an arbitrary value, the usual way of reasoning again holds true; if the amplitude is stable, $\delta \, T$ is negligible. As however the initial circumstances for the sea-apparatus make a_s zero at the beginning of the observation, the initial value of λ is also zero in this case and we cannot apply the conclusion, so that we have to look at the question from another view-point.

As $[R \, \ddot{\alpha}]$ is the mean vertical acceleration of the knife-edge with regard to the centre of the apparatus, we may notice that the first term of the formula for $\delta \, T$ has the same shape as the first term of formula ($9 \, A$), which gives the effect of the vertical accelerations of the whole apparatus, with an additional factor, which does not alter the order of magnitude. We may further be sure that the accelerations of the whole apparatus are considerably larger than the relative accelerations, which are considered here, and as the effect of those accelerations is negligible, as has been indicated in paragraph 3, we may safely conclude, that the effect of the relative accelerations, which is the subject of this paragraph, is also negligible.

Appendix IV to Chapter I.

Investigation into three pendulums swinging together in case the apparatus is used on land.

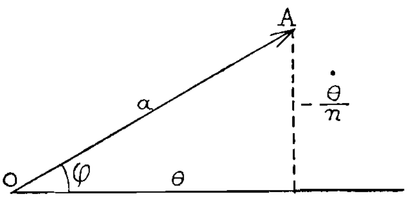
We suppose that the knife-edges are not subject to other horizontal accelerations than the accelerations, which correspond to the sway of the apparatus by the horizontal stresses by the pendulums, and we assume, that there are no other disturbances of the regular pendulum movement save the damping. This implies, that the conclusions of the following investigation are only rigorously true for infinitely small amplitudes. For the amplitudes, which are practically used, the periods are slightly variable corresponding to the reduction to infinitely small amplitude and this changes somewhat the course of the phenomenon. We will further assume, that the differences of the periods are small with regard to the periods themselves.

We will begin with the general problem of an arbitrary number of pendulums, swinging together.

§ 1. *General case of m pendulums swinging together.*

We will introduce complex quantities in the same way as has been done in Appendix I, so that each pendulum motion is characterised by the quantity

$$q = \theta - \frac{i}{n}\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$



corresponding to a revolving vector, of which θ is the horizontal projection, a the length and φ the phase.

The equation of motion of the undisturbed pendulum is

$$\dot{q} = i\,n\,q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43\,A)$$

and if the pendulum is subject to damping

$$\dot{q} = i\,r\,q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43\,B)$$

with

$$r = n + i\,k$$

in which n is the undisturbed phase-velocity (the angular velocity of the pendulum-vector) and k the damping-coefficient. The damping causes a logarithmic decrease of the amplitude a.

In case there is sway, caused by some second pendulum, which we will indicate by the suffix 1, there is a disturbance term in the formulae (43 A) and (43 B), which we can prove to be proportional to q_1, (See: Observations de Pendule dans les Pays-Bas, page 22 a. f., e. g. form. 20 C).

$$\dot{q} = i\,r\,q - i\,d\,q_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43\,C)$$

If the internal friction of the apparatus can be neglected, as is always the case, d is a real constant; it is dependent on the elasticity of the support and on the moment of mass of the second pendulum with regard to the knife-edge.

If we want the effect of the sway caused by the pendulum itself, we have to suppress the suffix 1 and we get for the pendulum swinging alone

$$\dot{q} = i\,(r - d)\,q = i\,(n - d + i\,k)\,q$$

that is to say that the phase-velocity of the pendulum-vector is diminished with d. If u is the corresponding increase of the period of the pendulum, that is to say the ordinary correction for sway, we have with sufficient approximation

$$d = \frac{n}{T}\,u = \frac{\pi}{T^2}\,u \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (44)$$

which gives the relation between d and the correction for sway.

If m pendulums are swinging together, each equation of motion gets m disturbance-terms, corresponding to the sway, caused by the m pendulums. Supposing that the pendulums have about the same moment of mass with regard to the knife-edges, so that the constants d are practically equal, we get:

$$\begin{array}{llll} \dot{q}_1 = i\,(r_1 - d)\,q_1 & - i\,d\,q_2 & \ldots\ldots - i\,d\,q_m & r_1 = n_1 + i\,k_1 \\ \dot{q}_2 = - i\,d\,q_1 & + i\,(r_2 - d)\,q_2 & \ldots - i\,d\,q_m & r_2 = n_2 + i\,k_2 \\ \ldots\ldots\ldots & & & \ldots\ldots \\ \dot{q}_m = - i\,d\,q_1 & - i\,d\,q_2 & \ldots\ldots + i\,(r_m - d)\,q_m & r_m = n_m + i\,k_m \end{array} \qquad (45)$$

The general solution of this set of m differential equations of the first degree and the first order can be written in the following shape

$$\begin{array}{l} q_1 = A_1\,e^{i x_a t} + B_1\,e^{i x_b t} + \ldots\ldots + M_1\,e^{i x_m t} \\ q_2 = A_2\,e^{i x_a t} + B_2\,e^{i x_b t} + \ldots\ldots + M_2\,e^{i x_m t} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ q_m = A_m\,e^{i x_a t} + B_m\,e^{i x_b t} + \ldots\ldots + M_m\,e^{i x_m t} \end{array} \qquad (46\,A)$$

in which x_a, x_b, ... x_m are the m roots of the equation

$$\begin{vmatrix} x - r_1 + d & d & \ldots\ldots & d \\ d & x - r_2 + d & \ldots\ldots & d \\ \ldots & \ldots & \ldots & \ldots \\ d & d & \ldots\ldots & x - r_m + d \end{vmatrix} = 0 \quad . \qquad (46\,B)$$

which can also be written in the form

$$\frac{1}{x - r_1} + \frac{1}{x - r_2} + \ldots\ldots + \frac{1}{x - r_m} = - \frac{1}{d} \;. \qquad (46B')$$

The constants A_1, A_2, ... A_m of (46 A) may be expressed in one constant A and in x_a

$$A_1 = \frac{A}{x_a - r_1} \quad A_2 = \frac{A}{x_a - r_2} \;\ldots\ldots\; A_m = \frac{A}{x_a - r_m} \quad . \qquad (46\,C)$$

and in the same way B_1, B_2, ... B_m in a constant B and in x_b, and so on for the other constants. All these constants will generally be complex quantities.

The constants A, B, ... M depend on the conditions at the beginning, e. g. the values of q_1, q_2, ... q_m for $t = 0$

$$\begin{aligned} q_1^\circ &= A_1 + B_1 + \dots\dots + M_1 \\ q_2^\circ &= A_2 + B_2 + \dots\dots + M_2 \\ &\dots\dots\dots\dots\dots\dots\dots \\ q_m^\circ &= A_m + B_m + \dots\dots + M_m \end{aligned} \qquad \dots (46\,D)$$

They may for instance be determined by introducing m sets of coefficients $\alpha_1, \alpha_2, \dots\dots \alpha_m$; $\beta_1, \beta_2, \dots\dots \beta_m$; etc., so that

$$\begin{aligned} A &= \alpha_1 q_1^\circ + \alpha_2 q_2^\circ + \dots\dots + \alpha_m q_m^\circ \\ B &= \beta_1 q_1^\circ + \beta_2 q_2^\circ + \dots\dots + \beta_m q_m^\circ \\ &\dots\dots\dots\dots\dots\dots\dots \\ M &= \mu_1 q_1^\circ + \mu_2 q_2^\circ + \dots\dots + \mu_m q_m^\circ \end{aligned} \qquad \dots (47\,A)$$

The m^2 constants

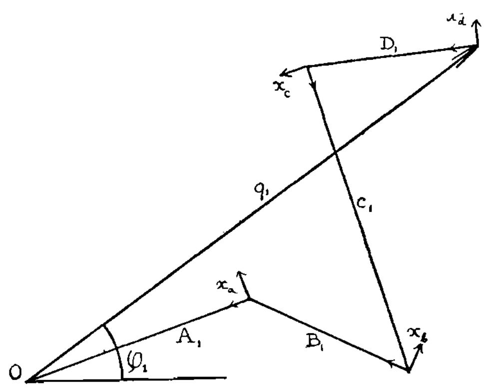
$$\begin{matrix} \alpha_1 & \alpha_2 \dots\dots \alpha_m \\ \beta_1 & \beta_2 \dots\dots \beta_m \\ \dots & \dots\dots\dots \\ \mu_1 & \mu_2 \dots\dots \mu_m \end{matrix}$$

are obviously found by taking the sub-determinants of

$$\begin{vmatrix} \frac{1}{x_a - r_1} & \frac{1}{x_a - r_2} & \dots\dots & \frac{1}{x_a - r_m} \\ \frac{1}{x_b - r_1} & \frac{1}{x_b - r_2} & \dots\dots & \frac{1}{x_b - r_m} \\ \dots & \dots & \dots & \dots \\ \frac{1}{x_m - r_1} & \frac{1}{x_m - r_2} & \dots\dots & \frac{1}{x_m - r_m} \end{vmatrix} \qquad \dots (47\,B)$$

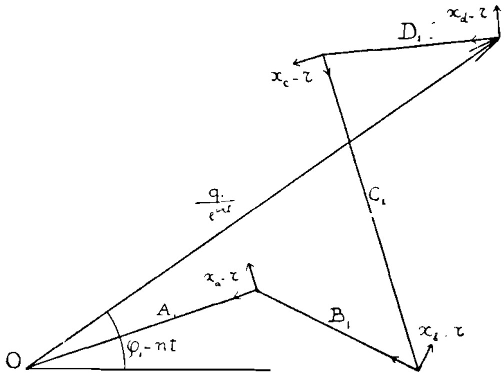
which correspond to their positions, and dividing them by the complete determinant.

The values of $x_a, x_b, \dots\dots x_m$ and of the coefficients $\alpha, \beta, \dots\dots \mu$ may be determined once for all for a certain set of pendulums and a certain value of the sway. By means of the equations (47 A) we can then determine the constants $A, B, \dots\dots M$ for an arbitrary initial condition and (46 C) gives the values of $A_1, A_2, \dots\dots A_m$; $B_1, B_2, \dots\dots, B_m$; etc. All the elements of the solution (46 A) are then known.



The graphical representation of the complex quantities by vectors is apt to give a better insight in the solution than the formulae can do. Formula (46 A) shows that each pendulum-vector is the sum of m vectors revolving with angular velocities, given by the real parts of $x_a, x_b, \dots\dots x_m$. The lengths of these vectors decrease logarithmically corresponding to damping coefficients, given by the imaginary parts of x_a, x_b, ... x_m.

In stead of representing the vectors q_1, q_2, ... q_m themselves we can also represent these vectors divided by e^{irt}, in which r is an arbitrary constant, for which we will choose the mean value of r_1, r_2, ... r_m. The differences $x_a - r$, $x_b - r$, ... $x_m - r$ being small with regard to x_a, x_b ..... x_m, this representation has the advantage that the figure is only slowly variable, in stead of the previous representation, in which the vectors are revolving quickly: Each vector is now revolving with a velocity, equal to the difference of its original angular velocity minus the real part of r, while its length is decreasing corresponding to a damping equal to the difference of the original damping minus the imaginary part of r.



This representation may give a good idea of the phenomenon; the phase-differences of the pendulum-vectors and the ratio of the amplitudes may for instance be immediately found in the figure as they remain unchanged by the division by e^{irt}.

We may notice a remarkable feature, which is shown by the solution. The values of the constants A, B, ... N depend on the initial circumstances; we may by a proper choice of initial conditions, reduce all the constants, save one to zero. For this it is necessary that the initial pendulum-vectors q_1°, q_2°, ... q_m° have the same ratio as A_1, A_2, ... A_m (see 46 C) or as one of the other sets; obviously (46 D) is then satisfied by putting all the constants, save one vertical row, zero.

Supposing this row to be the row of the constants A_1, A_2, ... A_m the general solution (46 A) gets the simple shape

$$
\begin{aligned}
q_1 &= A_1 \, e^{i x_a t} \\
q_2 &= A_2 \, e^{i x_a t} \\
\cdots & \quad \cdots\cdots\cdots \\
q_m &= A_m \, e^{i x_a t}
\end{aligned}
\qquad (48)
$$

This means for the graphical representation, that the pendulum-vectors coincide with the vectors A_1, A_2, ... A_m.

We see that in this case the pendulum-vectors have the same angular velocity, that is to say that all the pendulums have the same period and that their phase-differences are constant. The ratio of their amplitudes remains also the same during the whole duration of the observation: the pendulums have the same damping. We have however to bear in mind according to the statement on page 35, that this special situation is only maintained during an indefinite time if the reduction to infinitely small amplitude is negligible.

For a set of m pendulums there are m different special situations of this kind, corresponding to the m terms, of which the pendulum-vectors $q_1, q_2, \ldots . q_m$ are composed. We made allusion to these situations in § 12, page 26.

We will not further discuss here the properties of the general solution and the special cases, which may occur; only one of them will be considered more in detail, i. e. the case that all the pendulums have the same damping coefficient.

§ 2. *m pendulums with equal damping, swinging together.*

This is practically always the case, when similar pendulums are swinging together; the difference of the damping coefficients is generally so small, that it can be neglected.

In this case $r_1, r_2, \ldots . r_m$ and their mean value r have the same imaginary part, ik. The differences $r_1 - r$, $r_2 - r$ etc. are therefore real quantities and the same may be said of the differences $x_a - r$, $x_b - r, \ldots .$ $x_m - r$. We can see this at once by examining the equation ($46 B'$); if $x_a - r$ has any imaginary part, all the quantities $x_a - r_1$, $x_a - r_2, \ldots . x_a - r_m$ must have this same imaginary part, so that the reciprocal values of these last quantities must have imaginary parts which have the same sign. As however their sum equals the real quantity $-\frac{1}{d}$, we see that this is impossible, so that we may conclude that all the quantities in the left member of ($46 B'$) are real and therefore also the quantity $x_a - r$. The same reasoning holds true for the other qantities $x_b - r$, $x_c - r, \ldots . x_m - r$.

These quantities as well as the quantities $r_1 - r$, $r_2 - r, \ldots . r_m - r$ are obviously small with regard to r. Each represents the difference between the angular velocity of one of the vectors x_a, $x_b \ldots . r_1, r_2 \ldots .$ and the angular velocity n of r, n being the real part of r and therefore the mean of $n_1, n_2, \ldots . n_m$. We might profitably introduce in the formulae these small differences of angular velocity instead of the complex quantities $x_a, \ldots . x_m$ and $r_1, \ldots . r_m$ themselves but it is still more useful to introduce the differences of the periods, which correspond to these differences of the angular velocities, because the periods are more immediately the subject of the observations.

We see at once that the difference s_1 of the period, which corresponds for instance to the difference $r_1 - r$ of the angular velocity, is with sufficient approximation

$$s_1 = -\frac{T}{n}(r_1 - r) = -\frac{T^2}{\pi}(r_1 - r)$$

and so we will introduce the following two sets of quantities

$$\begin{array}{ll}
s_1 = -\frac{T^2}{\pi}(r_1 - r) & y_a = -\frac{T^2}{\pi}(x_a - r) \\
s_2 = -\frac{T^2}{\pi}(r_2 - r) & y_b = -\frac{T^2}{\pi}(x_b - r) \\
\ldots\ldots\ldots & \ldots\ldots\ldots \\
s_m = -\frac{T^2}{\pi}(r_m - r) & y_m = -\frac{T^2}{\pi}(x_m - r)
\end{array} \qquad (49 A)$$

of which the meaning is given by:

$$\begin{array}{ll} s_1 = T_1 - T & y_a = T_a - T \\ s_2 = T_2 - T & y_b = T_b - T \\ \dots\dots\dots & \dots\dots\dots\dots \\ s_m = T_m - T & y_m = T_\mu - T \end{array} \quad . \; . \quad (49\,B)$$

T_1, T_2, ... T_m are the undisturbed periods of the pendulums and T their mean value; T_a, T_b, ... T_μ are the periods of the m different terms, of which each pendulum-vector q is composed according to formula (46 A).

Lastly we will introduce the correction for sway u instead of d (see form. 44).

By these substitutions the solution given by the formulae (46) and (47) gets the following shape

y_a, y_b . . . . . . y_m are the roots of

$$\frac{1}{y - s_1} + \frac{1}{y - s_2} + \dots\dots + \frac{1}{y - s_m} = \frac{1}{u} \, . \qquad (50\,A)$$

The terms of this equation as well as the roots are all real quantities.

Each pendulum-vector q is the sum of m components, revolving with periods T_a, T_b, ... T_μ given by (49 B). The first components of the m pendulum-vectors, A_1, A_2, ... A_m, depend in the following way on one vector A

$$A_1 = \frac{A}{y_a - s_1} \quad A_2 = \frac{A}{y_a - s_2} \quad \dots\dots \quad A_m = \frac{A}{y_a - s_m} \, . \qquad (50\,B)$$

As y_a and s_1, s_2, ... s_m are real, the components A_1, A_2, ... A_m have all the same direction, that is to say the same phase, as A.

In the same way the second components B_1, B_2, ... B_m depend on y_b and on a vector B and have the same phase, and so on.

The vectors A, B, ... M depend on the initial circumstances in the way given by (46 D); they may be determined by means of the coefficients

$$\begin{array}{lll} \alpha_1 & \alpha_2 & \dots \; \alpha_m \\ \beta_1 & \beta_2 & \dots \; \beta_m \\ \dots & \dots & \dots \\ \mu_1 & \mu_2 & \dots \; \mu_m \end{array}$$

and the equation (47 A). The coefficients α, β, . . . μ are found by taking the corresponding sub-determinants of

$$\begin{array}{cccc} \dfrac{1}{y_a - s_1} & \dfrac{1}{y_a - s_2} & \dots\dots & \dfrac{1}{y_a - s_m} \\ \dfrac{1}{y_b - s_1} & \dfrac{1}{y_b - s_2} & \dots\dots & \dfrac{1}{y_b - s_m} \\ \dots & \dots & \dots & \dots \\ \dfrac{1}{y_m - s_1} & \dfrac{1}{y_m - s_2} & \dots\dots & \dfrac{1}{y_m - s_m} \end{array} \quad . \; . \; . \; . \; . \; . \quad (51)$$

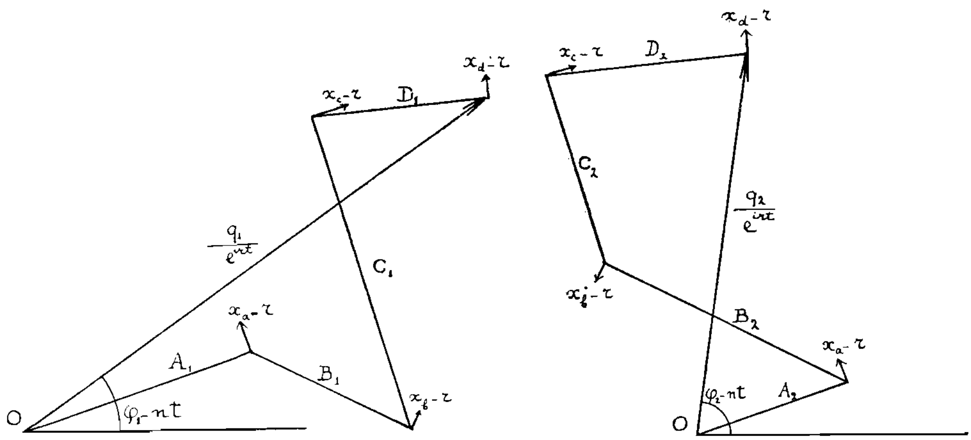
and dividing them by the complete determinant. As all the terms of this determinant are real, the coefficients α, β . . . . . μ are also real in this case.

The graphical representation of the pendulum-vectors $q_1, \ldots.. q_m$ has the same character as that of the previous paragraph. Each vector consists of m components. They revolve with periods $T + y_a$, $T + y_b, \ldots.. T + y_m$ and decrease logarithmically corresponding to the same damping coefficient k. The corresponding components are parallel in this case.

If we choose the second way and represent $\frac{q_1}{e^{irt}}$, $\frac{q_2}{e^{irt}}$ etc. in order to obtain a more stable figure, the m components of each vector have a constant length and revolve with the small angular velocities $x_a - r$, $x_b - r$ etc.; the periods of the components are therefore

$$\frac{\pi}{x_a - r} = -\frac{T^2}{y_a}; \quad -\frac{T^2}{y_b}; \quad \ldots. \quad -\frac{T^2}{y_m} \quad \ldots\ldots \quad (52)$$

in which a negative period means a rotation in contrary direction. The corresponding vectors for different pendulums are again parallel. As the figure gives immediately the phase-differences and the ratio of the amplitudes of the pendulums, (52) provides the means to deduce the periods of these quantities.



The special situations, which have been mentioned on page 38 and which are characterized by constant phase-differences and a constant ratio of the amplitudes for all the pendulums, occur when the initial conditions are such, that all the constants A, B, ... M save one are zero, that is to say when all the components of the pendulum-vectors are zero save one set. We see that in this case the pendulum-vectors have to be parallel, that is to say that the pendulums have to swing in the same or in contrary phase. The necessary ratio of the amplitudes is given by the formula (50 B); a negative ratio corresponds to a contrary phase.

§ 3. *Three pendulums with equal damping.*

The formula (50 A) gives the following equation for y

$$y^3 - 3\,u\,y^2 - 3\,s^2 y + 3\,s^2 u - s_1\,s_2\,s_3 = 0 \quad \ldots. \quad (53)$$

in which we have introduced s, defined by

$$s^2 = \frac{s_1^2 + s_2^2 + s_3^2}{6} \quad \ldots\ldots\ldots \quad (54)$$

The roots of (53) may be expressed in the angle ψ, given by

$$\cos \psi = \frac{u^3 + \frac{1}{2} s_1 s_2 s_3}{(u^2 + s^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots \quad (55\,A)$$

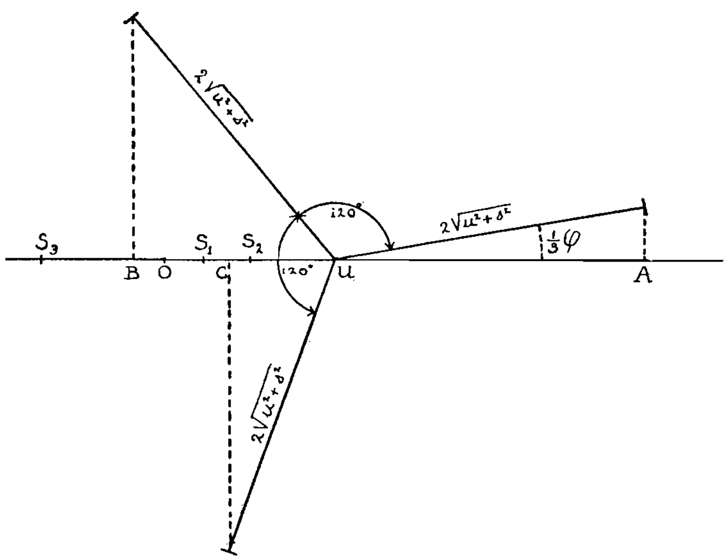
and we find

$$y_a = u + 2\sqrt{u^2 + s^2} \cos \tfrac{1}{3} \psi \quad \ldots\ldots\ldots \quad (55\,B)$$

$$y_b = u + 2\sqrt{u^2 + s^2} \cos (\tfrac{1}{3} \psi + 120^\circ) \quad \ldots\ldots \quad (55\,C)$$

$$y_c = u + 2\sqrt{u^2 + s^2} \cos (\tfrac{1}{3} \psi + 240^\circ) \quad \ldots\ldots \quad (55\,D)$$

This may be represented graphically



OS_1, OS_2, OS_3, and OU represent s_1, s_2, s_3, and u; OA, OB, and OC the three roots y_a, y_b, and y_c. This representation is of course only a constructional figure, it has nothing to do with the representation of complex quantities.

According to formula ($50B$) the ratio of the constants A_1, A_2 and A_3 is given by the reciprocal values of S_1A, S_2A and S_3A and in the same way the reciprocal values of S_1B, S_2B and S_3B give the ratio of B_1, B_2 and B_3, and those of S_1C, S_2C, and S_3C the ratio of C_1, C_2 and C_3. These three sets of ratio give the ratio of the amplitudes for the three special situations; a positive ratio means that the pendulums swing in the same phase while a negative ratio corresponds to contrary phase. We need not repeat that for these situations the ratio keep constant during the whole duration of the observation if the amplitudes are so small that the reduction to infinitely small amplitude may be neglected.

The determination of the coefficients α, β and γ by means of the determinant (51) does not present any special features. If they have been computed, the components A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 of the three pendulum-vectors may easily be determined for any initial conditions by means of (47 A) and (50 B). According to (49 B) their periods are found by adding T to y_a, y_b and y_c.

Not wishing to pursue the general discussion of this case we will restrict our investigation to the special case, which occurs, if the apparatus for sea observations is used, i. e., the case that the correction for sway u is great with regard to the differences of the periods of the pendulums.

§ 4. *Three pendulums with equal damping swinging together; the differences of the periods are small with regard to the correction for sway.*

As the differences of the periods are supposed to be less than 50 10^{-7} sec. and the correction for sway to be several hundreds of 10^{-7} sec. — a determination made with the new apparatus gave for instance 810 10^{-7} sec. —, this will normally be the case.

Neglecting all higher powers of the ratio $\frac{s_1}{u}$, $\frac{s_2}{u}$, $\frac{s_3}{u}$ and $\frac{s}{u}$ we find by applying formulae (55)

$$\cos \varphi = 1 - \frac{3}{2} \frac{s^2}{u^2} \ldots\ldots$$

and therefore

$$\varphi = \sqrt{3}\, \frac{s}{u} \ldots\ldots$$

$$\cos \frac{1}{3} \varphi = 1 + \ldots\ldots$$

$$\cos \left(\frac{1}{3} \varphi + 120^\circ\right) = -\frac{1}{2} - \frac{1}{2} \frac{s}{u} + \ldots\ldots$$

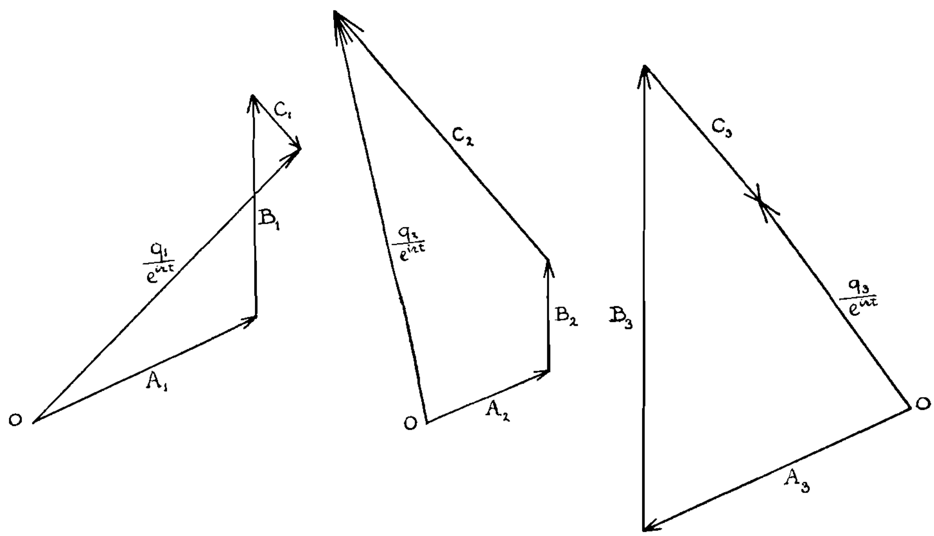
$$\cos \left(\frac{1}{3} \varphi + 240^\circ\right) = -\frac{1}{2} + \frac{1}{2} \frac{s}{u} + \ldots\ldots$$

so that we get

$$\begin{aligned} y_a &= 3\,u \\ y_b &= -s \qquad \ldots\ldots\ldots\ldots \qquad (56) \\ y_c &= +s \end{aligned}$$

in which terms of the order of $\frac{s^2}{u}$ and smaller are neglected.

We will first examine the result, which ensues for the periods of the phase-differences and of the ratio of the amplitudes. We can do this for instance by making use of the graphical representation of the vectors $\frac{q_1}{e^{irt}}$, $\frac{q_2}{e^{irt}}$ and $\frac{q_3}{e^{irt}}$, which gives insight into the relative variations of the three pendulum-vectors.



By applying formula (52) we see that the three components, of which each vector is composed, have the half-periods

$$+\frac{T^2}{3u} \qquad -\frac{T^2}{s} \qquad +\frac{T^2}{s} \quad \ldots\ldots \quad (57A)$$

so that obviously the phase-differences and the ratio of the amplitudes have the half-periods

$$T_p = \frac{T^2}{3u} \quad \text{and} \quad T_q = \frac{T^2}{2s} \quad \ldots\ldots \quad (57B)$$

T_p = half-period of the relative movement of the components A and B or C.
T_q = half-period of the relative movement of the compononts B and C.

This result shows that the periods are strongly different. The first half-period T_p, which depends solely on the correction for sway, is not a very long time: for the value of u of 810 10^{-7} sec., which is mentioned above, it is for instance 18 minutes, that is to say that the components A_1, A_2 and A_3 of the vectors of the figure make half a revolution in that time. The second half-period T_q, which depends solely on the differences of the pendulum-periods, is a long one. For the Dutch pendulums, for which s is about 6 10^{-7} sec. this half-period is for instance 58 hours.

This result has an important bearing on the execution of observations for a longer time, which subject has been investigated in paragraph 12. As has been indicated there, we wish to avoid making a continuous record of the middle pendulum in order to be able to compute the corrections, which depend on the movement of this pendulum. We try therefore to make this movement so regular, that the record of the initial and final parts and perhaps of a few intermediate moments is sufficient. Obviously we can do this by realising one of the special situations, but it is practically difficult to realise them exactly and even if this were not the case, the situation would not be maintained indefinitely because the amplitudes are not infinitely small.

Besides we wish to have more liberty for choosing the most effective ratio of the amplitudes in view of other requirements.

We may see now, by examining the figure, how this last point may be obtained: Although it is necessary, in order to get a regular movement for the middle pendulum, to make the first component A_2, which causes a variation with the short half-period T_p, zero, it is not necessary to make also one of the other components B_2 or C_2 zero, which would mean the realisation of one of the special situations. The second half-period T_q is so long with regard to the duration of the observation of some two hours, that the relative movement of B_2 and C_2 during that time is small, so that we may assume that the small variation of the sum of B_2 and C_2, is linear with the time. We may therefore find its mean value simply by taking the mean of the initial and final values, and so we may adopt any combination of B_2 and C_2 without having to expect trouble with the computations. This gives us more liberty in choosing appropriate initial conditions. We need not add of course, that this reasoning is only valid as long as the period T_q is great i. e. as long as s is small.

We noticed above that the first component A_2 has to be made zero as nearly as possible in order to avoid variations of q_2 having the short half-period T_p. We will see hereafter, that this condition is satisfied by the condition indicated in paragraph 12, i.e. by making the initial sum of the three pendulum-vectors: $q_1^0 + q_2^0 + q_3^0$ zero. It is of course not possible to realise this condition exactly, and so A_2 will generally have some small value, causing slight variations of the regular movements of the middle pendulum. We may however eliminate their effect by taking at the beginning of the observation the mean of two records of the middle pendulum, made with a time-interval of the half-period T_p and by doing the same at the end of the observation. The effect is obvious; the variations of the movement of the middle pendulum for two moments at this time-interval will be equal with opposite sign, so that they are eliminated by taking the mean value.

We will now get back to the investigation of the further features of the solution. According to (50 B) and (56) the ratio of the amplitudes for the three special situations are given by

$$\frac{1}{3u - s_1} : \frac{1}{3u - s_2} : \frac{1}{3u - s_3} \quad . \; . \; . \; . \; . \; . \quad (58A)$$

$$\frac{1}{s + s_1} : \frac{1}{s + s_2} : \frac{1}{s + s_3} \quad . \; . \; . \; . \; . \; . \quad (58B)$$

$$\frac{1}{s - s_1} : \frac{1}{s - s_2} : \frac{1}{s - s_3} \quad . \; . \; . \; . \; . \; . \quad (58C)$$

in which again a negative ratio means opposite phase. We see that for the first situation the pendulums have to get about equal amplitudes in the same phase; these ratio are nearly independent of the differences of the pendulum-periods. This is not the case for the ratio corresponding to the other two special situations.

For the determination of the coefficients α, β and γ we have the following determinant (see (51)) in which we have neglected the second terms of the denominators in the first line corresponding to the neglections made in the beginning of this paragraph

$$\begin{vmatrix} \frac{1}{3u} & \frac{1}{3u} & \frac{1}{3u} \\ -\frac{1}{s+s_1} & -\frac{1}{s+s_2} & -\frac{1}{s+s_3} \\ \frac{1}{s-s_1} & \frac{1}{s-s_2} & \frac{1}{s-s_3} \end{vmatrix} \quad \ldots\ldots \quad (59)$$

We find in this way

$$\begin{array}{lll} \alpha_1 = u & \alpha_2 = u & \alpha_3 = u \\ \beta_1 = \frac{(s+s_2)\,(s+s_3)}{6s} & \beta_2 = \frac{(s+s_3)\,(s+s_1)}{6s} & \beta_3 = \frac{(s+s_1)\,(s+s_2)}{6s} \\ \gamma_1 = \frac{(s-s_2)\,(s-s_3)}{6s} & \gamma_2 = \frac{(s-s_3)\,(s-s_1)}{6s} & \gamma_3 = \frac{(s-s_1)\,(s-s_2)}{6s} \end{array} \quad (60)$$

Finally we get by substituting these coefficients in ($47\,A$) and by applying ($50\,B$), the following formulae for $A_1, \ldots\, C_3$. For simplification we introduce the quantities v_1, v_2 and v_3

$$v_1 = \frac{s_1}{s} \qquad v_2 = \frac{s_2}{s} \qquad v_3 = \frac{s_3}{s} \quad \ldots\ldots \quad (61)$$

$$\begin{aligned} A_1 &= \frac{1}{3}\,(q_1^0 + q_2^0 + q_3^0) \\ B_1 &= +\frac{1}{6}\,(2 - v_1)\,q_1^0 - \frac{1}{6}\,(1 + v_3)\,q_2^0 - \frac{1}{6}\,(1 + v_2)\,q_3^0 \\ C_1 &= +\frac{1}{6}\,(2 + v_1)\,q_1^0 - \frac{1}{6}\,(1 - v_3)\,q_2^0 - \frac{1}{6}\,(1 - v_2)\,q_3^0 \\ \\ A_2 &= \frac{1}{3}\,(q_1^0 + q_2^0 + q_3^0) \\ B_2 &= -\frac{1}{6}\,(1 + v_3)\,q_1^0 + \frac{1}{6}\,(2 - v_2)\,q_2^0 - \frac{1}{6}\,(1 + v_1)\,q_3^0 \qquad (62) \\ C_2 &= -\frac{1}{6}\,(1 - v_3)\,q_1^0 + \frac{1}{6}\,(2 + v_2)\,q_2^0 - \frac{1}{6}\,(1 - v_1)\,q_3^0 \\ \\ A_3 &= \frac{1}{3}\,(q_1^0 + q_2^0 + q_3^0) \\ B_3 &= -\frac{1}{6}\,(1 + v_2)\,q_1^0 - \frac{1}{6}\,(1 + v_1)\,q_2^0 + \frac{1}{6}\,(2 - v_3)\,q_3^0 \\ C_3 &= -\frac{1}{6}\,(1 - v_2)\,q_1^0 - \frac{1}{6}\,(1 - v_1)\,q_2^0 + \frac{1}{6}\,(2 + v_3)\,q_3^0 \end{aligned}$$

We see that the vectors A_1, A_2 and A_3 are identical and that they become zero if $q_1^0 + q_2^0 + q_3^0$ is zero. This confirms what has been said before. We must however draw attention to the fact, that the formulae of this paragraph are all approximate so that the real initial conditions for making A_1, A_2 and A_3 zero will be slightly different; but if the differences of the pendulum-periods are below the limit, mentioned in the beginning, these deviations of the initial ratio are so small, that they may safely be neglected.

The formulae (62) provide the means to determine the vectors A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 for an arbitrary initial situation and so all the elements of the graphical representation of the figure of page 44 are known. The angular velocities having already been given in (57 A), the whole course of the phenomenon is determined and may easily be followed in this figure.

We will finish the investigation by applying the formulae (62) to the two initial situations, which have been adopted in § 12 for the practical execution of the observations on land

I. $a_1 = a_3 = a$; $a_2 = 0$; $\varphi_1^0 = 0$; $\varphi_3^0 = \pi$.

We have therefore to substitute

$$q_1^0 = a;\ q_2^0 = 0;\ q_3^0 = -a$$

and we find

$$A_1 = 0 \qquad A_2 = 0 \qquad A_3 = 0 \tag{63}$$

$$B_1 = +\frac{1}{6}(3 - v_1 + v_2)\,a \quad B_2 = +\frac{1}{6}(v_1 - v_3)\,a \quad B_3 = -\frac{1}{6}(3 + v_2 - v_3)\,a$$

$$C_1 = +\frac{1}{6}(3 + v_1 - v_2)\,a \quad C_2 = -\frac{1}{6}(v_1 - v_3)\,a \quad C_3 = -\frac{1}{6}(3 - v_2 + v_3)\,a$$

II. $a_1 = 2\,a_2$; $a_3 = 3\,a_2$; $\varphi_1^0 = \varphi_2^0 = 0$; $\varphi_3^0 = \pi$.

so that we have to substitute

$$q_1^0 = 2\,a_2;\ q_2^0 = a_2;\ q_3^0 = -3\,a_2.$$

We find

$$A_1 = 0 \qquad A_2 = 0 \tag{64}$$

$$B_1 = +\frac{1}{6}(6 - v_1 + 4\,v_2)\,a_2 \qquad B_2 = +\frac{1}{6}(3 + 5\,v_1 + v_2)\,a_2$$

$$C_1 = +\frac{1}{6}(6 + v_1 - 4\,v_2)\,a_2 \qquad C_2 = +\frac{1}{6}(3 - 5\,v_1 - v_2)\,a_2$$

$$A_3 = 0$$

$$B_3 = -\frac{1}{6}(9 + 4\,v_1 + 5\,v_2)\,a_2$$

$$C_3 = -\frac{1}{6}(9 - 4\,v_1 - 5\,v_2)\,a_2$$

The formulae may serve to study the course of the phenomenon for these initial conditions, in case we might wish to do this. It will perhaps be superfluous to add, that this is not necessary for obtaining the results of the observations, so that we will not have recourse to these formulae in Chapter III, where the methods for the computations are treated.

CHAPTER II.

The apparatus and the execution of the observations.

We will successively take up:

§ 1. Description of the apparatus;
  A. the pendulums.
  B. the pendulum apparatus.
  C. the recording apparatus.
  D. the gimbals.

§ 2. Adjustment of the apparatus.

§ 3. Execution of the observations at sea and on land.

§ 4. Supplementary observations:
  A. ship's position.
  B. ship's velocity.
  C. sea-depth.

§ 5. Determination of the sway on land.

§ 1. *Description of the apparatus.*

The apparatus consists of two parts: the pendulum apparatus with three pendulums and the photographic recording apparatus. These two parts are combined in one unit, which is suspended in gimbals.

A. *The pendulums.*

The pendulums are half-second pendulums of the Sterneck-Stückrath type. The knife-edges are agate; their central part is used for the observations while the lifting levers act at the ends. The headpiece of the pendulum is forked in order to leave the central part of the knife-edge free. At each side it is provided with a mirror, of which the centre is about 2 c.m. above the knife-edge. The mirrors have their reflecting surface on the outside in order to avoid secondary reflections; this is desirable because the light-beams do not strike the mirror normally.

In case the apparatus is used in a submarine it is thought to be preferable to use only non-magnetic pendulums because the magnetic conditions in these ships are variable. Brass pendulums meet this requirement and are stable at the same time. They have of course a large temperature coefficient but this difficulty can be met by a good temperature insulation of the apparatus.

The pendulums have to be made as nearly as possible isochronous. This may be done in one way by drilling a hole in the centre of the bottom of the bulb. As the taking away of material at this spot affects the period only slightly, it is not very difficult to produce in this way small changes of period. We can easily find experimentally how far we have to go to obtain the desired isochronism (maximal difference of period for inst. $50\ 10^{-7}$ sec.).

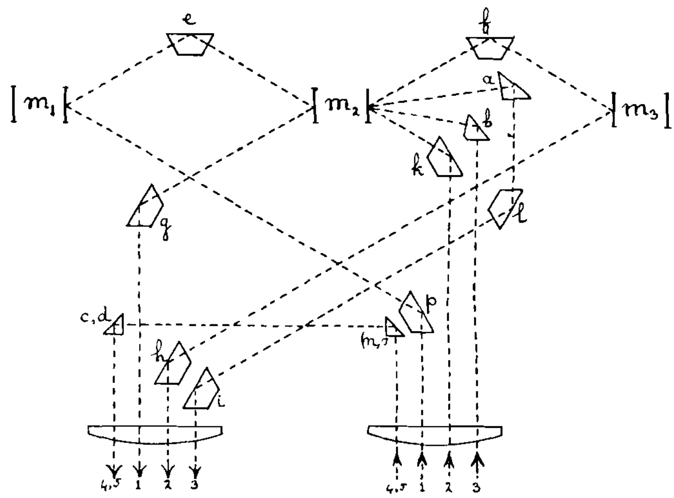
It may be noticed that the pendulums have to be isochronous at normal temperature and at atmospheric air-pressure; this does not necessarily mean that the reduced periods have to be equal: if the temperature and air-density constants are somewhat different, this is not the case.

B. *The pendulum apparatus.*

The three pendulums swing in the same plane at distances apart of 13 c.m. Numbering them 1, 2 and 3, we make a photographic record of $\theta_1 - \theta_2$, $\theta_2 - \theta_3$ and θ_2. The first records are the records of two fictitious pendulums, which are undisturbed by the horizontal movements of the apparatus; these are the principal records. The record of pendulum No. 2 is used only for the computation of some corrections. It is made by recording the angle of elongation of this pendulum with regard to the position of an auxiliary pendulum which can move in a plane parallel to the swinging-plane and which is so strongly damped that oscillations are prevented.

Besides these three records there is a record made of the temperature of the air inside the apparatus and the position of the swinging-plane with regard to the vertical. To obtain this last record, the apparatus is provided with a second auxiliary pendulum, similar to the above, which can move in a plane perpendicular to the swinging-plane and which is likewise prevented from swinging, by a damping device. The movement of this auxiliary pendulum with regard to the apparatus is recorded.

All these records are made by light-beams, which enter the apparatus horizontally through a lens which makes them parallel. The figure gives a sketch of the course of these light-beams; m_1, m_2 and m_3 are the pendulum-mirrors.



No. 1 records $\theta_1 - \theta_2$;

No. 2 records $\theta_2 - \theta_3$;

No. 3 records θ_2; the prisms *a* and *b* are fixed to the auxiliary pendulum, which can move in a plane parallel to the swinging plane of the principal pendulums;

No. 4 records the temperature of the air; prism *c* is fixed to a temperature recording device;

No. 5 records the position with regard to the apparatus of the second auxiliary pendulum, which can move in a plane perpendicular to the swinging-plane; prism *d* is fixed to this pendulum.

The horizontal projection of the rays Nos. 4 and 5 coincide; the other prisms have a height of 30 m.m. but *c*, *d*, *n* and *o* have a height of only 12 m.m. and are one above the other.

The prisms *e*, *f*, *g*, *h*, *i*, *k*, *l*, *n*, *o* and *p* and the lenses are fixed to the top-plate of the apparatus.

The rays leave the apparatus through a second lens with a focal distance of 111 c.m. at which distance the photographic paper receives the images.

With the exception of the reflections on the pendulum-mirrors, the reflections are all effected by prisms. This reduces the loss of light to a minimum and favours the keeping of the reflecting properties in good condition. For each beam inside the apparatus one of the prisms has an adjusting device in both horizontal and vertical sense (prisms *e*, *f*, *l*, *n* and *o*). By means of the first, the beams can be adjusted so as to follow the right path as given by the figure; by means of the last, the images can be brought at the desired position on the photographic paper.

Besides this precise adjustment the other prisms mounted on the top-plate of the apparatus can be turned round a vertical axis by applying a stress by means of a screw-driver: their fixing in this regard is frictional. In this way these prisms can be brought into the right positions.

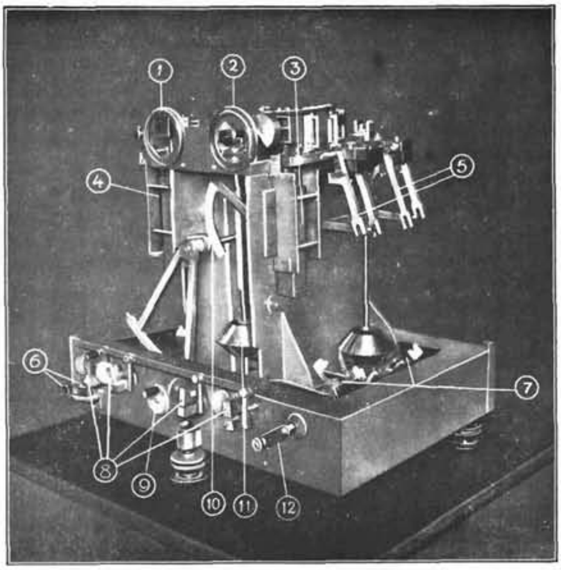


Fig. 1. Pendulum apparatus (opened) front view.

1. Outgoing lens. 2. Ingoing lens. 3. Damped pendulum swinging in parellel plane. 4. Second damped pendulum. 5. Levers for lifting and fixing knife-edges. 6. Outside levels. 7. Clutches for fixing bulbs. 8. Coupled levers with graduated screws for giving amplitude. 9. Wheel for lifting knife-edges. 10. Hair-hygrometer. 11. Thermometer in dummy pendulum. 12. Lever for fixing bulbs.

For the auxiliary pendulums a damping system fixed to the apparatus has been avoided, because in this way the fluctuations of the apparatus would affect the position of the pendulum: each one is damped by a self contained system. To this end each one has a second pendulum inside with the knife-edge in the same line as that of the outer pendulum. The periods of these pendulums are quite different. As the inner pendulum has a tail-piece which moves in a small oil-tank fixed in the outer pendulum, the movements of the pendulums are coupled by this damping mechanism.

The investigation of the equations of motion of these two pendulums leads to the result that, if the moments of mass and the moments of inertia, both taken with regard to the knife-edge, are inversely proportional, the movements of the pendulums will be regularly damped, that is, the amplitudes of the pendulums will decrease logarithmically in the same way as any pendulum that is ordinarily damped. Their damping-coefficients are equal and can easily be made so great that the movement practically disappears in a few seconds. This damping device has given entire satisfaction.

The inner pendulum swings in the outer pendulum by means of steel knives in steel grooves. The outer pendulum is suspended in the apparatus by means of a vertical steel point at each side; at one side this point rests in an agate cup, at the other side in an agate groove. The steel points can be renewed if necessary. The necessity of renewal may be noticed in the records: if the axis of the curve given by the middle pendulum (the lower curve on the record) shows irregularities, this may be considered as an indication that the suspension of the corresponding damped pendulum is no longer satisfactory and it will be wise to renew at the same time also the points of the second damped pendulum. The points ought to last for several thousand hours of use.

For renewing the points, it is necessary to take out the damped pendulums; this can easily be done by unscrewing the fastening screws.

The damped pendulums as well as the principal pendulums have to be lifted from their knife-edges after the observation is finished. This is done with one mechanism, i. e. by turning a wheel at the bottom of the front-side of the apparatus, which operates the lifting levers of all the pendulums.

For the damped pendulums this lifting operation serves to lift both the inner and the outer pendulum. This is effected by lifting the inner pendulum, which at a certain point takes up the outer pendulum by means of projecting pins; by continuing the lifting, the outer pendulum is also lifted. When the operation is finished, the inner pendulum is fixed in the outer pendulum and the outer pendulum is fixed in the apparatus; both fixings are effected by means of projecting pins which, during the upward movement, slip into slots.

For the lifting of the principal pendulums an important point has to be considered; it is desirable that the pendulums be left inside the apparatus during an entire voyage, because inserting the pendulums shortly before the observation would disturb the temperature conditions and if there were no special provisions for the fixing of the pendulums, they could not be inserted long before, as this would imply the necessity of small ship's movements during this time, i.e. the submerging of the submarine during a considerable time for each observation.

To fix the pendulums requires two things: first, the knives of the pendulums have to be fixed with springs in order to lessen the disturbing effect of the ship's vibrations on the pendulums, and second, the bulbs of the pendulums have also to be fastened. This last point will be considered later.

The fixing of the knife by means of springs attached to the lifting-lever, would involve the risk that when the pendulum was lowered, it would not always have exactly the same position on the agate planes. This difficulty has been met by the introduction of a second pair of levers which have a slower lifting movement. These levers are provided with metal grooves, which take up the knife very slowly at the beginning of the lifting operation. At a certain point the springs of the other levers, which move at a quicker rate, come into contact with the knife and lift it from the grooves. The lowering operation is of course the reverse: The knife is first lowered onto the grooves of the slowly moving lever and afterwards this lever lowers it onto the agate plane.

The grooves of the slowly moving levers are provided with screws which permit them to be adjusted in such a way, that the knife-edge is put down parallel to the plane and that this is done at the right moment during the lowering operation.

In order to hold the knife always in the same position laterally, the quickly moving levers are provided with two small metal strips fastened with screws, which fill up the space between the knife-ends and the levers in their raised position. In the lowered position these strips are so far down that the knives are free. If other pendulums are used of which the knives differ somewhat in length, other strips can be introduced.

The clutching of the bulbs must likewise be done by springs in order to lessen the effect of the ship's vibrations on the pendulums. These springs are fastened to levers which are all moved simultaneously by means of a single handle at the outside of the apparatus. This movement has not been coupled to the lifting operation of the knives, as this would involve difficulties in the insertion of the pendulums in the apparatus.

The fastening of the clutching springs to the levers can be slightly adjusted in order to adapt the mechanism to pendulums of slightly different length.

The lifting and the clutching operations are interlocked by means of a lever on the inside of the apparatus. This works in such a way that two things are impossible: first, the pendulums cannot be clutched while the knife-edges are in contact with the agate planes, and second, the wheel for lowering the knives cannot be turned when the bulbs are clutched. Obviously, both might have disastrous results if they were possible: the knife-edges would be damaged if the bulbs were clutched while they rested on the agate planes, and if the lowering wheel could be operated when the bulbs were clutched, the pendulums would be prevented from following the lowering movement and they would fall on their knife-edges when the clutches were released.

The handle which governs the clutching mechanism can be locked in the position which corresponds to clutched pendulum-bulbs; unlocking can only be

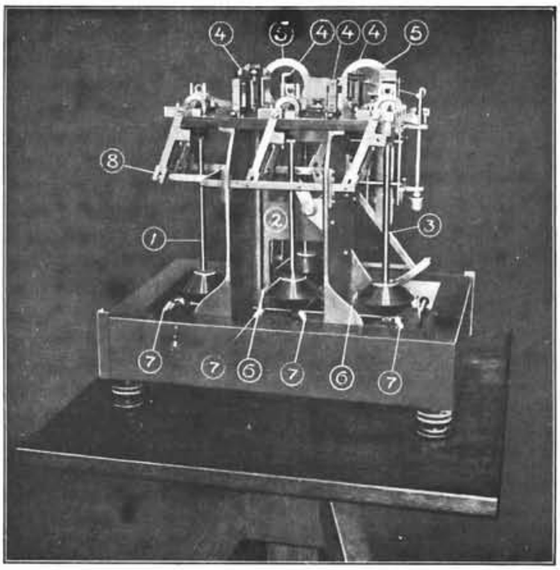


Fig. 2. Pendulum apparatus (opened) without recording apparatus, back view.

1. Pendulum Nr. 1. 2. Pendulum Nr. 2. 3. Pendulum Nr. 3. 4. Prisms. 5. Lenses. 6. Levers for giving amplitude. 7. Clutches for fixing bulbs. 8. Levers for lifting and fixing knife-edges.

done by means of a small key. As the lowering mechanism is then also locked by the above interlocking device, the pendulums can neither be lowered nor unclutched without using this key. This device for making the pendulums safe has been introduced, because the apparatus is used on board of ships where it may be troublesome to guard continuously against tampering.

The method of giving amplitude to the pendulums is in principle the same as for the Stückrath apparatus. Against each pendulum-bulb a lever can be pressed, of which the position may be regulated at the outside by means of a graduated screw, and which can be turned back in order to set the pendulum swinging.

A coupling lever makes it possible to set either all three pendulums or two of them swinging at the same time, but it is not necessary to use this lever: the pendulums can also be started separately. As we generally wish to give two of the pendulums opposite phase, we can only profit by this coupling mechanism if the movement of some of the amplitude-levers is reversed, so that, when the coupled levers at the outside are turned in the same direction, the levers at the inside act on the pendulums from opposite sides. This reversal is effected by the interposition of cog-wheels.

In order to give all the possibilities which we desire to have, the lever of pendulum No. 2 has a reversed movement, the lever of pendulum No. 3 a direct movement, while pendulum No. 1 is provided with two levers of which one has a direct and the other a reversed movement; we are free to choose which one shall be used.

In this way many possibilities may be realized, e. g. the initial conditions given in §§ 11 and 12 of chapter I. Pendulums 1 and 3 may be started in opposite phase with pendulum 2 hanging free, or pendulums 1 and 2 may be started in the same phase with $a_1 = 2\, a_2$ and pendulum 3 in opposite phase with $a_3 = 3\, a_2$.

The apparatus is provided with two thermometers: one is inserted in a dummy pendulum in the same way as in the Stückrath apparatus, and the other is a recording thermometer in the shape of a double metal strip fastened at one end, of which the deformation is recorded by means of a prism fastened to the other end. Both thermometers are mounted in such a way as to be insulated from the metal frame of the apparatus, and so are supposed to give the air-temperature in the apparatus with a certain lag, which will be smaller for the recording thermometer as its mass is much less than that of the dummy pendulum.

It is thought probable that the temperature of the real pendulums will differ less from that of the dummy pendulum, than will that of the recording thermometer, so that this last difference may give an indication of the upper limit of the temperature error. For the computation of the observation, the data given by the thermometer of the dummy pendulum are used.

This thermometer is read through a window in the front of the apparatus which can be closed with a piece of cork when not in use. Through the same window the scale of a hair-hygrometer can be read. This hygrometer is mounted inside, because the apparatus is kept closed — although not air-tight — during the whole voyage, so that the hygrometric conditions inside might well deviate from those in the ship. For long voyages it is of course desirable to take also along a psychrometer for controlling from time to time the indications of the hair-hygrometer. To this end the apparatus is opened by taking off the top-cover and the psychrometer is hung near the hair-hygrometer.

In order to keep the temperature as uniform as possible inside the apparatus, it is provided with an insulating mantle consisting of two brass covers, of which the intervening space is filled with sheep's wool. This temperature insulation may cause however the following difficulty in case the observations are carried out on board of submarines. When a submarine dives, the temperature inside sometimes increases quickly by several degrees, and so the apparatus has then a temperature which is considerably below that of the circum-ambient air. The insulation prevents it from following this increase of temperature at once and so we have a continuous small increase of temperature during the whole observation, which might possibly give a systematic error in the temperature data.

To meet this difficulty the apparatus has been provided with a heating-coil in the bottom-part, of which the connecting-knobs are visible at the foot of the front. Some four hours or more before the observation, this coil is put into action and the apparatus is heated so far, that after submerging for the observation, it has about the same temperature inside, as that of the air in the ship. We may expect that during the intervening time the temperature in the apparatus will have become uniform enough for ensuring satisfactory regularity during the observation. With some practice we can easily reduce in this way the variation of the temperature during the observation to a few hundredths of a degree.

We may finish the description of the pendulum apparatus by mentioning the levelling devices. The apparatus is provided with two small levels at the outside, for bringing it into the right position when it hangs in the gimbals; this is done by adjusting the position of the counterpoises at the bottom.

When the apparatus hangs in the gimbals, the use of the footscrews is reduced to bringing the knife-edges of the pendulums to the same level with the knife-edges of the gimbals. From time to time however we have to set up the apparatus on land on a good foundation and to bring the agate planes into the level position in order to control the zero point of the curve given by the damped pendulum, which records the deviation of the swinging-plane from the vertical. For this levelling operation which is carried out with the footscrews, the small outside levels are not sufficient. For levelling the agate planes in a direction perpendicular to the swinging-plane and for controlling their parallellism in this direction (see § 4, Chapter I) we have to use the pendulum-level, which is supplied with the apparatus. For levelling in the direction of the swinging-planes, a second level is used which is put down cautiously on two of the agate planes

After the apparatus has been levelled in this way, the position of the small outside levels is noted for use during the subsequent sea observations and a record is made for fixing the zero point of the curve made by the damped pendulum.

C. *Recording apparatus.*

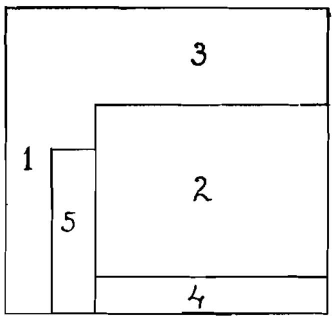
The recording apparatus which is mounted above the pendulum apparatus consists of three parts: the light part, the recording part and the clock-work.

The light part is located in the box at the left side, which is represented in fig. 3 after opening the doors which shut it off from the outside. This box contains also the handles for working the apparatus; these handles have to be operated by means of flexible cords screwed into the corresponding holes.

The recording part is located in the principal compartment, which is shown in fig. 4 after opening the double doors at the front.

The clock-work is located at the back of the apparatus.

Horizontal plan.



1. light part.
2. recording box.
3. clock-work.
4. space between double doors.
5. rays coming back from pendulum apparatus.

The light part.

The light which is used in the apparatus is an electric arc lamp. If direct current is available, an Edison Swan lamp (Edison Swan Electric Co. Ltd., Ponders End, Middlesex, England) may be put in, which is adapted to 100, 110, 200, 220 and 250 Volts; if the current is alternating, a Philips Wolfram arc lamp (Philips Gloeilampen Fabrieken, Eindhoven, Holland) of the smallest type (1.3 Amp, 220 Volts) may be used.

By means of the lens A an image of the light is made on the diaphragm B after the rays have been reflected by an adjustable prism C. The diaphragm is a slit, 5.0×0.1 m.m., which is nearly parallel to the front side of the apparatus. After loosening a tiny screw it can be turned in order to adjust it in such a way that the images on the cylinder-lens, which is put before the photographic paper in order to concentrate the light, are perpendicular to the axis of this lens. At the bottom side of the diaphragm a movable wedge-shaped diaphragm makes possible a shortening of the slit, which may be useful in case the light on the photographic paper is too strong.

The slit of the diaphragm can be shut by means of shutters D and E, each of which is operated by an electrical current of 4 to 6 Volts, passing through a pair of coils with a resistance of 140 Ohms. The springs which draw the shutters back when the current is interrupted, can be adjusted by means of screws.

A high precision chronometer is put into each circuit. To eliminate the spark in the chronometers, an arrangement, which is supplied with the apparatus, is applied in parallel with each chronometer.

The left shutter is provided with a small slit parallel to the slit of the diaphragm. The amplitude of the movement of this shutter may be varied by moving up and down the mechanism *F*, which is adjustable in order to ensure the shutter's functioning well. In this way we can realize three possibilities:

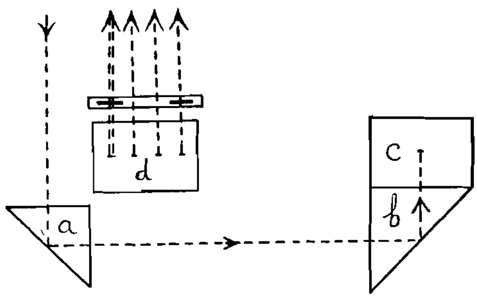
1. The shutter goes past the diaphragm and the light is only momentarily interrupted;

2. the shutter shuts off the diaphragm as long as the circuit is closed, and opens when the circuit is broken, i. e. the light is interrupted each time about half a second;

3. the amplitude of the movement of the shutter is so small, that it remains before the slit of the diaphragm, but the slit in the shutter passes before it and so each half second a flash of light is emitted.

This last arrangement has not been used during the past sea or land observations.

After passing through the diaphragm, the light-beam leaves the apparatus through a hole in the bottom and enters the pendulum apparatus through the left part of the hole in the top-cover. It strikes then the prisms *a*, *b* and *c*, which are mounted on the inside of a plate, fastened with screws to the front plate of the pendulum apparatus; prism *c* projects the rays backwards towards the entry lens of this apparatus (focal distance 616 c.m.). In reality prisms *b* and *c* form one piece of glass.



All these prisms have 45° angles with the exception of *d* of which the lower angle is 44° 50′ and the upper 45° 10′. This prism projects the outgoing rays back upwards into the recording apparatus. The slight deviation of the angles of this prism is introduced in order to compensate the chromatic dispersion which is caused by the fact, mentioned later, that these rays do not strike normally some of the prisms in the recording apparatus.

Of the prisms *a*, *b*, *c* and *d* the last has a precise adjustment devi 'or turning it round an axis perpendicular to the frontplate; the other prisms can be adjusted roughly by loosening the screws.

Directly after passing through prism *d*, the outgoing rays pass through a frame, in which small diaphragms can be inserted from the outside, which serve to diminish the light-intensity of the outer light-beams, viz., the light-rays at the left side which record the temperature and the position of the damped pendulum and the rays at the right side which record the middle pendulum (see figure of page 50). It is well to do this because these beams have only

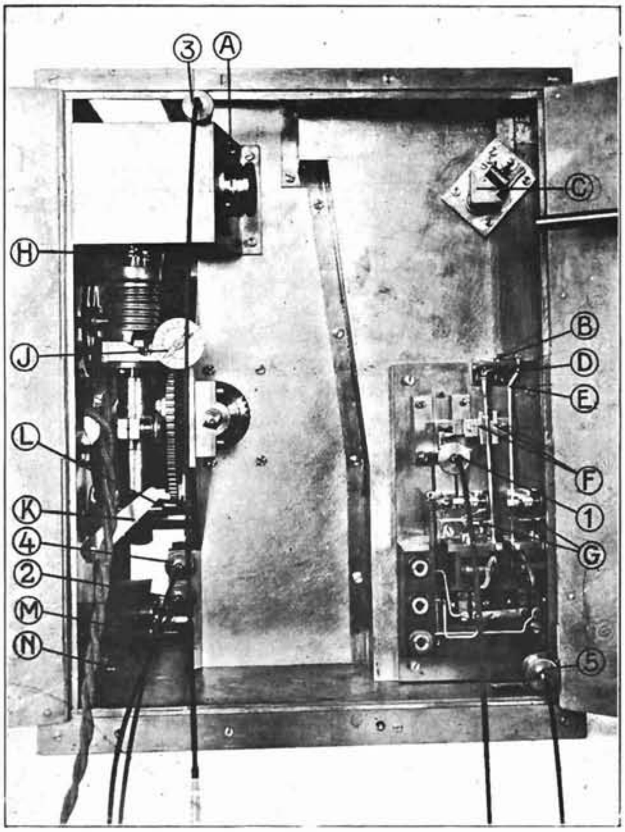


Fig. 3. Light-box of the recording apparatus.

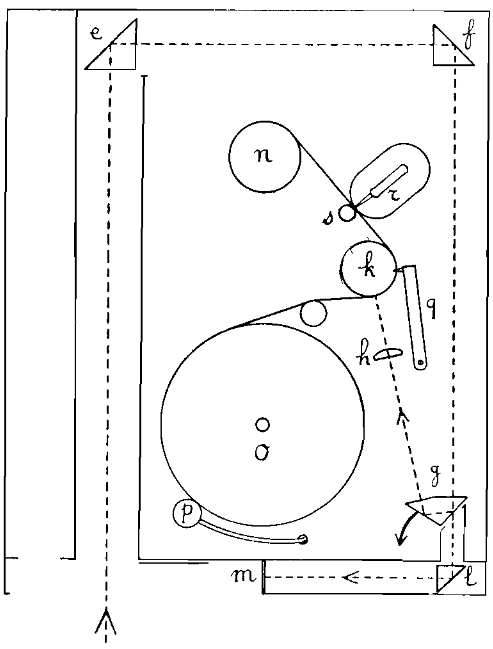
A Lens, *B* Diaphragm, *C* Adjustable prism, *D* and *E* Shutters, *F* Amplitude regulation of the movement of *D*, *G* Adjustable springs of shutter movement, *H* Arc-lamp, *J* Clock-hand indicating paper movement, *K* Starting lever of clock-work, *L* Speed regulating screw, *M* Paper-store indicator, *N* Clutch for fixing paper-store indicator.

Flexible cords:

N⁰. 1 for regulating the amplitude of the movement of shutter *D*.
N⁰. 2 for turning prism *g* (fig. page 57).
N⁰. 3 for changing paper velocity.
N⁰. 4 for giving marks on the paper.
N⁰. 5 for opening control window.

a slight movement in the pendulum apparatus, and so describe a much shorter curve on the photographic paper than the beams which record the fictitious pendulums. If these last beams give therefore a satisfactory photographic record, the first would be too intense. For the left beam, this is always the case except when the apparatus undergoes so strong angular movements that the damped pendulum records large deviations. For the right beam this is only the case when the middle pendulum has a very small amplitude during the whole observation, which occurs for most of the observations on land. For sea observations we do not insert a diaphragm at this side. The inserted diaphragms must of course be adapted to the circumstances.

The light-beam leaves the pendulum apparatus through the right part of the window in the top-cover and reënters the recording apparatus through a vertical tube, which may be shut at the bottom when the apparatus is not in use in order to prevent stray light entering the recording box. At the top of the tube the beam is reflected by prism *e* and enters the recording box, it is further reflected by prisms *f* and *g*, passes through the cylinder-lens *h*, and strikes the photographic paper at *k*. Prism *f* has a precise adjustment device for turning it round an axis perpendicular to the front-plate.



The optical distance from the outgoing lens of the pendulum apparatus to the photographic paper equals the focal distance of this lens. The real distance is 4.2 c.m. greater than this in order to compensate for the path of the beam through prisms in which the light-velocity differs from that in the air. For the same reason the distance from the diaphragm of the light-box to the entry lens of the pendulum apparatus has been made 3.7 c.m. greater than the focal length of this lens.

Prism *g* can be turned round its axis by a mechanism which is governed by the flexible cord No. 2 in the light-box (see fig. 3) and it can be fixed temporarily in this upturned position by an arrangement at the end of this cord. The light-beam passes then towards the prism *l* and strikes the ground glass window *m*, which is at the same optical distance as the photographic paper, so that the images of the diaphragm-slit are focussed on this window. The window being provided with a scale of 12 c.m. corresponding to the breadth of the photographic paper, this device permits bringing the images on the right spot of the paper by means of the vertical adjustment of the adjustable prisms inside the pendulum apparatus.

The photographic paper is moved by the clockwork, which moves the axis *n* on which the strip winds up. The clockwork will run somewhat more than an hour without re-winding. *o* is the supply roll which can hold 75 metres; these are of standard dimensions, manufactured, for instance, by Palaphot G. m. b. H., Heilbronn a/N, Germany.

To keep the paper straight, a friction roll *p* presses against the supply roll; the pressure can be regulated by inserting the small winding-screw in the aperture at the front side below the doors. On the axis of rotation of this pressure-roll, an indicator is mounted which moves along a scale at the left side of the light-box. This indicator gives roughly the quantity of paper which is still unused.

In order to enable the observer to control the movement of the paper during the observation, the rotation of the roll *k* is transmitted to a little clock-hand, which rotates before a circular scale in the light-box. One division of this scale corresponds to a paper-length of 5 m.m.

The clock-work can be made to give two velocities to the paper, of which one is one third of the other. Changing from one to the other is done by means of the flexible cord No. 3 of fig. 3. If by some unhappy chance, this mechanism does not work satisfactorily, it can be set right by turning the cog-wheel in the right upper corner of the back partition of the recording-room. This will however involve the stopping of the observation as it can not safely be done till after the pendulums have been lifted. The paper-velocity may be adjusted by means of a screw at the left-side of the light-box (L of fig. 3).

In order to enable the observer to separate the observations in the dark room before they are developed, the paper-strip can be marked by means of a lever *q*, which gives pin-pricks at the border; this lever is operated with the flexible cord No 4 of fig. 3.

When paper has to be taken out, which may for instance be done after some three or four observations have been made, it can be cut off by a cutting mechanism *r*. For doing this, the ring at the back is pulled forwards while the paper is pressed slightly with the fourth finger of the same hand, towards the horizontal rod *s*. The cutting mechanism is provided with a safety razor blade, so that the cutting part can be renewed at any moment. This is done by pulling out the ring and unscrewing the top part to which the ring is fastened.

After the exposed part of the paper is cut off, it can be loosened from the axis *n* and taken out of the apparatus by giving the screw-handle at the top

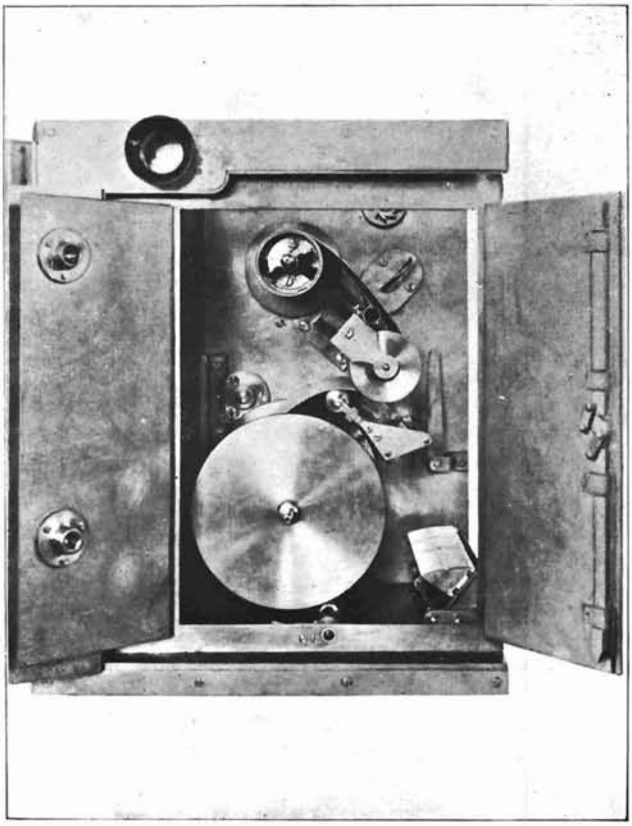


Fig. 4. Recording-box after opening double doors.

For explanation see fig. on page 57.
At the top, control window.
At the bottom, screw for regulating the pressure which acts on the paper roll.

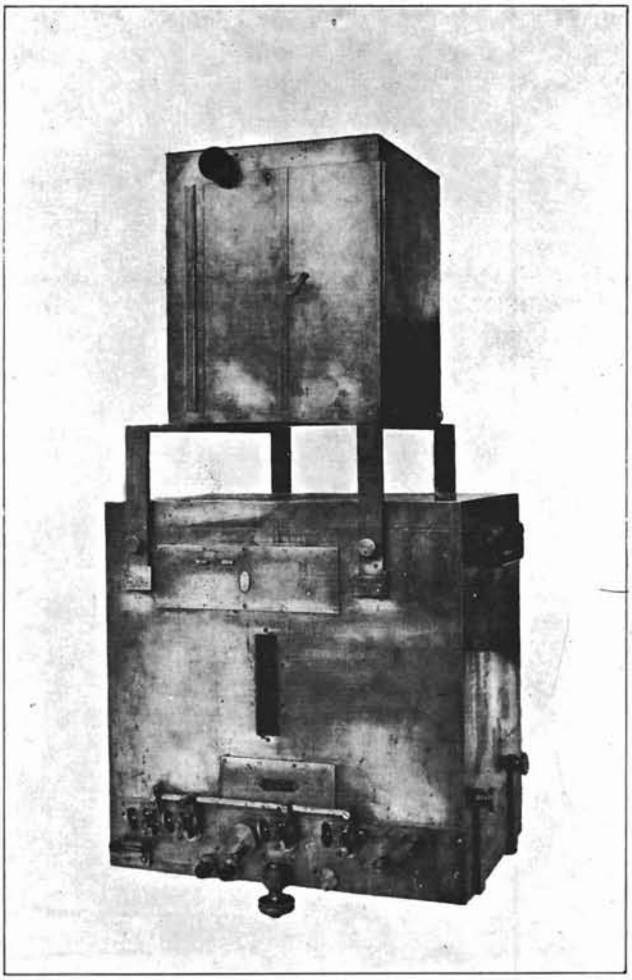


Fig. 5. General view of pendulum apparatus with recording apparatus.

of n a small twist against the direction of the clock-movement; the roll can then be taken out and the paper slipped off and put in a box for keeping it till it can be developed. To attach the paper again, in order to prepare for the next observation, the roll is put back on the axis n, and the end of the paper is inserted in the slit of this roll and fastened by twisting back the screw-handle till it stops. By giving the roll a complete turn, we ensure that the paper is winding properly.

When the supply roll is used up, a new roll has to be put in, which may be done by screwing off the top flange. During this operation, the pressure-roll can be fastened by pushing it as far back as possible. To put it back into action, a small handle at the left bottom corner of the light-box is pushed in, so that the indicator is loosened.

After putting in the paper, the end is unrolled and brought through the recording box in the way as shown by fig. 4. The end is fastened to the roll n as has been described. One supply roll is sufficient for some 50 to 60 observations.

The operations in the recording-box, such as taking out the records and fastening again, may be executed without taking the apparatus to a dark room, because it is provided with a black cloth in the space between the double doors, which has two circular holes for introducing the hands. The necessary practice for doing this without looking inside, is soon obtained. For putting in a new supply roll, a somewhat larger piece of cloth is used; as the roll is too big for introducing it through the holes in the cloth, the roll is put inside the cloth before the cloth is fastened in the space between the doors.

As it is desirable to control the functioning of the apparatus during the observation, without interrupting the photographic recording, a window has been made at the left upper corner of the front-plate, through which the movement of the light-images on the paper may be observed. It opens by means of the flexible cord No. 5 of fig. 3. In order to prevent stray light striking the photographic paper, it ought not to be opened before the eye is put before the window and it ought to be shut before the eye is taken away.

D. *The gimbals.*

The combination of the pendulum apparatus and the recording apparatus can be put in a cradle, which is suspended in gimbals. The cradle may be levelled by means of adjustable counterpoises at the bottom. The gimbals are supported by steel knives. To prevent unnecessary wear, the cradle can be lifted by means of four screws, which are supported by movable metal strips fastened to the fixed support. In this way the gimbals are only used during the observations. For the lifting operation, only two of the screws need be turned as the supporting strips of the other two can be put under the screws by slightly dipping the cradle.

The fixed support may be fastened to a wall or to an iron structure, which is supplied with the apparatus and which is put on the floor; this structure can be roughly levelled by means of foot-screws.

§ 2. *The adjustment of the apparatus.*

The adjustment of the apparatus is made once for all and needs only to be checked from time to time, when there is reason to be doubtful about some point. It comprises:

A. *The adjustment of the slow lifting levers of the pendulums* in such a way, that the three knife-edges are lowered parallel to the agate planes and that they touch the planes at the same moment during the lowering operation, e.g. after turning the wheel about two turns and a half.

B. *The adjustment of the prisms.*

In order to carry this out, the apparatus is put in a darkened room, so that the light-beams may easily be followed. It is mounted in the cradle, of which the screws are not loosened. The pendulums are inside, the slit of the diaphragm in the light-box is opened as far as possible and the diaphragms in the front-plate of the pendulum apparatus are taken out.

The first step is to make the rays enter the pendulum apparatus in the right way, viz. so that prism *a* of the figure on page 56 is completely covered. This is done by adjusting the height of the lamp together with prism *C* of fig. 3. At the same time the distance of the light to the lens is adjusted in such a way that the image of the light is focussed on the diaphragm. Care has to be taken that the bulb of the lamp does not touch metal.

The light will now strike the entrance lens in the right way. After taking off the top-cover of the pendulum apparatus, we may make sure of this by inserting a piece of paper near to the lens, which ought to show an evenly lighted rectangle.

The five different beams inside the apparatus are now separately followed from the entrance lens to the outgoing lens. Each prism is successively adjusted in such a way, that the next prism is completely lighted. As only part of the prisms has an adjustment device, this has to be done for most of them by means of a screw-handle in order to overcome the friction with which they are fastened.

For the adjustment of the rays after passing the outgoing lens, the top-plate of the recording apparatus is unscrewed and carefully taken off in order not to damage the two prisms which are fastened to it. By looking downwards in the tube through which the light comes upwards, we may assure ourselves that the five beams are properly directed. If this is not the case they may be corrected by turning the last prism for each beam inside the pendulum apparatus and by making the exact adjustment with the adjustable prisms; for the two left rays (temperature and damped pendulum) the last prisms are fixed, so the adjustment has to be done entirely with the adjustable prisms. If the whole set of beams is too much to the right or to the left, it can be remedied by means of the adjustment of prism *d* of the figure of page 56.

Before screwing on the top-plate again, we make sure that prism *e* of the figure on page 57 is in the right position, viz., with its sides parallel and perpendicular to the plate. Having fastened the top-plate, the prism *f* is adjusted through the hole in the top-plate so that the rays strike the prism *g* and the cylinder lens *h*, correctly which can be observed by opening the double doors.

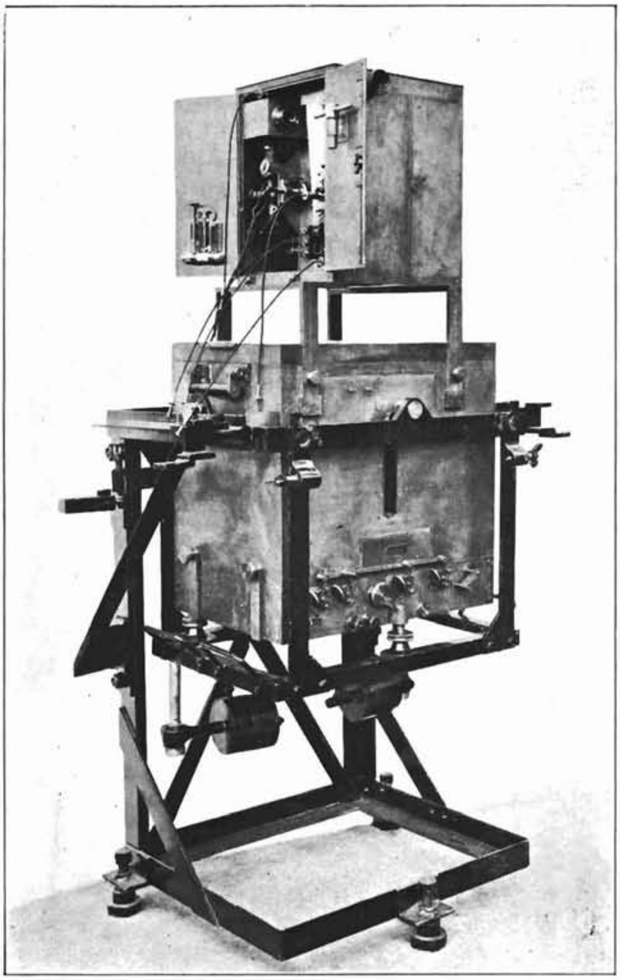


Fig. 6. General view of complete apparatus and gimbals, in working condition.

In order to bring all the images well centred on the cylinder-lens, the adjustable prisms on the top-plate of the pendulum apparatus are finally readjusted.

When prism *g* is turned up, the images ought to appear in a horizontal line on the ground glass-scale at the bottom of the recording apparatus. We must now examine these images to see if they are exactly vertical. If this is not the case, they can be corrected by turning the slit of the diaphragm of the light-box, which may be done after the two tiny screws are loosened.

We should now examine the positions of the images on the scale in a horizontal sense. These positions may be adjusted by means of the vertical adjustment of the adjustable prisms on the top-plate of the pendulum apparatus. A satisfactory arrangement is for instance:

the axis of the record of pair Nos. 3 and 2 at 9.6 c.m.,
the axis of the record of pair Nos. 1 and 2 at 5.4 c.m.,
the axis of the record of pendulum No. 2 at 2.0 c.m.,
the zero-line for the record of the angle β at 8.5 c.m.,
the temperature-line for a mean value at 4.5 c.m.

Before finishing the light adjustment, we have to investigate if each fictitious pendulum record really gives exactly the difference of the two angles of elongation of the pendulums involved. In order to do this, the cradle is loosened, the pendulums are lowered without giving them amplitude and the cradle is gently dipped round an axis perpendicular to the front-plate of the apparatus. In this way the pendulums are given equal elongation with regard to the apparatus, and so the fictitious pendulums ought not to have any amplitude, and the corresponding images on the scale ought not to move.

If there appears any movement, this is caused by the fact that the angles which the beam makes with the normals on the pendulum-mirrors, are not exactly equal. By slightly turning the adjustable prism of this beam and examining if the effect increases or decreases, we may find out in which direction the rays diverge. We may then take away the effect by slightly turning the prism which precedes the first pendulum reflection and by bringing the image back by means of the adjustable prism. A few trials will generally be sufficient to perfect the adjustment. This adjustment is of course limited to a small amount, as a greater adjustment of the prism which precedes the first pendulum reflection would throw the beam off the mirror.

C. *Adjustment of the electrical shutters in the light-box.*

The adjustment of the springs of the electrical shutters has to be done in such a way, that the shutters do not have too strong vibrations, because in that case the slit of the diaphragm would not only be covered by the passage of the shutter, but also by its subsequent vibrations. It is difficult to examine this visually; the photographic record is the only sure means of checking it. At the same time this adjustment is made we should adjust the amplitude of the shutter movement, and also make sure that the slit of the shutter is parallel to the slit of the diaphragm.

D. *Adjustment of the pressure of the roll which presses against the supply-roll of photographic paper.*

This is effected by introducing the small winding-key, which is put away in the light box, in the aperture below the inner doors in the front of the recording apparatus. It is done in the same way as giving tension to the spring of a clock-work.

E. *The adjustment of the paper-velocity.*

This is done by means of a screw at the left side of the light-box, which can be turned with an adjusting pin.

§ 3. *Execution of the observations at sea and on land.*

The apparatus is put up on board ship in the following way.

The cover of the pendulum apparatus is taken off and the three pendulums with the dummy pendulum put in their places; for doing this it is necessary slightly to lower the lifting mechanism for the pendulums, but care should be taken not to lower it so far that the knife-edges would touch the agate planes. After this the pendulums are lifted completely and their bulbs fixed with the clutches. After making sure that the screws at the bottom of the damped pendulums, which fix them during transportation, have been loosened and that the indicator of the hair-hygrometer is released, the apparatus is put in the cradle of which the counterpoises have been taken off and the cradle is hung in the gimbals which have previously been fixed in the ship. After this the apparatus is further built up by putting on the cover, fixing the supports of the recording apparatus, putting this last apparatus in its place and screwing on the counterpoises. The electrical wiring for the Edison Swan lamp and for the light-shutters being made, the apparatus is ready for the observations.

During a voyage the apparatus and the chronometers should be left in their places in the ship and the pendulums are kept inside the apparatus.

The execution of the observations at sea comprises:

1. Opening the light-box of the recording apparatus and screwing on the flexible cords, opening the light-entry in the bottom-plate of this apparatus,
2. winding the clock-work,
3. connecting the electrical wiring for the lamp and for the shutters,
4. loosening the gimbals and levelling the apparatus by means of the counterpoises at the bottom. The last part of this operation may easily be perfected by putting a small weight on the top of the pendulum apparatus and moving it,
5. unclutching the pendulums,
6. pressing the amplitude-handles of the outer pendulums (the right one of the left pendulum) against the bulbs,
7. projecting the light-rays on the ground glass-scale,
8. lowering the pendulums,
9. giving amplitude by turning back the coupling lever,

10. starting the clock-work, marking the photographic paper and switching back the prism *g* of the recording box,
11. reading the thermometer, the hygrometer, the barometer, and the temperature of the locality,
12. five or six minutes after No. 10: switching over the clock-work from the greater to the smaller velocity and changing the shutter movement from momentary interruptions to half-second interruptions,
13. half-way the observation (i. e. 15 minutes after No. 10): reading again the thermometer, the hygrometer, the barometer and the temperature of of the locality,
14. five or six minutes before the end: switching back the clock-work from the smaller to the greater velocity and changing over again to momentary interruptions by the shutter,
15. just before the end: final reading of the thermometer, the hygrometer, the barometer and the temperature of the locality,
16. ending the observation by lifting and clutching the pendulums, marking the paper, shutting off the light, stopping the clock-work, fastening the gimbals, screwing off the flexible cords, opening the shutter-circuits and shutting the light-box and the light-entry of the recording apparatus.

Further data which are necessary, are: the mean depth of the ship below the surface of the sea during the observation, the mean latitude and longitude during this interval, the ships velocity and course, the sea-current and the depth of the sea during the observation.

During the observation the positions of the light-beams may be inspected by looking through the window at the left top-corner of the recording apparatus. We have to take care however that the eye is always before the window when it is open, in order to prevent stray light striking the photographic paper. We can also observe the paper-movement and the paper-velocity by looking at the little clock-hand in the light-box.

The time at the beginning and at the end of the observation, during which the greater paper-velocity has been used, ought to be long enough to give at least two passages of the chronometer-marks through the axis of the record, i. e. these intervals should be somewhat more than double the coincidence-interval.

About operation No. 9 of the above list, some further details may be given. We may first mention, that the deductions of § 11 of Chapter I have shown that the most appropriate initial conditions are: giving the outer pendulums equal amplitudes in opposite phase and loosening the middle pendulum without giving it amplitude. It is however somewhat difficult to realize these conditions. The position of the amplitude-lever, which presses against the bulb of a pendulum, can be adjusted by means of a graduated screw at the outside of the apparatus, but this only corresponds to a certain amplitude of the pendulum if the apparatus is level when the lever is turned back. If this is not the case, the pendulum gets another amplitude; in the case for instance where the outer pendulums get amplitude by means of levers working in contrary sense, one of the pendulums gets too much amplitude and the other too little.

Before starting the pendulums we have therefore to look to the small outside levels of the apparatus and we have to turn back the lever when the bubble passes through zero position, or better still a fraction of a second before this moment, in order to take into account the lag of the movement of the bubble. We can observe at once on the glass-scale if the operation has been successful: if the amplitudes differ too much, e. g. more than one third of their value, we should repeat it until the result is satisfactory.

For a land observation, nearly all the operations are the same. The principal difference is, that the observation lasts longer, viz., some two hours or more, and that it is not necessary to make a continuous record during the whole time. We may for instance begin with a record of some five or six minutes at the larger paper-velocity (see the above remark about this time), then change over to the small paper velocity and the half second light-interruption during some three or four minutes, then stop the paper, start it again after an interval of T_p (see form. 57 B of App. IV to Chapter I and page 84 of Chapter III) for three or four minutes, stop again and then apply the reversed programme for the second half of the observation. The intermediate recordings have to include at least one moment for which the determination of the quantity b is easy (see page 81 of Chapter III), i. e., one moment that the chronometer-mark of the fictitious pendulum-records has a maximum elongation. They have therefore to be somewhat longer than the coincidence-interval.

The observations on land can also of course be executed with the apparatus standing on a firm foundation, instead of being mounted in the cradle, or if it is mounted in the cradle, the cradle can be fastened with the screws. In these cases the sway will differ from the sway in the loosened cradle, so that the half period T_p is different. The levelling can now be done more accurately by means of the pendulum-level, so that the line recorded by the damped pendulum need not be used. Otherwise these possibilities do not present any special features.

The investigation of § 12 of Chapter I has shown that the most appropriate initial conditions for land observations are the same as those for sea observations, i.e. to give the outer pendulums equal amplitudes in opposite phase and to loosen the middle pendulum without giving it amplitude. The realization of these conditions does not present any difficulties now; the apparatus does not swing, so that it is continually in a level position, and so the pendulums really get the amplitudes, which have been set by means of the graduated screws of the amplitude-handles.

We have mentioned in § 12 of Chapter I that it is well to fill up the whole interval between two time signals with pendulum observations and that one or two of this series of observations may profitably be executed with other initial conditions than those which have been mentioned, in order to get data about the middle pendulum. These conditions are: giving the left pendulum twice as much amplitude as the middle pendulum in the same phase and giving the right pendulum thrice as much but in contrary phase.

By means of the coupling lever, these initial conditions are easily realized.

Up to now we have considered the land observations which are necessary as a supplement to the sea observations, i. e., the observations at the base-station and at intervening ports. We will now discuss briefly the possibility of using the apparatus also for land gravity surveys. If it proves successful it would considerably reduce the time needed for the survey.

The idea is to mount the apparatus in a truck which is specially adapted to the purpose, and to leave the pendulums inside the receiver for the whole duration of the observation tour. The car should be provided with a wireless receiving set, connected in such a way, that the timesignals appear on the photographic record of the pendulum apparatus. The precision of this photographic recording of the signals ought to be 0.001 sec. or at least the lag ought to be constant to this extent.

If the radiostation gave two timesignals in the morning and two in the afternoon at an interval of about one hour and a half, we could arrange the following programme. The car is stopped at a station shortly before the first signal of the morning pair, the pendulums are set swinging, the signal is received on the record, the observation is continued till the second signal is recorded and the pendulums are fastened again. The car is then driven to the next station and in the afternoon the programme is repeated for that second station. At the same time, observations for the same programme are executed at the base-station, in order to get the necessary data about the time signals. In this way we could dispense with chronometers and two stations could be observed each day.

Returning to the sea observations, attention should be drawn to an important point. Besides making observations at the base-station before and after the voyage for standardizing the pendulums, it is also necessary to get controldata for the zero-line of the record of the damped pendulum, which gives the deviation of the swinging-plane from the vertical. From time to time, therefore, we must put the apparatus on a good foundation, level the agate planes carefully and make a record for getting the position of this zero-line with regard to the axes of the two fictitious pendulum records. These data are somewhat more accurate than the position of this line with regard to the paper-strip, because this strip may undergo small movements between the flanges of the rolls in the recording apparatus, so that its position is not always exactly the same.

We need not add that special care has to be taken that the adjustable prisms on the top-plate of the pendulum apparatus are not readjusted in vertical direction during the voyage because this would change the relative position of the axes of the above records. The prisms in the front-plate of the apparatus and in the recording apparatus may of course be readjusted if necessary, as this does not affect this relative position.

For taking out the photographic record of the observations, which may, for instance, be done after some three or four have been made, the outer front doors of the recording apparatus are opened and the two hands and a little tin box for the record are inserted through the two holes of the black-cloth.

This cloth remains inside the apparatus during the whole voyage. After making sure that the cloth fits well round the wrists, the inner doors are opened, the paper-strip is wound up by giving the roll *n* one turn and is cut off by pulling out the ring, while gently pressing the paper against the tube, which is beside the ring. After turning round the screw at the top of *n*, we can take off the roll, slip off the paper, put it in the tin box, put on the roll again, insert the end of the unused paper in the slit of this roll, turn the screw in order to fasten it and give the roll a complete turn in order to make sure that the paper winds well.

We may then shut the inner doors, take out the box with the records and shut the outer doors.

For introducing a new supply-roll, we begin by fastening the paper-indicator at the left side of the light-box by pressing it downwards. We then take out the black cloth by pressing the knobs at the bottom and insert a larger piece of cloth after having put the new roll inside. We introduce the hands in the holes of this cloth, take off the paper-cover of the new roll, open the inner doors, screw off the top-flange of the roll *o*, take out the old paper, put on the new roll, making sure that it touches the back flange, screw on the top-flange till it stops by metallic contact, bring the end of the paper through the apparatus and fasten it to the roll *n* as explained above after having cut it off straight with the cutting mechanism.

Having shut the inner doors, the black cloths are interchanged again, the outer dours shut and the paper-indicator in the light-box set into action by pressing the little lever. The apparatus is then ready for further observations.

§ 4. *Supplementary observations.*

In order to obtain the data, which have been mentioned on page 63, supplementary observations have to be made, about which a few remarks follow.

A. *The ship's position.*

A knowledge of the ship's position is necessary for two reasons for computing the gravity anomalies.

First, the normal value of gravity depends on the latitude; this dependency is less for low and high latitudes than for middle latitudes and at the equator and the poles the variation practically disappears, so that a latitude error affects the gravity anomaly whenever the observation is not made near the equator or near the poles. At 45° latitude 1′ corresponds to 0.0015 c.m. gravity variation.

Secondly the topographic and isostatic reductions depend on the position of the ship with regard to the topography; in this respect, an error in the position has of course more effect if the observation is made near great topographic irregularities, than if it is made at a great distance from any irregularity.

So it is desirable to get the ship's position often with the utmost accuracy which may be obtained and it is well to make use of all opportunities for getting more data. Under favourable conditions, such as a clear sky and well trained observers, a mean error of one mile, i. e. one latitude minute of arc, may be obtained, but usually it will be greater. If bearings can be taken it is in general easy to get the position with sufficient accuracy.

B. *The ship's velocity.*

For computing the EÖTVÖS effect we need the east-west component of the ship's velocity. This velocity is composed of two parts: the ship's velocity with regard to the water and the velocity of the current.

The first part, the ship's velocity with regard to the water, may be determined with sufficient accuracy. First, we have the data, which are experimentally determined for every ship, giving the velocities corresponding to different numbers of revolution of the screw. For most ships these data are redetermined often.

Still circumstances may occur for which these data are somewhat doubtful, e. g. when the ship is making a long voyage without the hull being cleaned. In that case the friction is greater and the velocity is less than it ought to be. If in these circumstances there is no opportunity to determine the ship's velocity experimentally, we may recur to the pendulum apparatus for determining it. To this end, two observations are made, the first while steering eastward and the second while steering westward back along the same track. The difference of the observed values of the gravity gives double the EÖTVÖS effect, so that formula 81 may be used to deduce the ship's velocity from this difference. It may easily be seen that this method is also valid if there is a sea-current, provided it is the same during both observations; although this current affects the value of the EÖTVÖS effect for each observation, the difference of these effects remains the same as if there were no current.

It is not so easy to get the east-west component of the current with sufficient accuracy. Actually the only way to get data about the current is to determine the ship's position both before and after the observation and compute the total effect of the current during the interval. In this way we get the mean current during that time, which, however, may easily deviate from the actual current during the observation itself; generally the time interval between the determinations of the ship's position is several hours, while the observation takes only half an hour. Moreover the observation is made at a certain depth and the ship is at the surface during the remainder of the time. A smaller time interval between the two determinations would not be of much use, as the accuracy of these determinations is limited. Only when good bearings can be taken immediately before and after the observation, is it possible to get a satisfactory value.

In mid ocean the resulting uncertainty is less than near to coasts and islands, because the current conditions are more regular. We think therefore that in midocean it is generally well to use the value for the current which has been deduced from the ship's positions and to apply a certain fraction dependent on the depth, e. g. one half for a depth of 30 meters. For observations near to coasts or islands, however, the current data will often be so doubtful that it is better not to apply them at all, unless they are obtained by good bearings shortly before and after the observation.

C. *The sea-depth.*

It is desirable to supplement the gravity survey with a survey of the topography of the sea-bottom. In the first place the depth should be determined during the observation, but often an extension of this survey to the surroundings will be of use. We may distinguish two cases: the topography of the sea-bottom is already known or it is not or only imperfectly known.

In the first case the determination of the sea-depth may be a valuable expedient for determining the position of the ship during the observation with regard to the topography, so that better data are obtained for the computation of the topographic and isostatic reductions. It depends on the circumstances whether these data will also be taken into consideration for the final determination of the latitude of the observation, which is necessary for the computation of the normal value of gravity. In case the direct determination of the ship's position is doubtful, these data may be useful, provided the location of the topography on the chart is reliable.

In the second case where the topography of the sea-bottom is either not known or only imperfectly known, it is of course still more desirable to make a depth-survey, in order to obtain the data for the computation of the topographic and isostatic reductions.

§ 5. *Determination of the sway.*

For the observations with the apparatus which have been described in this chapter, the determination of the sway has only secondary importance.

For the observations at sea, it has obviously no use at all, because the ship's movements are much greater. For the observations on land, the value of the sway is not needed for the deduction of the results, but it may be useful to have an approximate value for determining the half period T_p (see form. 57 *B*, page 44) of the fluctuations shown by the record of the middle pendulum; we wish to known T_p for determining the moments at which intermediate recordings have to be made.

We will indicate two methods for determining the sway, which have proved satisfactory for the purpose.

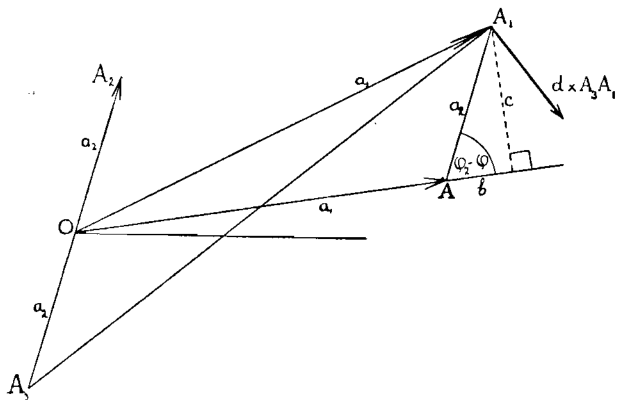
The first method is to determine immediately the period of the fluctuations, shown by the record of the middle pendulum. To this end we make a continuous record of one of the ordinary land observations, which has been started with the outer pendulums having equal amplitudes in opposite phase and with the middle pendulum having a very small amplitude. We find in this way T_p, and u may be found by means of formula 57B of page 44.

The second method is a more direct and more accurate way of determining the sway. It is similar to the ordinary two pendulum method. We start by giving amplitude to one of the outer pendulums, e.g. No. 1, and by giving no amplitude to the middle one, No. 2; the third pendulum is kept fixed by the amplitude-lever. As it is difficult to realize exactly a zero amplitude for pendulum No. 2, we will slightly vary the method and we will assume only that its initial amplitude is small; this can of course easily be realized by lowering the pendulum without giving amplitude.

The formula for this method is derived from the formulae 45 of page 36:

$$\dot{q}_1 = i\,r\,q_1 - i\,d\,(q_1 + q_2) \qquad d = \frac{\pi}{T^2}\,u$$
$$\dot{q}_2 = i\,r\,q_2 - i\,d\,(q_1 + q_2)$$

of which the first term of the right side gives the undisturbed pendulum movement and the second term the disturbance by the sway. These formulae may be represented graphically in the same way as has been done in Chapter I:



OA_1 is the pendulum-vector of No. 1, corresponding to q_1, OA_2 of No. 2, corresponding to q_2, OA of the fictitious pendulum, corresponding to $q = q_1 - q_2$ and A_3A_1 represents $q_1 + q_2$.

The effect of the sway as given by the above formulae, is a velocity of A_1 perpendicular to the vector A_3A_1 and equal to $d \times A_3A_1$. As may be expected, the position of the fictitious pendulum is not affected by the sway.

We may generally neglect the deformation of the triangle $O\,A_1\,A$ which is caused by the deviation from isochronism (see also page 7), first because the duration of the observation is small, i. e. some 5 to 10 minutes, and second because the amplitude a_2 is small. So the movement of A_1 is the only deformation which has to be taken into account. If we indicate the components of $A\,A_1$ in the sense of $O\,A$ by b and perpendicular to $O\,A$ by c, we may write the following formula for the differential coefficient of c according to the time

$$\dot{c} = -(a + 2\,b)\,d$$

so that we get

$$u = -\frac{T^2}{\pi}\frac{\dot{c}}{a + 2\,b}.$$

If we do not wish to neglect the effect of the deviation from isochronism, we may easily deduce the following formula which takes into account its principal part.

$$u = -\frac{T^2}{\pi}\frac{\dot{c}}{a + 2\,b} + (T_1 - T_2)\frac{b}{a + 2\,b}.$$

The quantities a, b and c may be found from the record. a is proportional to the amplitude of the record of the fictitious pendulum, b and c may be derived from the record of the second pendulum. This pendulum is doubly recorded, first by means of the damped pendulum and second by means of pendulum No. 3 of which the mirror acts as a fixed mirror. For the computation we will choose the last record; the dimensions of the first would have to be multiplied by 0.88 in order to be comparable to those of the other records (see page 81).

The quantities b and c may be measured directly. By keeping in view that b and c are the components of the pendulum-vector of No. 2 parallel and perpendicular to the fictitious pendulum-vector, we see that b is the elongation of No. 2 at the moment that the fictitious pendulum has its greatest elongation, and c the elongation of No. 2 at the moment that the fictitious pendulum has zero elongation (see § 3 of the next chapter, where this point is treated more in detail). So we may substitute for b the ordinate of the chronometer-mark of the record of No. 2 for the second for which the mark of the fictitious pendulum record has maximum elongation, and for c the value of this ordinate for the second that the mark of the fictitious pendulum record passes through the axis of the record. Both ordinates are in the same ratio proportional to b and c as a is to the amplitude of the fictitious pendulum record, so that we need not multiply by this ratio when substituting these quantities in the formula.

An appropriate way of making the computation is to measure c at the moments of three or more consecutive passages of the mark of the fictitious pendulum record and to measure a and b in the moments of greatest elongation of this mark between. The formula for u is applied to each passage-interval and so we obtain several values for u, of which we take the mean value.

For deriving $\dot{c}$ from the values c_b and c_e of c at the beginning and at the end of the passage-interval, we have to take into account the damping. If the damping-coefficient is k and the duration of the interval t, we have

$$\dot{c} = \frac{(1 + {}^1\!/_2\,kt)\,c_e - (1 - {}^1\!/_2\,kt)\,c_b}{t} = \frac{c_e - c_b}{t} + k\,\frac{(c_e + c_b)}{2}$$

which may be introduced in the formula for u. For a and b we introduce the values measured midway between c_b and c_e.

For the damping-coefficient k we may take a value for a mean atmospheric pressure, which may be determined once for all for a certain set of pendulums or we may derive a value from the values a' and a'' of the amplitude of the fictitious pendulum at the beginning and at the end of the observation.

If a' and a'' are a time t' apart, we have, because t' is supposed not to be great

$$k = \frac{2\,(a'' - a')}{t'\,(a'' + a')}.$$

For making the chronometer-marks visible in the record of pendulum No. 2 in order to have no difficulties in measuring b and c, we have to adjust the shutter in the light-box in the middle position, i. e. so that the light is interrupted for half a second. The paper may be run at its lowest speed during the whole observation.

CHAPTER III.

The computation of the results.

We will successively examine:

§ 1. The deduction from the record of the periods of the fictitious pendulums, in case one chronometer is used;

§ 2. The same in case two chronometers are used;

§ 3. The deduction from the record of the correction for deviation of isochronism and of the reduction to infinitely small amplitude in case the record refers to a sea-station;

§ 4. The same in case the record refers to a land-station;

§ 5. The complete reduction of the observed period;

§ 6. Appendix to the previous paragraphs; short summary of the methods of computation of the results given by the ordinary four pendulum Sterneck apparatus;

§ 7. The computation of the result for the gravity;

§ 8. The reduction of the observed gravity for the effect of the east-west component of the ship's velocity (Eötvös effect);

§ 9. The reduction of the observed gravity to the surface of the sea;

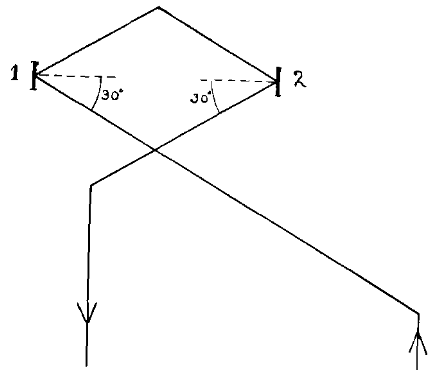
§ 10. The computation of the free-air, the Bouguer, and the isostatic anomalies; reduction of Bowie.

§ 1. *The deduction from the record of the period of the fictitious pendulum in case one chronometer is used.*

The records are made by light-beams entering the pendulum-apparatus horizontally through a lens, which makes them parallel, and leaving it through a lens with a focal distance of 111 cm. which coincides with the optical distance of the photographic paper. The angular deviations, which are imparted to the rays in the apparatus, are therefore recorded multiplied with a base-length of 111 cm. As this distance is great with regard to the maximum linear deviation on the record of 6 cm. (= half of the breadth of the photographic strip), we may assume, that the deviation on the record x is proportional to the angular deviation α imparted by the apparatus

$$x = 1.938\,\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65\,A)$$

The record of the fictitious pendulum, e. g. that which is derived from the pendulums Nos 1 and 2, is effected by a light-beam which is successively reflected in the apparatus by the mirrors of the pendulums 1 and 2; these reflections take place under angles of 30° with the normals to the mirrors



and so the beam gets a deflection, which may be represented by the formula

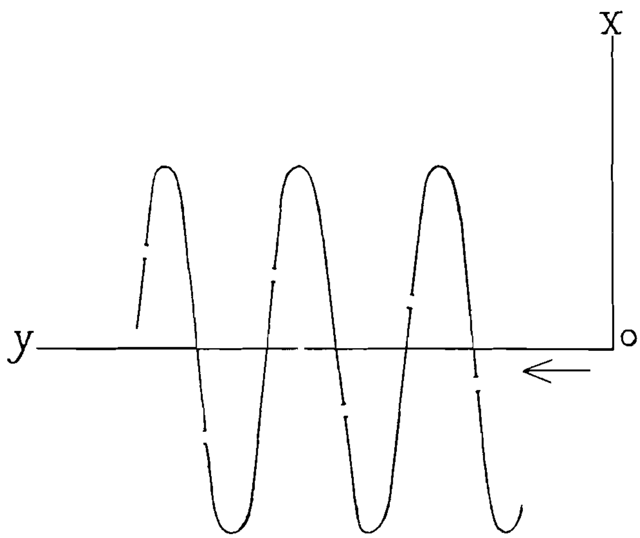
$$\alpha = 2(\theta_1 - \theta_2)\cos 30^\circ = 2\,\theta\cos 30^\circ$$

in which θ is the angle of elongation of the fictitious pendulum, and the record shows a deviation

$$x = 3.35\,\theta \quad . \quad . \quad (65\,B)$$

(θ in degrees, x in cm.), if we choose the Y axis corresponding to $\theta =$ zero.

The following curve gives a sketch of a record, in which the y-dimension has been enlarged for the sake of clearness

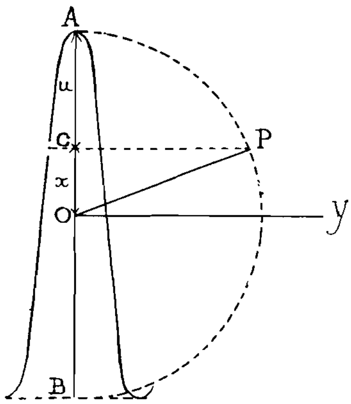


Each second the chronometer shuts an electrical current, which causes a short interruption of the light and about half a second afterwards the current is opened again and the light is again momentarily interrupted. The record shows therefore two series of interruptions, each series at intervals of exactly one second, while the marks of one series are about midway between the marks of the other. We will use only one series for our computations and we will choose the most regular of the two. Generally this will be the series caused by the opening of the current.

The ordinate x of a chronometer-mark gives the angle of elongation of the fictitious pendulum corresponding to the moment of the mark. The position of the pendulum-vector at this moment may at once be found from

$$\cos\varphi = \frac{OC}{OP} = \frac{x}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66\,A)$$

We may repeat here that the pendulum-vector is a vector, which represents the pendulum movement. For an undisturbed pendulum it has a constant length, the amplitude a, and rotates with constant angular velocity n.



The projection on the X axis is the angle of elongation of the pendulum, the angle φ between the vector and the X axis is the phase of the pendulum and the period of the pendulum is half of the time in which the vector makes a complete revolution, so that $T = \pi / n$.

Another formula for determining φ, which may be useful, is:

$$tg\ {}^1/_2\ \varphi = \frac{CP}{CB} = \sqrt{\frac{AC}{CB}} \quad . \qquad (66\,B)$$

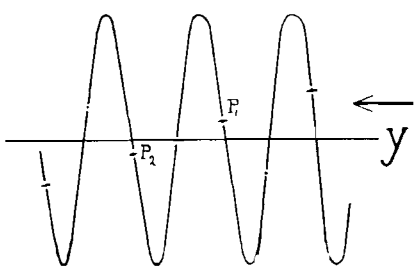
If the period of the pendulums is somewhat more than half a second, the successive positions of the pendulum-vector, corresponding to the chronometer-marks, recede slowly. The time-interval in which it makes half a revolution will be called as usual the coincidence-interval; it may be described as the time in which the pendulum-phase loses 180° with regard to the phase of some hypothetical pendulum, which is isochronous with the rhythm of the chronometer. If the receding movement of the pendulum-vector has an angular velocity N, the coincidence-interval c is

$$c = \frac{180^\circ}{N} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (67\,A)$$

For the computation of the period T the usual formula may be used

$$T = 0{,}5 + \frac{0{,}25}{c - 0{,}5} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (67\,B)$$

The simplest way to find c is to determine the moments that φ is 90° or 270°, i.e. that the pendulum-vector passes through the Y axis; c is the interval between those moments. We may do this by determining the chronometer-second, corresponding to the mark P_1 which precedes the passage and by making an estimate of the fraction of the second by taking the ratio of the x-ordinates of P_1 and P_2. This method is the same as the ordinary coincidence method for the Sterneck apparatus. If we identify the flash with the chronometer-mark, the observation of the passage through the middle thread is exactly the same proceeding.



In special circumstances, however, it may occur that the determination of these moments gives difficulties; we can then resort to a more general method for determining c, which is made possible in our case by the photographic recording. By means of the formulae (66) we may deduce the position of the pendulum-vectors at some arbitrary moment in the beginning and at the end of the observation, and this gives the angular velocity N and therefore c.

We will successively examine both methods.

A. Determination of c by deducing the moment of the passage through the Y axis.

For getting c in this way we will compute a group of a few passage-times at the beginning of the observation and also at the end; by taking the difference of the mean of each group and by dividing by the number of coincidence-intervals, included between these means, we find c.

Each passage-time may however be determined more accurately than can be done by means of visual observation with the coincidence apparatus of the Sterneck apparatus. Not only is it possible exactly to measure the ratio of the distances to both sides of the Y axis of the marks preceding and succeeding to the passage, so that we can make a better estimate of the fraction of the second, but we can use a great many more marks before and after the passage, so that each passage is deduced from a great number of chronometer-marks.

In these circumstances it is not necessary to compute many passages at the beginning and at the end of the observation. We will, however, if possible, not take less than two for each group, in such a way, that we observe two passages in opposite directions, i. e. for $\varphi = 90°$ and for $\varphi = 270°$. If more are taken, we will compute as many passages in one direction as in the other.

In this way we eliminate the error in the location of the Y axis, which is the consequence of wrongly estimating the position of the upper and the lower points A and B of the curve. This error is possible because of the blurring of the record at these parts of the curve, which is caused by the excess of light. If those errors are equal at the upper and lower points, the Y axis is located rightly, but we cannot be sure of this and so it is prudent to eliminate an eventual error.

If in a special case it is not possible to get passages in both directions, we still may eliminate the greatest part of this error by taking the passage at the beginning of the observation in the same direction as at the end.

The case that only one passage is available may occur for instance when part of the record has gone wrong or when the pendulum is nearly isochronous with the chronometer, so that the marks pass only once through the Y axis during the first and last parts of the observation.

We mentioned already that each passage may be deduced from a great number of chronometer-marks. This is more urgent for sea-observations than for land-observations. For the last, we have only to do this in order to lessen the effect of the slight fluctuations, caused by the imperfect working of the electrical shutter; for the first it is also necessary, because we have to lessen in this way the effect of the vertical accelerations, which may cause much greater fluctuations of the marks (see fig. 7 and § 3 of chapter I).

In the following way we may deduce a formula for determining the passage-time from a number of consecutive marks. If we suppose the pendulum movement to be regular during this time, the mean ordinate x_m of m consecutive marks is

$$x_m = \frac{a}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} \cos\varphi \, d\varphi = a \frac{(\sin\varphi_1 - \sin\varphi_0)}{\varphi_1 - \varphi_0} = a \frac{\sin\frac{1}{2}(\varphi_1 - \varphi_0)}{\frac{1}{2}(\varphi_1 - \varphi_0)} \cos\frac{1}{2}(\varphi_1 + \varphi_0)$$

in which φ_0 and φ_1 are the phases of the first and of the last mark, so that $\frac{1}{2}(\varphi_1 + \varphi_0)$ is the phase for the mean of the seconds, which correspond to the marks. The x ordinate for that mean time is therefore

$$x_g = a \cos\frac{1}{2}(\varphi_1 + \varphi_0) = \frac{\frac{1}{2}(\varphi_1 - \varphi_0)}{\sin\frac{1}{2}(\varphi_1 - \varphi_0)} x_m \quad . \quad . \quad . \quad (68\,A)$$

So we have to multiply the mean ordinate x_m with a factor e in order to find the ordinate for the mean time x_g, e being given by

$$e = \frac{\frac{1}{2}(\varphi_1 - \varphi_0)}{\sin\frac{1}{2}(\varphi_1 - \varphi_0)}$$

and if we introduce the time-interval Δ_{10} between the first and last mark of the series, we get

$$\varphi_1 - \varphi_0 = N\Delta_{10} = \pi \frac{\Delta_{10}}{c}$$

and therefore

$$e = \frac{\frac{\pi}{2}\frac{\Delta_{10}}{c}}{\sin\frac{\pi}{2}\frac{\Delta_{10}}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68\,B)$$

We can make a table for e with the entry $\frac{\Delta_{10}}{c}$.

Applying this to the determination of a passage, we may compute the x-ordinate of the mark, which precedes the passage, from a series of consecutive marks, of which this mark is the middle one. The mean of the ordinates, multiplied with e, gives the ordinate of that middle mark. Divided by the vertical velocity of the mark, which near the Y axis is obviously given by Na, i.e. the angular velocity of the pendulum-vector corresponding to the mark, multiplied by the length of that vector, we find the fraction of the second, which the passage is later than the second corresponding to the middle mark. It can easily be seen, that it is not necessary that the middle mark is the mark preceding the passage, but that any mark near the Y axis may be used; if this mark corresponds to the second T_g, the time of the passage is given by

$$T_g + \frac{x_g}{Na} = T_g + \frac{c\,e\,x_m}{\pi\,a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

We have of course to take care that T_g does not differ too much from the time of the passage, because otherwise we would not be justified in

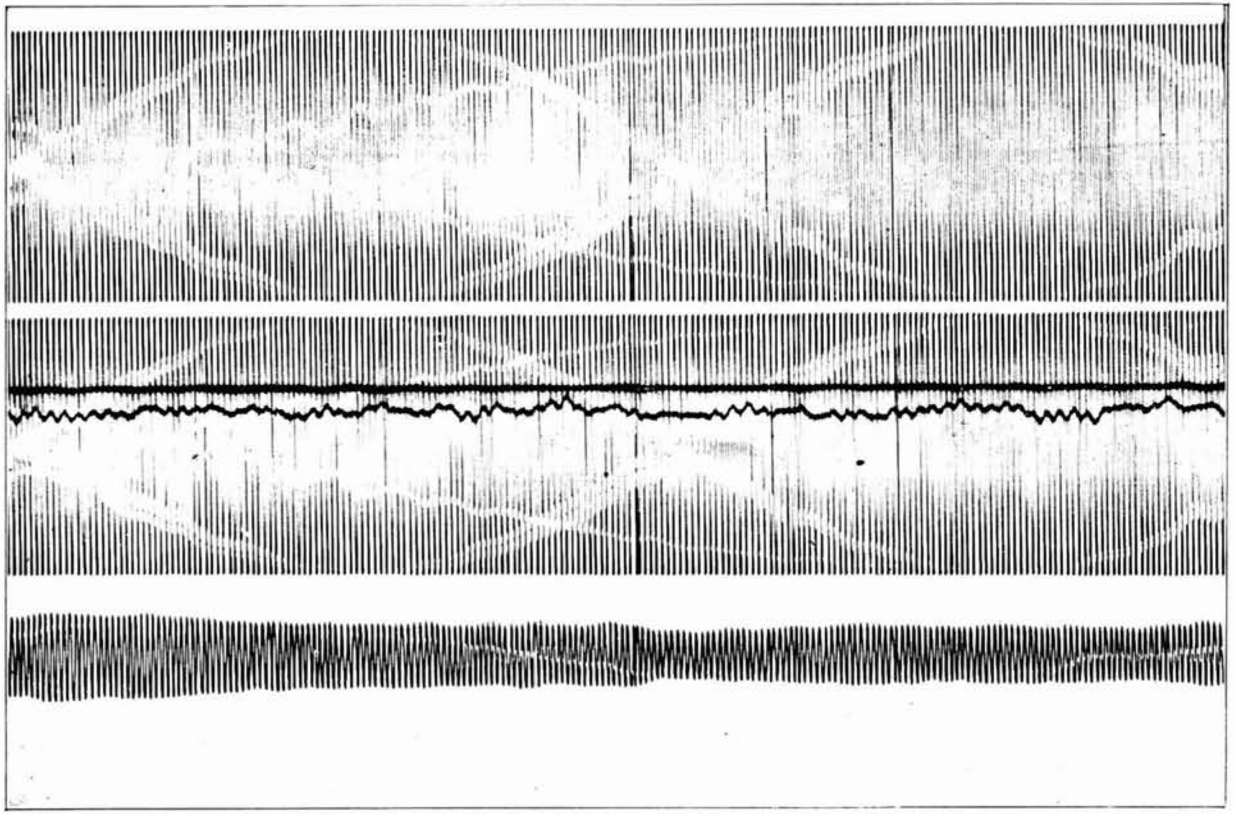


Fig. 7. Part of the record of fig. 9.

First part of an ordinary oceanic record on true scale; vertical accelerations and therefore fluctuations of the chronometer-marks.

Upper record of pair Nos. 3 and 2, middle record of pair Nos. 1 and 2, lower record of No. 2 apart, in the centre record of the angle of deviation β of the swinging-plane and record of the temperature of the air.

applying a constant vertical velocity Na. The vertical velocity for a phase-angle φ is the projection $Na \sin \varphi$ of the velocity Na of the extremity of the corresponding pendulum-vector on the X axis and so we may only suppress the factor $\sin \varphi$ if φ is nearly 90° or 270°.

Objection might be made that c is not known beforehand, so that we would not be able to compute e and the second term of (69), but as this term is small, this objection is not serious; we can introduce a rough estimate of c without making an appreciable error.

Instead of (69) we may use another formula, which we find by substituting in the formula of e

$$\sin \frac{1}{2} (\varphi_1 - \varphi_0) = \frac{(\cos \varphi_0 - \cos \varphi_1)}{2 \sin \frac{1}{2} (\varphi_1 + \varphi_0)}$$

and as $\frac{1}{2} (\varphi_1 + \varphi_0)$ is nearly 90°, we may suppress the sine in the denominator. We get in this way for the passage-time instead of formula (69)

$$T_g + \frac{c (\varphi_1 - \varphi_0)\, x_m}{a (\cos \varphi_0 - \cos \varphi_1)\, \pi} =$$

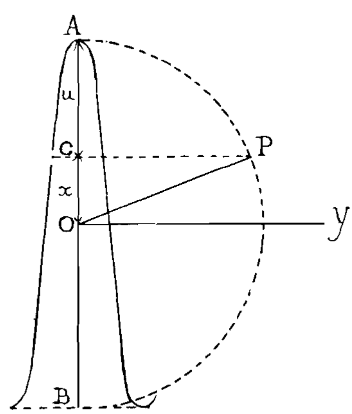
$$= T_g + \frac{x_m}{(x_0 - x_1)} \Delta_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

in which x_0 and x_1 are the ordinates of the first and last mark, and therefore $x_0 - x_1$ their distance in the x direction.

This formula is slightly less accurate than formula (69) because $x_0 - x_1$ is influenced by the fluctuations of the marks, caused by the vertical accelerations, but as we suppose the second term of (69) and (70) to be small, this causes no error of any moment.

We have up to now supposed that the marks are consecutive. We can however see easily that the formulae remain valid for any series of marks at equal intervals; the interval need not be one second. As it is desirable, for eliminating as well as possible the effect of the fluctuations of the marks, that the series includes several complete fluctuations, we may appropriately choose the interval greater than one second, so that for a certain number of marks, the series extends over a greater number of fluctuations.

For the computation of the Dutch observations we have therefore chosen an interval of 2 seconds. If the fluctuations were considerable a series of 20 marks was taken and otherwise less, for instance 10 marks. In order to get an impression of the uncertainty in the passage-time resulting from the fluctuations of the marks, we did not take at once the mean of all the marks, but we combined first Nr. 1 with Nr. 20, then Nr. 2 with Nr. 19 and so on. The comparison of these values gives a good idea of the effect of the fluctuations. In order to obtain a figure, we can consider



these values as independent observations of the mean time and determine the mean error of the mean of these values. In reality this mean value has a smaller mean error, as a more careful consideration shows, so that we may consider the figure which has thus been found as a maximum value.

In this way we get the following computation. We do not measure x, as this would imply the drawing of an Y axis in the record, but the distance AC, which will be indicated by u; CB could of course also have been chosen. We afterwards subtract $AO = a$.

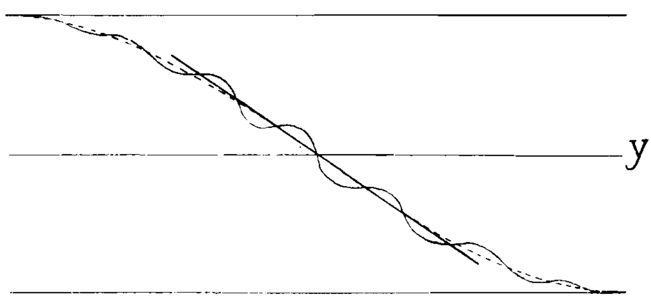
No.	time 2^{m}	u	No.	time 2^{m}	u	$^1/_2\,(u_1 + u_{20})$ etc.	dev. p of mean	p^2
1	1^s	9.5	20	39^s	28.9	19.2	— 0.2	0.04
2	3	10.5	19	37	28.0	19.2	— 0.2	0.04
3	5	12.0	18	35	27.0	19.5	+ 0.1	0.01
4	7	13.0	17	33	26.0	19.5	+ 0.1	0.01
5	9	13.8	16	31	24.9	19.4	0.0	0.00
6	11	14.9	15	29	23.8	19.4	0.0	0.00
7	13	15.9	14	27	22.7	19.3	— 0.1	0.01
8	15	17.0	13	25	22.3	19.6	+ 0.2	0.04
9	17	17.6	12	23	21.9	19.8	+ 0.4	0.16
10	19	18.8	11	21	20.1	19.4	0.0	0.00
$2a = 38.00$						19.43	90	0.31
						19.00		0.0034
						$x_m = 0.43$ (for $2^m\ 20^s$)		$\sqrt{\ } = 0.06$

$$\text{passage-time} = 2^m\ 20 - \frac{0.43}{19.4} \times 38 = 2^m\ 19^s.2 \qquad \text{(formula 70)}$$

$$\text{mean error} = 0.06 \times \frac{38}{19.4} = 0^s.1$$

This is the computation of the passage-time of the sidereal time chronometer for the middle record of fig. 7.

We may mention a much simpler way of determining the passage-time and eliminating the effect of the fluctuations. We can draw a straight pencil-line on the record through the central part of the curve of the marks, i. e. we draw as well as possible the tangent to the sine-curve, which it ought to be if there were no fluctuations, in the point $x = 0$. We then make an estimate of the second, which corresponds to the intersection of this line with the Y axis.



This partly graphical way is of course less accurate than the previous method, but with some practice, we may obtain accuracy enough, and it saves time. It is not so easy to judge of the error which is made, unless the periods of the pendulums and of the chronometer are strongly different, so that a number of passages are available in the first and last parts of the observation. In that case we may deduce the mean error by determining all these passages and comparing their intervals.

In the last case this graphical method is surely indicated as we can get enough accuracy because of the number of the passages, but in case the periods of pendulums and chronometer are less different, so that only two passages can be determined at each end of the record, we will have to study the question by applying both methods to some of the records and by comparing the results. In this way we may obtain the mean error, which results for the determination of the pendulum-period. We will require this error to be less than $5 \; 10^{-7}$ sec.

If the graphical method comes up to this requirement, we will certainly adopt it as a greater accuracy is unnecessary in view of the other errors, e. g. the errors of the rate-correction and of the correction for the EÖTVÖS effect. It saves much time.

For computing the mean error m_T of the pendulum-period T, we can make use of the following formula, deduced by means of $(67\,B)$

$$m_T^2 = \frac{2\,\mu^2}{p\,q^2} \times \frac{0.25^2}{c^4} = \frac{\mu^2}{8\,p\,t^2 c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

in which μ = mean error of one passage-time,
p = number of computed passages at the beginning and at the end,
q = number of coincidence-intervals during the observation,
$t = q\,c$ = duration of the observation.

B. Determination of c by deducing the angular velocity N of the pendulum-vector.

In special cases, when there cannot be determined a passage through the Y axis, e. g. because part of the record has gone wrong, we may still determine c if there is a sufficient number of marks clearly visible. We may measure for each of these marks the distances $A\,C$ and CB (see fig. of page 74) and find the corresponding phase-angle φ by applying $(66\,B)$. We take the mean of a group at the beginning, so that we get a phase-angle φ_g' corresponding to a mean time t_g', and we do the same for a group at the end of the observation, so that we get φ_g'' for a time t_g''. The angular velocity N is obviously found by dividing the difference $\varphi_g'' - \varphi_g'$, increased with as many times 360° as the vector has made complete revolutions, by the time-interval $t_g'' - t_g'$, so that we find for c

$$c = \frac{180}{N} = \frac{180\,(t_g'' - t_g')}{\varphi_g'' - \varphi_g' + n \times 360^\circ} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

If possible we will choose the marks in such a way that by taking the mean of the group we will eliminate the effect of eventual errors in the estimate of the ends A and B of the curve. This is obtained when the mean value of $tg\,\frac{1}{2}\,\varphi$ as well as of $cotg\,\frac{1}{2}\,\varphi$ for the whole group is about zero.

§ 2. *Two chronometers.*

If there are two chronometers, the computations can of course be made independently for both chronometers, so that we get the two results for the period of the fictitious pendulum in the same way as in the previous paragraph. We can however take a shorter way if one of the two chronometers marks sidereal time and the other mean time. We can then determine the moments of coincidence of the chronometers in the beginning and at the end of the observation, and in this way find their difference in rate during the observation; we may safely neglect the fact that the coincidence in the beginning will generally not agree exactly with the mean of the passage-times, which must be considered as the beginning of the computed observation and that an analologous disagreement will occur at the end of the observation, so that the observation interval will not exactly coincide with the interval for which the difference of the rate is determined.

The moment of coincidence of the chronometers may be found in the following way by determining the intersection of the two mark-curves. One of the shutters, which cause the chronometer marks, is provided with a narrow slit, so that the corresponding marks show a black spot in the middle. As a first approximation we may say that the second (or the mean of the seconds) for which this black spot is obscured by the mark of the second chronometer, is the time of coincidence; at this moment both shutters pass together before the slit of the diaphragm. If we wish to have the time of coincidence more accurately, we have to determine the passage of the marks of the second chronometer through the line connecting the black spots, and we may do this in an analogous way as has been described in the previous paragraph for determing the passage-time through the Y axis, by measuring for a series of seconds the distance between the black spot and the central point of the mark of the second chronometer. In this way we get the time of coincidence easily with a mean error not exceeding a tenth of a second. For an observation of half an hour this corresponds to a mean error of the difference of the daily rates of 0.02 sec., which is equivalent to $1\ 10^{-7}$ sec. in the pendulum period. The above approximation leads to a mean error which is about five times greater.

If we find in this way the value Δ for the difference of the daily rate during the observation of the first chronometer minus that of the second chronometer and if Δ_d is the difference of the mean daily rates during the whole interval between the time-signals, we have to add a correction

$$\frac{T}{86400}\,(\Delta - \Delta_d)\ \text{sec.} = 58.1\ (\Delta - \Delta_d)\ 10^{-7}\ \text{sec.} \quad \ldots \quad (73)$$

to the period of the pendulum T given by the first chronometer in order to get the period as it would have been given by the second chronometer, if that had been used for the computation.

§ 3. *The deduction from the record of the correction for deviation from isochronism and of the reduction to infinitely small amplitude in case the record refers to a sea station.*

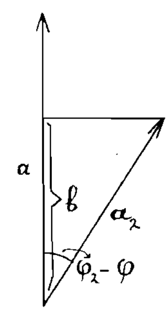
To compute these corrections and reductions we have to apply the formulae 19 A and 21 A or 21 B. For a certain set of two pendulums the quantity U_{21} (see formula 19 C) may be tabulated for different temperatures, while we may assume that the difference $d_2 - d_1$ of the density constants is so small that the variation of the third term of formula 19 C may be neglected because the observations are made at the atmospheric pressure in the ship, so that the differences of the density of the air are restricted between narrow limits.

The amplitude a of the fictitious pendulum can at once be found from the record of this pendulum; a is half of the total breadth of the record.

In the same way the amplitude a_2 of the second pendulum may be deduced from the record of this pendulum, i. e. the lower record on the strip of photographic paper. We have only to keep in mind that the light-beam which causes this record does not make an angle of 30° with the normal of the pendulum-mirror, but an angle of 10°, so that we have to multiply all the vertical dimensions of this record by $\frac{\cos 30^\circ}{\cos 10^\circ} = 0.88$ in order to bring them back to the same scale as that of the vertical dimensions of the record of the fictitious pendulums.

The last quantity which we have to determine is $a_2 \cos (\varphi_2 - \varphi)$. We may find this quantity, which we will indicate by b, in two ways. Firstly we may compute it after determining a_2 and $\varphi_2 - \varphi$ separately and secondly we may determine it directly. As this last way is the quickest we will not consider the first method.

In order to find b directly, we may notice that this quantity is the projection of the pendulum-vector of pendulum No. 2 on the fictitious pendulum-vector. It follows that at the moment that the fictitious pendulum-vector passes through the X axis, b is the X component of the pendulum-vector No. 2. As the X component of a pendulum-vector is identical with the angle of elongation, we may conclude that b is the angle of elongation of pendulum No. 2 at the moment that the fictitious pendulum has its maximum elongation.



This procures the means of measuring b in the record. The X component of the chronometer-mark in the lower record at the moment that the chronometer-mark of the fictitious pendulum-record has its greatest elongation, is the quantity b at that moment.

The record shows two fictitious pendulum-records for each of which the quantity b has to be determined. If the fictitious pendulums are exactly in opposite phase, which is the case if the right initial situation has been successfully realized, the moments of the greatest elongation of the chronometer-marks coincide and the two corresponding quantities b are equal with opposite sign. If the phases are not exactly opposite, we have to measure the quantities b separately.

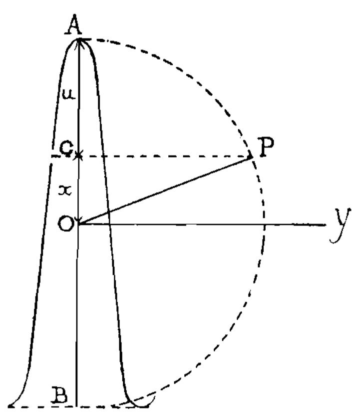
Regarding the phases of the fictitious pendulums we have to keep in mind that for the pair of pendulums Nos. 1 and 2 the angle of elongation $\theta_1 - \theta_2$ of the fictitious pendulum is rightly recorded — this is the middle record of the strip — but that for the pair Nos. 3 and 2 the angle $\theta_3 - \theta_2$ is not recorded but the angle $\theta_2 - \theta_3$ — this is the upper record of the strip — so that we have to reverse the sign and therefore the phase of this record in order to make it comparable to the sign and the phase of the other records.

Instead of measuring in the lower record the distance b of the chronometer-mark to the Y axis of the record, we may also measure $2b$ by taking the distance between the chronometer-mark and the corresponding mark of the second series of light-interruptions; this saves us the trouble of drawing the Y axis of the record. This is of course only correct if this second series is just half way in time between the principal series, because only in this case are the curves of both series exactly symetrical with regard to the Y axis.

b is an irregularly varying quantity, so that it is not sufficient only to measure b in the beginning and at the end of the observation in the same way as this might be done for the regularly varying amplitude of an undisturbed pendulum. We have to measure a series of values for b, regularly distributed throughout the whole observation, for instance at all the moments that the chronometer-mark of the fictitious pendulum has its greatest elongation. This will generally suffice to get the necessary data for a sufficiently accurate mean value of the corrections during the observation. We may refer here to the remarks about this computation on page 24.

Neither in $19A$, nor in $21A$ or $21B$ will it in general do to substitute the mean values for b and a in order to get the mean values of these formulae during the observation; for an exact computation of these mean values it is necessary to compute the values of the formulae for each value of b and to take the mean of these values. If however U_{21} is small, e. g. not above $50\ 10^{-7}$ sec., and if the duration of the observation is not more than half an hour, so that the amplitude a of the fictitious pendulum does not diminish too much during that time, we may simplify the computation of $19A$, i. e. the correction for deviation of isochronism, by substituting the mean values of a and b.

If the lower record is strongly variable, as may for instance be the case for observations at the surface of the sea, it might occur that the number of values for b, which is got in this way, is not sufficient for a computation with the necessary accuracy. In this case we may introduce supplementary determinations of b midway between the values which have been got in the above way, that is to say at the moments that the fictitious pendulum-vector passes through the Y axis. At these moments b is the Y component of the pendulum-vector of pendulum No. 2 and so we may deduce from the figure:



$$b = \sqrt{AC \times CB} \quad \ldots\ldots\ldots\ldots \quad (74)$$

b may therefore easily be computed for these moments.

If it is necessary still to increase the number of values for b, we may determine it at further intermediate moments, for which the formulae can be deduced without difficulty by starting from the definition that b is the projection on the fictituous pendulum-vector of the vector of pendulum No. 2. The X and Y projections of this last vector are given by the x coordinate of the chronometer-mark and by formula (74).

This necessity of computing more values of b will also occur if the pendulum and the chronometer are nearly isochronous, so that the moments of maximum elongation of the chronometer-marks are far apart. We will therefore try to avoid this by taking pendulums of which the period differs enough from that of one of the chronometers, e.g. more than 0.001 sec.

We have seen that for the computation of b it is necessary not only to have a continuous record of the amplitude of the second pendulum but also of its phase, so that we need the chronometer-marks in the curve. To make these marks clearer, which is especially necessary in case the amplitude of pendulum No. 2 is small because in this case the interruptions may become practically invisible, the device for changing the light-interruptions is used; instead of interrupting for a small moment, the interruption is continued from the moment of shutting the electrical current to the moment of opening, i.e. for $\pm$ half a second. This is done during the whole central part of the observation during which no passage-times need be determined. During this same time-interval the velocity of the paper is diminished to one third of its original value in order to save photographic paper. These two alterations change the general appearance of the record from the character given in fig. 7 into that, given by fig. 8.

During each sixtieth second the current is kept opened by the chronometer, which enables us to identify the chronometer-marks. This causes the white vertical lines in the record.

A complete record is given on a lesser scale in fig. 9.

The formulae for the practical computation are found by introducing 65 B and the factor 0.88 in 19 A and 21 B. Supposing $T = 0.502$ we get

Deviation of isochronism: $$\delta T = -0.88 \frac{2b}{2a} U_{21} \, 10^{-7} \text{ sec.} \quad \ldots\ldots \quad (75)$$

Amplitude: $$\delta T = +2.12 A \, 10^{-7} \text{ sec.} \quad \ldots\ldots \quad (76A)$$

with: $$A = (2a + 1.32 \times 2b)^2 + 0.774 \left[(2a_2)^2 - \frac{1}{4} (2b)^2 \right] \quad (76B)$$

which leads to the following example of a complete computation (computation of the record represented by figs. 7, 8 and 9). The sign of $2b$ in the third column refers to the pair Nos 1 and 2; for the pair Nos 3 and 2 this quantity has the same value but with opposite sign.

In taking the mean, half weight has been given to the first and last line.

No.	$2a_2$	$2b$	$2a_{12}$	$(2a_{12}+1.32\times 2b)^2$	$2a_{32}$	$(2a_{32}-1.32\times 2b)^2$	$(2a_2)^2$	$(2b)^2$
1	0.8	—0.6	3.8	9.1	3.9	22.0	0.6	0.4
$\frac{1}{2}\times$ 1		—0.3	1.9	4.6	2.0	11.0	0.3	0.2
2	1.1	—0.6	3.7	8.5	3.9	22.0	1.2	0.4
3	1.2	—0.2	3.7	10.9	3.9	18.5	1.4	0.1
4	1.4	—1.0	3.6	5.2	3.8	26.2	2.0	1.0
5	1.4	—0.9	3.6	5.8	3.7	23.9	2.0	0.8
6	1.2	—1.1	3.5	4.2	3.7	26.5	1.4	1.2
7	1.3	—1.3	3.5	3.2	3.6	28.3	1.7	1.7
8	1.5	—1.4	3.4	2.4	3.6	29.7	2.2	2.0
9	1.2	—0.9	3.4	4.9	3.5	22.0	1.4	0.8
10	0.9	—0.8	3.3	5.0	3.5	20.8	0.8	0.6
11	0.8	—0.6	3.3	6.3	3.4	17.6	0.6	0.4
12	0.8	—0.6	3.2	5.8	3.4	17.6	0.6	0.4
13	1.1	—1.1	3.2	3.1	3.3	22.6	1.2	1.2
14	1.0	—1.0	3.2	3.5	3.3	21.3	1.0	1.0
$\frac{1}{2}\times$15		—0.4	1.6	2.1	1.6	9.5	0.8	0.3
15	1.3	—0.8	3.1	4.2	3.3	19.0	1.7	0.6
Mean:		—0.87	3.44	22.68	3.59	5.39	1.33	0.86
				0.86		0.86	—0.22	
				$A_{12} = 23.54$		$A_{32} = 6.25$	$\frac{1.11}{0.86}\times 0.774$	

For the pair 12: factor of $U_{12} = +0.88 \times \dfrac{0.87}{3.44} = +0.223$.

Amplit. corr. $\delta T = 2.12 \times 23.54 = 50 \quad 10^{-7}$ sec.

For the pair 32: factor of $U_{32} = -0.88 \times \dfrac{0.87}{3.59} = -0.213$.

Amplit. corr. $\delta T = 2.12 \times 6.25 = 13 \quad 10^{-7}$ sec.

§ 4. *The deduction from the record of the correction for deviation of isochronism and of the reduction to infinitely small amplitude in case the record refers to a land station.*

In this case the record of pendulum No. 2 is nearly regular if the initial situation does not differ too much from those treated in Chapter I, § 11. The small fluctuations of this pendulum show two periods; a long one with a half-period

$$T_q = \frac{T^2}{2\,s}$$

which will generally be so great that the variations are practically linear during the duration of the observation, and a shorter one with a half-period

$$T_p = \frac{T^2}{3\,u}$$

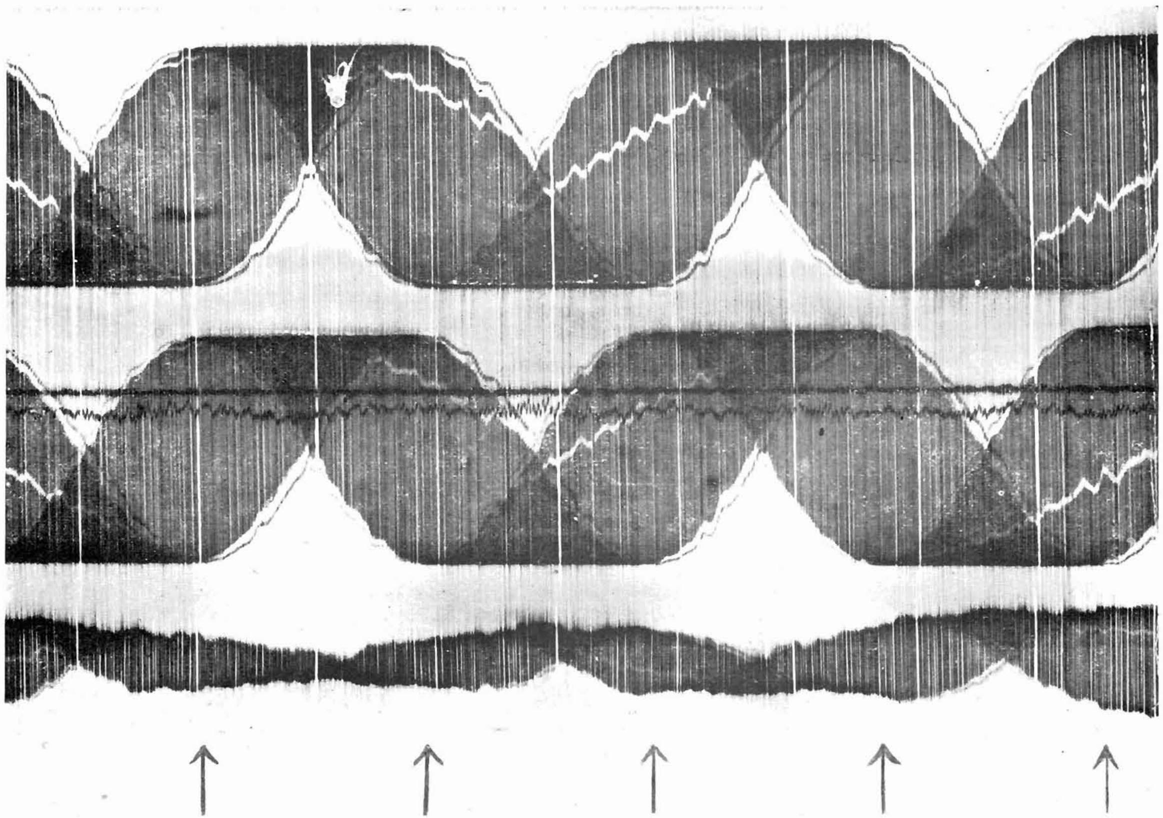


Fig. 8. Part of the record of fig. 9.

Central part of the record on true scale; paper-velocity one third of fig. 7; light-interruption changed. The arrows indicate the moments where b is measured.

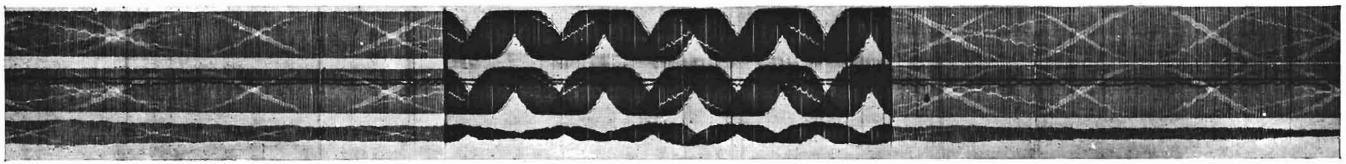


Fig. 9. Complete Oceanic record.

Scale about 1 : 5. The record reads from right to left; fig. 7 is part of the beginning, fig. 8 of the central part.

(u = the sway-correction, which is supposed to be roughly known, so that T_p is also known; see page 68).

If the quantities a_2 and b are determined at the beginning of the observation for two moments which are a time T_p apart, the mean values of a_2 and b are free from this last fluctuation. The same may be done at the end of the observation to get the final values of a_2 and b. For the way to determine a_2 and b or rather $2a_2$ and $2b$ from the record, we may refer to the previous paragraph.

We may now use these initial and final values of $2\,a_2$ and $2\,b$ to compute the formula (75) for the correction for deviation of isochronism at the beginning and at the end of the observation; the mean of these results will not much differ from the result which would have been found by means of a great many intermediate determinations of $2\,a_2$ and $2\ b$.

In the same way we may proceed to determine the mean reduction to infinitely small amplitude; we compute the values of formula (76) at the beginning and at the end of the observation and take the mean of those values. It is however desirable to apply a correction in order to take into account the logarithmic decrease of the quantities a, a_2 and b, caused by the damping. This correction may be found in the same way as for any ordinary amplitude reduction. If A_b and A_e are the values of the quantity A of formula (76) at the beginning and at the end of the observation and A_m the mean of those two values, we get for the mean value A during the whole duration

$$A = A_m \left[1 - \frac{1}{12} \left(\frac{A_b - A_e}{A_m} \right)^2 \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which terms of higher order are neglected. With this restriction this formula is valid for all quantities which decrease logarithmically.

So we may conclude that for land observations we do not need intermediate determinations of a, a_2 and b, even for observations of several hours. A continuous record is therefore unnecessary.

Example of computation:

	$2\,a_2$	$2\,b$	$2\,a_{12}$	$2\,a_{32}$	Pair 12	Pair 32	Factor of U
	0.03	0.00	3.43	3.48			
	0.23	—0.04	2.92	2.96			
Mean (begin)	0.13	—0.02	3.18	3.22	$A_b = 9.93$	$A_b = 10.58$	—0.006
	0.16	+0.08	2.60	2.66			
	0.14	+0.10	2.22	2.26			
Mean (end)	0.15	+0.09	2.41	2.46	$A_e = 6.42$	$A_e = 5.50$	+0.032
					$A_m = 8.18$	$A_m = 8.04$	+0.013
					$A_b - A_e = 3.51$	$A_b - A_e = 5.08$	

For the pair 12: factor of $U_{12} = + 0.013$,
corr. for amplit.: $\delta T = 2.12 \times 8.18 (1 - 0.015) = 17.0 \; 10^{-7}$ sec.
For the pair 32: factor of $U_{32} = - 0.013$,
corr. for amplit.: $\delta T = 2.12 \times 8.04 (1 - 0.033) = 16.4 \; 10^{-7}$ sec.

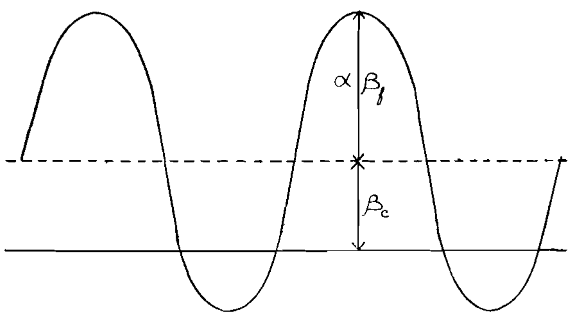
§ 5. *The complete reduction of the observed period. Check of the pendulums.*

Of the formulae of Chapter I, § 8, for the reductions of the observed period, we have already treated in the previous paragraph the formulae for the reduction to infinitely small amplitude and for deviation of isochronism: (75) and (76). Of the other formulae the first two for the reduction to zero temperature and zero air-density do not present any special features; they are ready for use, when the constants are known.

For the formula (17 *A*) for the correction corresponding to the angular movement of the swinging-plane, we may write:

$$\delta T = 1.02 \; (\beta_c^2 + C \alpha^2) \; 10^{-7} \text{ sec.} \quad . \; . \; . \; . \; . \; . \quad (78\,A)$$

in which β_c and α are deduced from the curve made by the damped pendulum; β_c is the constant part of the deviation and α the amplitude of the fluctuating part; both are expressed in m.m. For β_c^2 and α^2 we have of course to take the mean value for the whole duration.



The coefficient of (78 *A*) is computed for the pendulum-period $T = 0.502$ sec. In order to find the above figure we may notice that the beam of light-rays which reflects on the mirror of the damped pendulum, gets an angular deviation equal to the angular deviation of the swinging-plane.

The coefficient C depends on the half-period T_f of the gimbal suspension; it is given by (17 *B*)

$$C = 0.5 - \frac{0.25}{T_f^2} \quad . \; . \; . \; . \; . \; . \; . \quad (78\,B)$$

Lastly we may introduce $T = 0.502$ sec. in formula (22) for the the correction for the rate of the chronometer and we find

$$T = 0.581 \; r \; 10^{-7} \text{ sec.} \quad (r = \text{daily rate}) \; . \; . \; . \; . \quad (79)$$

The practical execution of the computation does not present any difficulties.

Example of the complete reduction for a sea-station (record of fig. 9, made on Feb. 11th, 1927, in the Indian Ocean south of Java):

Temp.	Bar.	Pair Nos. 2 and 3.	Pair Nos. 1 and 2.	Diff. V	Mean period.
30.81	778.5	+ 1442	+ 1405		
— 0.18	— 6.2	+ 607	+ 600		
30.63	— 10.2	+ 50	+ 13		
	762.1	+ 3	+ 3		
		— 2	+ 5		
	0.902	+ 154	+ 154		
		2254	2180		
		0.5022326	0.5022290		
		0.5020072	0.5020110	38	0.5020091
					0.5020088
					0.5020090
					0.5012506
					V_T = 7584

Temp.	Bar.	Pair Nos. 2 and 3.	Pair Nos. 1 and 2.	Diff. of pairs.	Mean period.
Thermometer	Barometer	Temp. corr.	Temp. corr.		
Therm. corr.	Bar. corr.	Density corr.	Density corr.		
Temperature	Corr. for Hygr.	Arc corr.	Arc corr.		
	Barometer	Corr. for β	Corr. for β		
		Corr. for dev. from isochron.	Corr. for dev. from isochron.		
	Air-density	Corr. for rate	Corr. for rate		
		Sum of corr.	Sum of corr.		
		Obs. period.	Obs. period.		
		Red. period.	Red. period.	V	Mean period acc. to chronometer I.
					Mean period acc. to chronometer II.
					Mean period.
					Mean period at central base station.
					Difference V_T.

The reduced periods of the fictitious pendulums, which are thus found, are the periods T_3 and T_1 of the outer pendulums, i. e., the mean period is $\frac{1}{2}(T_1 + T_3)$ and the difference V is $T_1 - T_3$. The mean period is used for the computation of the gravity (par. 7 of this chapter). The values of V provide a means of controlling the stability of the pendulums but we have to realize that, because of the introduction of U_{21} and U_{23} in the computations, V is not only dependent on the outer pendulums, but also on the middle one. If Δ_1, Δ_2 and Δ_3 are the variations of the three pendulums and if c_{21} and c_{23} are the factors with which U_{21} and U_{23} are multiplied to get the correction for deviation from isochronism in the above computation (formula 75), we get for the variation Δ of V (see page 23):

$$\Delta = \Delta_1 - \Delta_3 + c_{21}(\Delta_2 - \Delta_1) - c_{23}(\Delta_2 - \Delta_3) =$$
$$= \left[1 - \tfrac{1}{2}(c_{21} + c_{23})\right](\Delta_1 - \Delta_3) + (c_{21} - c_{23})\left[\Delta_2 - \tfrac{1}{2}(\Delta_1 + \Delta_3)\right].$$

If V has continually a constant value, i. e., $\Delta = 0$, we may conclude that the pendulums are stable; otherwise we will have to apply this formula for finding out the culprit. For the ordinary land observations with the initial conditions $a_1 = a_3$, $a_2 = 0$, $\varphi_1 = \varphi_3 + \pi$, c_{21} and c_{23} are very small, so that practically Δ equals $\Delta_1 - \Delta_3$, that is to say that one or both of the outer pendulums are responsible. For the ordinary sea observations, made wlth the same initial conditions, this is only true of the disturbances have been so small that the middle pendulum has got no big amplitude, in other words, if the factors c_{21} and c_{23} are small. Otherwise, we see that Δ may likewise be caused by variations of the middle pendulum. As the factors c_{21} and c_{23} will vary irregularly for different observations, we will soon be able to decide which pendulum is causing the variations of V.

The land observations with the initial conditions $a_1 = 2a_2$, $a_3 = 3a_2$, $\varphi_1 = \varphi_2 = \varphi_3 + \pi$, are specially made for controlling the middle pendulum: the factors c_{21} and c_{23} are about -1 and $+0{,}25$, so that the factor of $\Delta_2 - \frac{1}{2}(\Delta_1 + \Delta_3)$ in the above formula is more than 1 (see also page 23 a. f. of the first chapter). Substituting in this formula for $\Delta_1 - \Delta_3$ the value of the variation Δ given by the other observations of the series, the value of $\Delta_2 - \frac{1}{2}(\Delta_1 + \Delta_3)$ may be computed.

For these last observations the mean result $\frac{1}{2}(T_1 + T_3)$ is better deduced in a modified way in order to keep it free from the effect of eventual variations of the middle pendulum. If T_{12} and T_{32} are the reduced periods of the fictitious pendulums, we may find $\frac{1}{2}(T_1 + T_3)$ in the following way, corresponding to the method of computation given on page 28:

$$\tfrac{1}{2}(T_1 + T_3) = \frac{c_{23}}{c_{23} - c_{21}} T_{12} - \frac{c_{21}}{c_{23} - c_{21}} T_{32} - \frac{c_{21} + c_{23}}{2(c_{23} - c_{21})} V.$$

For V is substituted the value given by the other observations of the series.

This same method for computing $\frac{1}{2}(T_1 + T_3)$ may also be used for sea observations, if there is any reason specially to mistrust the middle pendulum and if c_{21} and c_{23} are not small. Otherwise, the above way of simply taking the mean is preferable as this does not introduce the uncertainty of estimating V.

§ 6. *Appendix to the former paragraphs.*
Short summary of the methods of computation of the results given by the ordinary four pendulum Sterneck apparatus.

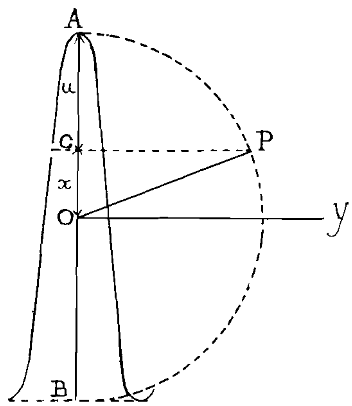
If the old type of apparatus is used, i. e. the four pendulum Sterneck apparatus, the pendulums are recorded separately, so that the methods of computation have to be different. We will give a short summary of them.

I. *The deduction from the record of the periods of the fictitious pendulums.*

The two pendulums, from which the fictitious pendulum is derived, being recorded separately, we may deduce from those records their pendulum-vectors. The difference of these vectors gives the pendulum-vector of the fictitious pendulum, so that we may determine in this way its position. By doing this at the beginning and at the end of the observation we can derive its angular velocity which leads immediately to the period of the fictitious pendulum.

The position of the fictitious pendulum-vector, corresponding to a certain chronometer-interruption, is found by determining its X and Y components, which are given by the difference of the X and Y components of the two original pendulum-vectors. The X components of these vectors are simply the X coordinates OC of the chronometer-marks and the Y components CP are given by:

$$CP = \sqrt{AC \times CB}.$$



In this way the position of the fictitious pendulum may be found for as many seconds at the beginning and at the end of the observation as we wish to take, so that we may find its angular velocity with the desired accuracy. Referring to § 1 of this chapter, we need not add here that it is necessary to compute it for several seconds in order to lessen the effect of the vertical accelerations.

The computations take a great deal more time than for the observations with the new three pendulum apparatus.

II. *The deduction from the record of the correction for deviation of isochronism and of the reduction to infinitely small amplitude.*

For the computation of these corrections the formulae 38, App. II to chapter I, may be applied, which does not present any special difficulty. As $a_2 - a_1$ is an irregularly varying quantity we have to determine it for a great many intermediate moments, regularly distributed throughout the observation, in the same way as has been mentioned in § 3 of this chapter, and by means of these data to determine the mean correction for the whole duration.

If the initial conditions described in App. II to Chapter I, have been successfully realized, the terms with ρ in formulae 38 will generally be negligible.

If each pendulum is recorded by means of the fixed system of mirrors and prisms of the Sterneck apparatus, we see easily that we do not record simply the angle of elongation θ of the pendulum, but the sum of θ and the angle of deviation α of the apparatus about an axis perpendicular to the swinging-plane. The axis of the photographic record shows therefore fluctuations representing this angle α.

As the two pairs of pendulums of the apparatus are swinging in two planes which are perpendicular to each other, the angle α recorded by one pair is identical with the angle of deviation β of the swinging-plane of the second pair about a horizontal axis in this plane, of which the effect has been determined in § 4, Chapter I (formula 17). So we see that it is not necessary, in order to be able to determine the correction corresponding to this effect, to make a separate record of β; β is given mutually by the record of the other pair. For the computation of this correction we may refer to § 5 of this chapter.

A few further details about the computations for the observations, made with this apparatus, are given in „Observation de pendule sur la mer", 1923, publication provisoire.

§ 7. *The computation of the result for the gravity.*

The deduction of the value of gravity does not present any special features. If V_T represents the difference between the mean of the two fictitious pendulum-periods at the station and at the central base-station, and if V_g represents the difference of gravity at both stations, we have

$$V_g = - C\,V_T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80\,A)$$

in which C is a factor given by

$$C = \frac{g_1 (T_1 + T)}{T^2}$$

T = the mean of the periods at the base-station,
T_1 = the mean of the periods at the observation-station,
g_1 = the gravity at the observation-station.

For a certain base-station we may tabulate C or the logarithm of C with the entry V_T.

If we de not wish to use a table for C, an easy way of computing it is for instance given by the following formula, which is derived from the above formula for C

$$C = \frac{2\,g}{T_1 + \frac{1}{2}\,V_T} \qquad (V_T = T_1 - T) \quad . \quad . \quad . \quad . \quad (80\,B)$$

g = the gravity at the base-station.

This formula is approximative, but the approximation is sufficient for all cases.

§ 8. *The reduction of the observed gravity for the East-West component of the ship's velocity (Eötvös-effect).*

Von Eötvös was the first to draw attention to the fact, that the East-West component of the ship's velocity affects the value of the centrifugal acceleration caused by the earth's rotation, so that we have to apply a reduction, in order to find the value of the gravity which would have been found if the ship had had no movement during the observation.

The change of the centrifugal acceleration, which acts in a plane perpendicular to the earth's axis, is

$$\frac{2\,V}{r}\,v$$

in which: r = the radius of the parallel,
V = the East-West velocity, caused by the earth's rotation,
v = the East-West component of the ship's velocity.

To get the component in the direction of the gravity, we have to multiply this disturbance by $\cos\varphi$. The reduction which has to be applied to the observed value of the gravity is therefore

$$d_g = \frac{2\,V}{r}\cos\varphi \times v = \frac{4\,\pi \times 100000}{86400 \times 3600}\cos\varphi \times v = 0.0040\,\cos\varphi \times v \text{ cm.} \qquad (81\,A)$$

in which v is the East-West component of the ship's velocity, expressed in kilometers per hour. If we express v in miles per hour we get

$$d_g = 0.0075\,\cos\varphi \times v \text{ cm.} \quad \ldots\ldots\ldots \quad (81\,B)$$

d_g is to be added to the observed gravity if the ship is going eastward.

The determination of the ship's velocity has been discussed in Chapter II.

§ 9. *The reduction of the observed gravity to the surface of the sea.*

If the gravity observations have been made on board of a submerged submarine, we have to apply two corrections to the observed gravity in order to get the value, which would have been found if the observations had been made at the surface of the water.

The first correction represents the difference of the attraction of the earth below a horizontal plane through the ship, at a point in the ship and at a point at the sea-surface; it is practically identical with the ordinary free-air reduction corresponding to the depth of the ship. The second takes into account the difference of the attraction of the layer of water above the ship; in the ship this attraction acts upwards, at the surface of the sea downwards, so that this correction is double the amount of this attraction.

If the depth of the ship is d and the earth's radius a, the first correction is

$$\delta_g = \frac{2\,g}{a}\,d = 0.308\,d\,10^{-3} \text{ cm. } (d \text{ in meters}) \quad \ldots \quad (82\,A)$$

For the computation of the second correction we assume that the layer of water has an infinite extension in horizontal direction. If the density of sea-water is σ_w its attraction at the surface is

$$2\,\pi\,k^2\,\sigma_w\,d$$

and as the formula for the earth's attraction gives

$$g = \frac{4}{3}\,\pi\,k^2\,\rho\,a$$

in which ρ is the mean density of the earth, we find for the correction

$$\delta_g' = 3 \frac{\sigma_w}{\rho} \frac{d}{a} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82\,B)$$

and by substituting $\sigma_w = 1.03$ and $\rho = 5.52$:

$$\delta_g' = 0.086\, d\; 10^{-3}\ \text{cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (82\,C)$$

If the station is near to important topographical features, this topography will of course have a disturbing effect on the way in which gravity varies with the depth. We may however neglect this effect if the case is not too extreme because d is always small; it will seldom exceed fifty metres.

Putting the formulae for the reduction of the observed gravity together we get

$$\begin{aligned} g_0 &= g + d_g - \delta_g + \delta_g' \\ d_g &= 7.5 \ \cos\varphi \times v \ 10^{-3}\ \text{cm.} \qquad (v = \text{W—E vel. in miles}) \qquad (83) \\ \delta_g &= 0.308\, d \qquad\quad ,, \qquad\qquad (d = \text{ship's depth in metres}) \\ \delta_g' &= 0.086\, d \qquad\quad ,, \end{aligned}$$

From the above deduction it follows that the last reduction is the double of the formula for the ordinary Bouguer reduction.

As we may generally neglect the effect of the deviation of the sea-surface from the geoid by the tides and the atmospheric disturbances, we may consider g_0 to be the value of gravity on the geoid.

§ 10. *The computation of the free-air, the Bouguer, and the isostatic anomalies; reduction of Bowie.*

The anomaly, which is got by subtracting the normal value of the gravity as it is given by one of the standard formulae, from the value g_0 at the sea-surface, will be called the free-air anomaly of the sea-station. This denomination is chosen because this anomaly has the same character as the free-air anomaly of a land-station; it is the anomaly on the geoid, determined without taking away the attraction of either the topography or the compensation.

It may further be desirable to determine the anomaly after taking away the effect of the topography. This may be done in the same way as is done for a land-station; the only difference is that the height of the topography at the station is negative and that its density is likewise negative; it is the density of sea-water minus the mean density of the earth-crust. The principal part of this reduction is as usual the Bouguer reduction, which takes into account the attraction of a layer of infinite horizontal extension, of which the negative height in this case is the depth of the sea at the station while the density is given by the above definition. If the sea-depth is d_s and the mean crust-density σ, the Bouguer reduction is therefore

$$\Delta_g' = \frac{3}{2} \frac{(\sigma - \sigma_w)}{\rho} \frac{g}{a} \times d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84\,A)$$

and by substituting $\sigma = 2.83$, $\sigma_w = 1.03$ and $\rho = 5.52$, we find:

$$\Delta_g' = 0.0754\, d_s\; 10^{-3}\ \text{cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (84\,B)$$

to be added to g_0.

In the same way as is the case for land-stations this Bouguer reduction ought to be supplemented by a topographic reduction, which takes into account the deviation of the real topography from this simplified assumption.

As has often been explained, the most important anomaly is the isostatic anomaly, which is found by not only taking away the effect of the topography, but also of the compensation. The computation of this anomaly for a sea-station according to one of the well-known systems, does not present any features on which we wish to enlarge in this short sketch.

There is however a point to which Dr. WILLIAM BOWIE drew attention and which we wish to mention here. If we reduce for the effect of the topography, this means that we compute the gravity-value which would be observed on an earth of which the topography had been taken away; for the oceans this amounts to filling them up with mass till they have the mean crust-density. This proceeding will however imply a change of the geoid, so that if we wish to determine the gravity-value which we would have found on the geoid of this regulated earth, we will have to apply a supplementary reduction corresponding to the distance from the original geoid to this new geoid.

For the isostatic reduction we have the same question. This reduction means the determination of the gravity on a still more regulated earth, of which not only the topography is taken away but also the isostatic compensation in the earth-crust. The corresponding additions and removal of masses will likewise change the geoid, so that we have again to apply a supplementary reduction in order to reduce the gravity to the new geoid. This reduction is the reduction of Bowie.

Before going into the methods of computation of this reduction, we wish to explain why this reduction, which apparently has nothing to do with the question if we have a sea or a land station, gets a special importance when sea-stations are introduced, while it did not matter formerly when we had only land-stations to compare. This is caused by the fact that the reduction of Bowie has systematically another sign for sea-stations than for land-stations so that, although it is so small that we may neglect it as long as only land-stations are considered, we have to take it into account if we compare land and sea-stations together. Any isostatic reduction ought then to be supplemented by the reduction of Bowie. *)

In his paper: „An approximate rule for the distance between the geoid and the spheroid on the assumption of complete local isostatic compensation of irregularities of mass in the earth's crust", **) WALTER D. LAMBERT gives an approximative formula, which may obviously also be applied to find the distance between the geoids before and after isostatic reduction; it represents the shift of the potential surface from a theoretical geoid to the spheroid, caused by the same process of taking away the topography and compensation of the earth's crust. The formula is

*) In the same way the Bouguer anomaly ought to undergo an analogous reduction, as has been indicated above. We will not go into this subject here.

**) To appear shortly in: „Bulletin Géodésique".

$$h_s = \frac{3}{4}\frac{\sigma}{\rho}\frac{H}{a}(1650 - 0.636\ d_m) \quad . \quad . \quad . \quad . \quad . \quad (85\ A)$$

$$\text{or} \quad h_l = \frac{3}{4}\frac{\sigma}{\rho}\frac{H}{a}(1650 + t_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85\ B)$$

in which h_s = the distance between the geoids at sea,
h_l = " " " " " on land,
H = depth of compensation,
a = earth's radius,
d_m = regional sea-depth expressed in metres,
t_m = regional height of the topography,
σ = mean crust-density,
ρ = mean density of the earth.

By regional depth and regional height are indicated the mean values for the region round the station.

By substituting $H = 100$ km., $\sigma = 2.83$, $\rho = 5.52$, $a = 6370$ km. we find in metres

$$h_s = 10.0 - 3{,}8\ d_m \quad (d_m \text{ in km.}) \quad . \quad . \quad . \quad . \quad . \quad (86\ A)$$

$$h_l = 10.0 + 6{,}0\ t_m \quad (t_m \text{ in km.}) \quad . \quad . \quad . \quad . \quad . \quad (86\ B)$$

The effect of the local topography on the distance h must be estimated separately and added to these values. In the above-mentioned paper Lambert gives a method for doing this. In general the effect is small.

For the Bowie reduction which reduces the gravity-values from the old geoid which is the outer surface of the regulated earth to the new geoid at a distance h below it, we must apply the free air reduction, $0.308\ h$, and we have to take into account the effect of the layer of mass of a thickness h between the geoids. As however this layer of mass, which may be considered as topography with regard to the new geoid, must be supposed to be compensated in the ordinary way by a compensating layer below the new geoid, and as h is small and slowly variable over the geoid, the total effect of both layers is negligible. *)

The Bowie reduction reduces therefore practically to the free air reduction over the distance h. We find thus to be added to g

*) Strictly speaking we ought to reduce for the attraction of all the differences in mass between the actual earth of which we have taken away the topography and the isostatic compensation and the theoretical earth, which would exist if these masses were not present, so that the equilibrium conditions would have given a slightly different shape to all the inside potential surfaces as well as to its geoid. Of these masses, the masses between the two geoids and their compensation are the principal part. For this short approximative investigation we will neglect the very small effect of the other mass-differences as well as the likewise very small discrepancy between the geoid of this theoretical earth according to this last definition and the assumed geoid which we derived from the actual geoid by determining the shift caused by the removal of the topography and the compensation.

for sea-stations: $\Delta_b = 3.1 - 1.2\ d_m\ 10^{-3}$ cm. . . . (87 *A*)
for land-stations: $\Delta_b = 3.1 + 1.8\ t_m\ 10^{-3}$ cm. . . . (87 *B*)
(d_m and t_m in km.)

To give an idea of the size of this reduction, we may mention that this gives + 0.0040 cm. for a land-station with a regional height of 500 metres and — 0.0029 cm. for a sea-station with a regional sea-depth of 5000 metres.

In general the formulae (87) are accurate enough for practical use. Only near to irregular topography a further investigation may be necessary to determine its effect on h in order to see if Δ_b has to be corrected accordingly.