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A CONNECTION BETWEEN GEOMETRIC AND GRAVIMETRIC GEODESY A FIRST SKETCH

by

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PREFACE

The present publication was originally intended to be a contribution to a special publication dedicated to professor ANTONIO MARUSSI on the occasion of his 70th birthday. Owing to personal circumstances, the deadline to submit a contribution could not be met, moreover it appeared that some missing links still required further investigation.
Nevertheless this publication – in spite of possible weak parts or errors – should be seen as a homage to professor MARUSSI, a scientist who by his own work, and by organizing the symposia on mathematical geodesy, has given a strong stimulus to geodetic research.
The idea of the theory presented here goes back to these symposia, and mainly to the presentation and discussion of A. BJERHAMMAR's paper "A General World Geodetic System" at the Second Symposium on Tridimensional Geodesy in Cortina d'Ampezzo, 1962.
It took more than a decade and a half to give shape to the idea. In the years 1964 to 1971 I had the indispensable assistance of my co-worker J. VAN MIERLO. His contribution to the mathematical formulation and to the discussion of results is gratefully acknowledged. I am also greatly indebted to J. E. ALBERDA for elaborating the English version of the manuscript.
The manuscript was finalized in the period July 1977 – July 1978. When preparing it for the press section 1.8 and some notes were added.

December, 1979 W. BAARDA

CONTENTS

1. INTRODUCTORY REMARKS

1.1 Introduction

The sketch given here evolved as a spare time activity from the presentation and discussion of the paper [BJERHAMMER, 1962]. The struggle with the subject concerned the main lines: not all mathematical details have been satisfactorily solved and the theory is not complete, hence the sketchy character of the treatment. The incentive to this investigation came from two sides:

a. In setting up a spatial theory of geometric geodesy the need was felt for a connection with gravimetric {or physical} geodesy that was independent of the classical ellipsoidal approach.
b. Since the lectures by F. A. VENING MEINESZ in 1938 and 1939, the field of physical geodesy has always fascinated me and held my interest. However, as the number of publications on this field grew, the theoretical structure became less and less clear to me. Spherical and non-spherical approximations followed each other in arbitrary order, just as the use of Poisson- and Green integrals. The use of approximate values was somewhat curious, leading, on the one hand, to a kind of physical interpretation such as the "telluroid", and on the other hand to a "fundamental equation of geodesy" which sometimes was a hindrance. Further, the whole theory seems to be due to an "ill-posed problem", although the application of the collocation technique removes this difficulty or leaves it aside. And, finally, why is the purpose of geodesy the determination of a vague concept like the "geoid", and not the determination of the topographical land and sea surface of the earth?

In the course of years I have in developing the present approach become more and more convinced that its main lines have a real significance. Many classical results can be recognized in the new theory; its basic thought is connected with the model theory on which I based the adjustment theory of geometric geodesy. Yet many questions and problems are left open: newly developed measuring processes have not found their place yet, methods of dynamical satellite geodesy have not been sufficiently analysed, the elaboration and interpretation of the relationships found are still somewhat problematic. It is hoped that criticism will provide a check on the results obtained.

Let the following summary precede the theory:
The core of the theory is the connection of

- results of geometric networks
- spirit levelling
- gravity {and vertical gravity gradient} observations

to the third integral identity of Green. The way in which this connection is established is determined by the analysis of measuring processes, leading to a connection via dimensionless compound difference quantities. Linearization of the integral equations necessitates a closer study of Poisson's integral; it appears that effects of Gauss's integral and Poisson's integral cancel each other in the linearized equations. The closed model of approximate values turns out to be of dominating importance; spherical approximation {Poisson's integral} requires an order of mag-

nitude of difference-quantities that is smaller than is now in use for anomalies. This leads to a stronger interaction with geophysical model hypotheses.
The solution of the linearized integral equations leads to Stokes-like integral formulae, the so-called Green integrals, with possibilities for regional application on land and at sea, including some aspects of satellite geodesy.
A guess is made with respect to the background of the inverses of these Stokes-like integral formulae, deduced by MOLODENSKII.
Concepts like the "fundamental equation of physical geodesy", "free air reduction", "geoid" and "height" are critically examined. The sketch concludes with some remarks on collocation theory, leading to a consideration of the choice between the use of collocation or of integral formulae in the present theory.

1.2 Approximate values and geophysics

The choice of good approximate values, forming a closed system, is essential for the theory. The geocentric cartesian coordinate system, usual in geodesy, is adhered to, but its origin P_M is chosen close to the, initially unknown, centre of mass P_C of the earth. This origin P_M will further be denoted as the geocentre, and will also be taken as the centre of a reference ellipsoid as currently used in geodesy.
For points P_i on or above the surface of the earth one then has to assume values for the three coordinates and for the gravity potential, respectively the gravitational potential:

$$X_i^{\text{appr}},\ Y_i^{\text{appr}},\ Z_i^{\text{appr}},\ W_i^{\text{appr}} \text{ respectively } V_i^{\text{appr}} \quad \ldots\ldots\ldots\ldots \quad (1.2.1)^*)$$

The angular velocity of the earth's rotation is assumed to be known.
All other quantities occurring in the theory are given approximate values deduced from (1.2.1). Examples are: the length of rays r_i from the geocentre to point P_i, the gravity g_i in point P_i {the magnitude of the gravity vector, the direction of this vector belonging to the domain of geometric geodesy}, etc.

Essential items are:
1. Approximate values are indispensable for the theory, in order to be able to linearize integral equations in such a way that terms which are difficult to compute vanish {compare the introduction of the anomalous potential T in the existing theory [HM, (2–137)]**)}
2. It follows that the theory is concerned with linearized integral equations, or difference equations. These serve as the functional model for the application of adjustment processes. Consequently, the difference equations will have to be applicable both as relations between correction quantities and as relations between difference quantities:

*) The notation "appr" will be omitted in coefficients of difference equations if there is no danger of confusion.

**)HM is the abbreviation which will be used for the excellent book [HEISKANEN and MORITZ, 1967], to which the treatment will be connected as much as possible.

$$\Delta x = x - x^{\text{appr}} \quad \ldots\ldots\ldots \quad (1.2.2)$$

This is in contrast with geometric geodesy, where the misclosures of condition equations are computed from the non-linearized relations from the functional model. The quantities x frequently are quotient-quantities such as r_i/r_j and g_i/g_j in pairs of points P_i, P_j on the surface of the earth, with standard deviations:

$$\sigma_x \simeq 10^{-6} - 10^{-8}$$

In this case we have for the correction-quantities:

$$\epsilon_x \simeq 10^{-6} - 10^{-8}$$

Hence for the difference-quantities:

$$\Delta x \gtrsim 10^{-6} - 10^{-8} \quad \ldots\ldots\ldots \quad (1.2.3)$$

This requires a very close approximation of x^{appr} to x. Then it is not sufficient any more to choose for W^{appr} the potential of the normal gravity field, but one will have to take into account the influence or the mass distribution in the earth, locally and regionally. This goes further than the computation of isostatic and other corrections in current use.

3. The requirement mentioned in item 2 is even more stringent if in the coefficients of difference equations the earth is assumed to be a homogeneous sphere. This makes it possible to apply Poisson integrals in interaction with difference-expansions in spherical harmonics up to the surface of the earth. In this situation the earth can indeed be considered as a sphere, which follows from the fact that for pairs of points P_i, P_j on the surface the following appears to hold:

$$\left| \frac{g_i}{g_j} - 1 \right| \simeq 2 \left| \frac{r_i}{r_j} - 1 \right| < 0.01 \quad \ldots\ldots\ldots \quad (1.2.4)$$

so that the number of significant digits of Δx-quantities, computed from difference relations in the present theory, can be put at 2 or 3, in agreement with (1.2.3).

4. Consequently, in geodesy one is in two ways faced with geophysical problems. *In the first place* the determination or the checking of crustal movements, as following from the theory of plate tectonics. This theory makes it possible to formulate alternative hypotheses with respect to the adjustment model of two or more geodetic measurements executed at different times [BAARDA, 1975]. Testing of these alternative hypotheses with respect to the adjustment model as null hypothesis is done by methods of mathematical statistics. *In the second place*, one is now confronted with a second type of alternative hypotheses, namely hypotheses describing the influence of local, regional and global mass distribution on W^{appr}. Whereas the first type of alternative hypotheses relates to the measured quantities themselves, the second type relates to the approximate values of measured quantities. In section 7.2, a possibility for testing this second type of hypotheses will be reverted to.

1.3 The computing model to be used in geometric geodesy.

Geometric networks are built up from measurements of directions, distance-measures*), pseudo-azimuths*) and pseudo-distances*) {also from geometric satellite techniques}, astronomical orientation measurements of latitude, longitude and azimuth, etc.
The elimination of local or regional length-scale factors and orientations is obtained by constructing the computing model from relations between differences of directions and/or pseudo-azimuths and between ratios of distance-measures and/or pseudo-distances. Consistent use of this type of length ratios means that the computing model must be based on a division algebra without zero divisors. For the spatial three dimensional situation the only available algebra is then quaternion algebra [BAARDA 1973, 1975].
We cannot here go into the adjustment and the computation of coordinates in the projected geodetic quaternion theory. But the coordinate definition will be discussed: Choose approximate coordinate values for all network points in the usual cartesian X,Y,Z-system of geodesy. The direction of the Z-axis is as near as possible parallel to "the" axis of rotation of the earth {a deviation has only a second-order effect on the following theory}. The origin {geocentre} is the fictitious point P_M, at most some tens of metres from the still unknown centre of mass of the earth P_C {a first-order effect in the following theory}. Choose two base points P_k, P_m of "the first type" and one base point P_n of "the second type" [BAARDA, 1975]. Now apply a similarity transformation to the computed set of coordinates of network points, in such a way that:

$$\begin{pmatrix} X_k \\ Y_k \\ Z_k \end{pmatrix} = \begin{pmatrix} X_k^{\text{appr}} \\ Y_k^{\text{appr}} \\ Z_k^{\text{appr}} \end{pmatrix}, \quad \begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix} = \begin{pmatrix} X_m^{\text{appr}} \\ Y_m^{\text{appr}} \\ Z_m^{\text{appr}} \end{pmatrix},$$

$$\{X_n, Y_n, Z_n\} \text{ in reference plane through } \left\{ \begin{matrix} X_k^{\text{appr}}, & Y_k^{\text{appr}}, & Z_k^{\text{appr}}, \\ X_m^{\text{appr}}, & Y_m^{\text{appr}}, & Z_m^{\text{appr}}, \\ X_n^{\text{appr}}, & Y_n^{\text{appr}}, & Z_n^{\text{appr}}. \end{matrix} \right\}$$

The transformed coordinates obtained are then defined in an $S_{k,m,n}$ system, with rank deficiency 7 for their covariance matrix**). See figure 1.3-1. The coordinate system thus defined is evidently fixed by the points P_k, P_m, P_n; length-scale factor, orientation and the coordinates of the origin of the coordinate system are non-stochastic. It is therefore very simple to derive the stochastic properties of the rays r_i and the ratios r_i/r_j. Length ratios are indeed invariant with respect to a similarity transformation and therefore independent of the coordinate definition chosen.

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*) For these quantities, see [BAARDA 1973, 1977]

**) Since the completion of the manuscript the meaning of such a coordinate system, which is based on spatial S-transformations, has become clearer, also what the new spatial geodetic techniques concerns. In addition J. VAN MIERLO succeeded in establishing a link with ideas of A. BJERHAMMAR, G. BLAHA, E. W. GRAFAREND, P. MEISSL, H. PELZER and A. J. POPE. Reference may be made to author's paper: "Mathematical geodesy in relation to the Netherlands Geodetic Commission" included in the memorial volume issued on the occasion of the 100th anniversary of this Commission, entitled "The Centenary of the Netherlands Geodetic Commission", Delft, 1979.

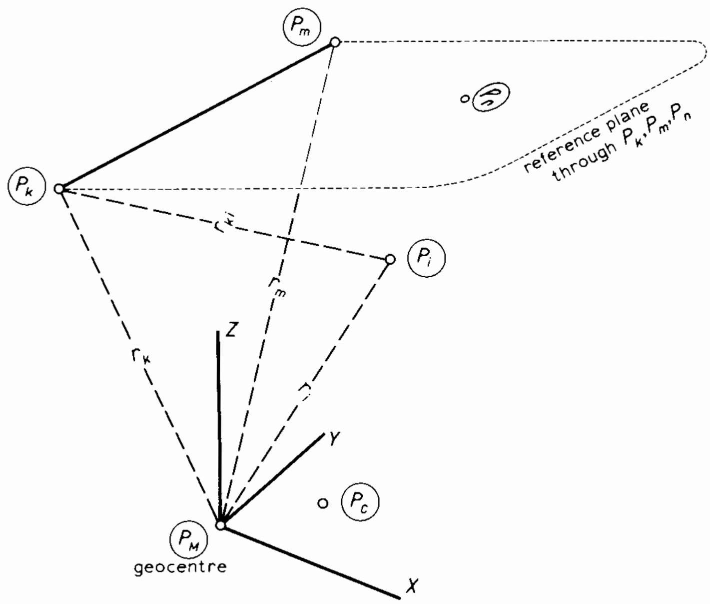


Fig. 1.3-1

The choice of the coordinate system takes account of the significance of the centre of mass and the rotation axis of the earth, both in classical theory and in the theory presented here. This is mainly connected with the centrifugal potential and the expansion in spherical harmonics of the gravitational potential of the earth, and quantities derived therefrom.
In quaternion theory all line elements are directed. For example, r_i is in fact the abbreviation of r_{Mi}, the magnitude of the vector $\overrightarrow{P_M P_i}$. The significance of the introduction of r_i is that position fixing on the earth by the methods of geometric geodesy is weaker {has a lower precision} in the direction normal to the surface of the earth than in a direction along this surface. In this way, gravimetric geodesy, by correcting the values r_i, can contribute to a better homogeneity in position fixing.

1.4 Notations

X_i, Y_i, Z_i	cartesian coordinates of point P_i in a geocentric coordinate system, with origin {geocentre} P_M near the centre of mass of the earth P_C:
$X_i^{\text{appr}}, Y_i^{\text{appr}}, Z_i^{\text{appr}}$	approximate values of X_i, Y_i, Z_i
X_{ij}	$X_j - X_i$; $X_{k,k+1} = X_{k+1} - X_k$
ΔX_i, etc	$X_i - X_i^{\text{appr}}$, etc.
$\underset{k,m;n}{\text{S}}$	S-system with base points of the first type P_k , P_m and base point of the second type P_n ; definition of coordinates X, Y, Z
r_i	magnitude of $\overrightarrow{P_M P_i}$ {see Figure 1.7-1}
r_{ij}	magnitude of $\overrightarrow{P_i P_j}$
S	surface of the earth

S^*	geosurface {see section 1.7}
$\overline{S}$	spherical approximation of S^*, with radius R and gravity G
M	mass of the earth including the gravitational constant k ; $M = k\bar{M}$, $\bar{M}$ mass of the earth as used in textbooks
W_i	gravity potential in P_i
V_i	gravitational potential in P_i
$\frac{1}{2}\omega^2(X_i^2+Y_i^2)$	centrifugal potential in P_i; $W_i = V_i + \frac{1}{2}\omega^2(X_i^2 + Y_i^2)$
Φ_i	$-r_i \dfrac{\partial V_i}{\partial r_i}$
Ψ_i	$r_i^2 \dfrac{\partial^2 V_i}{\partial r_i^2}$
∇^2	$\dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial Y^2} + \dfrac{\partial^2}{\partial Z^2}$
Ω_j	solid angle; $d\Omega_j = \sin\theta_j \, d\theta_j \, d\lambda_j$
n_j	outer normal to surface S in P_j
h_{ij}	height difference along plumb line between P_j and P_i
g_i	$-\dfrac{\partial W_i}{\partial h_i}$, magnitude of gravity vector in P_i
q_i	$\dfrac{\partial^2 W_i}{\partial h_i^2} = -\dfrac{\partial g_i}{\partial h_i}$
δ_{ij}	$\dfrac{r_j^2 - r_i^2}{r_{ij}^2}$; $(-\delta_{ij})\dfrac{r_j}{r_{ij}}$ Poisson's kernel [KRARUP, 1969, p. 43]
$S_{0i;j}^{(n-1),\mathrm{etc}}$	Stokes'-type function or kernel, sum of spherical harmonics:
$P_{0i;j}$	Poisson's-type function or kernel

Abbreviated notation of series of spherical harmonics:

According to [HM, (1–83′)] we have for $r_i > r_j$:

$$\frac{r_j}{r_{ij}} = \sum_{n=0}^{\infty} \cdot \sum_{m=0}^{n} \left(\frac{r_j}{r_i}\right)^{n+1} \left\{ \frac{\overline{R}_{nm}(\theta_i,\lambda_i)}{\sqrt{2n+1}} \cdot \frac{\overline{R}_{nm}(\theta_j,\lambda_j)}{\sqrt{2n+1}} + \frac{\overline{S}_{nm}(\theta_i,\lambda_i)}{\sqrt{2n+1}} \cdot \frac{\overline{S}_{nm}(\theta_j,\lambda_j)}{\sqrt{2n+1}} \right\} \tag{1.4.1}$$

with [HM, (1–74)] only the following relations $\neq 0$:

$$\frac{1}{4\pi} \iint_{\Omega_j} \left\{ \frac{\overline{R}_{nm}(\theta_j,\lambda_j)}{\sqrt{2n+1}} \right\}^2 d\Omega_j = \frac{1}{4\pi} \iint_{\Omega_j} \left\{ \frac{\overline{S}_{nm}(\theta_j,\lambda_j)}{\sqrt{2n+1}} \right\}^2 d\Omega_j = \frac{1}{2n+1} \tag{1.4.2}$$

Then it proves to be possible to use the following abbreviated notation:

$$\frac{r_j}{r_{ij}} = \sum_{n=0}^{\infty} \left(\frac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)} \,, r_i > r_j \quad \ldots \ldots \ldots \ldots \quad (1.4.3)$$

$$\frac{1}{4\pi} \iint Y_j'^{(n)} \, Y_j'^{(\bar{n})} \, d\Omega_j \begin{cases} = \dfrac{1}{2n+1} & , \bar{n} = n \\ = 0 & , \bar{n} \neq n \end{cases} \quad \ldots \ldots \ldots \ldots \quad (1.4.4)$$

If so desired, one can check all derivations executed with (1.4.3) and (1.4.4) by using (1.4.1) and (1.4.2). Of course, formulae like (1.4.3) have to be translated back to (1.4.1).

For derived notations $Y_i^{(n)}$ and $\Delta Y_i^{(n)}$, see section 3.1.

1.5 The function $\frac{1}{r_{ij}}$

$$r_{ij}^2 = r_i^2 + r_j^2 - 2 r_i r_j \cos(r_i, r_j) \qquad , \delta_{ij} = \frac{r_j^2 - r_i^2}{r_{ij}^2}$$

$$\frac{\partial}{\partial r_j} \left(\frac{1}{r_{ij}}\right) = - \frac{1}{r_j r_{ij}} \, \frac{1 + \delta_{ij}}{2}$$

$$\frac{\partial}{\partial r_i} \left(\frac{1}{r_{ij}}\right) = - \frac{1}{r_i r_{ij}} \, \frac{1 - \delta_{ij}}{2}$$

$$\frac{\partial^2}{\partial r_i \partial r_j} \left(\frac{1}{r_{ij}}\right) = \frac{1}{r_i r_j r_{ij}} \left[\frac{1}{2} \left\{ \left(\frac{r_i}{r_{ij}}\right)^2 + \left(\frac{r_j}{r_{ij}}\right)^2 - 1 \right\} + 3 \left(\frac{1 + \delta_{ij}}{2}\right) \left(\frac{1 - \delta_{ij}}{2}\right) \right] =$$

$$= \frac{1}{r_i r_j r_{ij}} \left[\left(\frac{r_j}{r_{ij}}\right)^2 + \frac{1 + \delta_{ij}}{2} \, \frac{1 - 3\delta_{ij}}{2} \right]$$

$$\frac{\partial^2}{\partial r_j^2} \left(\frac{1}{r_{ij}}\right) = \frac{1}{r_j^2 r_{ij}} \left[- \left(\frac{r_j}{r_{ij}}\right)^2 + 3 \left(\frac{1 + \delta_{ij}}{2}\right)^2 \right]$$

$$\frac{\partial^2}{\partial r_i^2} \left(\frac{1}{r_{ij}}\right) = \frac{1}{r_i^2 r_{ij}} \left[- \left(\frac{r_i}{r_{ij}}\right)^2 + 3 \left(\frac{1 - \delta_{ij}}{2}\right)^2 \right] \quad \ldots \ldots \ldots \quad (1.5.1)$$

Compare [MOLODENSKII et al, 1962, p. 47] and [PICK et al, 1973, p. 456]. In addition, from (1.5.1) follows:

$$- \left(r_i \frac{\partial}{\partial r_i} + r_j \frac{\partial}{\partial r_j} \right) \left(\frac{1}{r_{ij}}\right) = \frac{1}{r_{ij}}$$

$$\left(r_i \frac{\partial}{\partial r_i} + r_j \frac{\partial}{\partial r_j} \right)^2 \left(\frac{1}{r_{ij}}\right) = \frac{2}{r_{ij}}$$

1.6 Poisson's kernel and related functions

From (1.5.1): $-r_i \dfrac{\partial}{\partial r_i} \left(\dfrac{r_j}{r_{ij}}\right) = \dfrac{1-\delta_{ij}}{2} \dfrac{r_j}{r_{ij}}$

From (1.4.3): $-r_i \dfrac{\partial}{\partial r_i} \left(\dfrac{r_j}{r_{ij}}\right) = \sum\limits_{n=0}^{\infty} (n+1) \left(\dfrac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)}$

Using these relations one can derive:

	$r_i > r_j$	$r_i < r_j$
$\dfrac{r_j}{r_{ij}}$	$\sum\limits_{n=0}^{\infty} \left(\dfrac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)}$	$\sum\limits_{n=0}^{\infty} \left(\dfrac{r_i}{r_j}\right)^{n} Y_i'^{(n)} \, Y_j'^{(n)}$
$-\dfrac{1+\delta_{ij}}{2} \dfrac{r_j}{r_{ij}}$	$\sum\limits_{n=0}^{\infty} n \left(\dfrac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)}$	$-\sum\limits_{n=0}^{\infty} (n+1) \left(\dfrac{r_i}{r_j}\right)^{n} Y_i'^{(n)} \, Y_j'^{(n)}$
$\dfrac{1-\delta_{ij}}{2} \dfrac{r_j}{r_{ij}}$	$\sum\limits_{n=0}^{\infty} (n+1) \left(\dfrac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)}$	$-\sum\limits_{n=0}^{\infty} n \left(\dfrac{r_i}{r_j}\right)^{n} Y_i'^{(n)} \, Y_j'^{(n)}$
$-\delta_{ij} \dfrac{r_j}{r_{ij}}$	$\sum\limits_{n=0}^{\infty} (2n+1) \left(\dfrac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} \, Y_j'^{(n)}$	$-\sum\limits_{n=0}^{\infty} (2n+1) \left(\dfrac{r_i}{r_j}\right)^{(n)} Y_i'^{(n)} \, Y_j'^{(n)}$

(1.6.1)

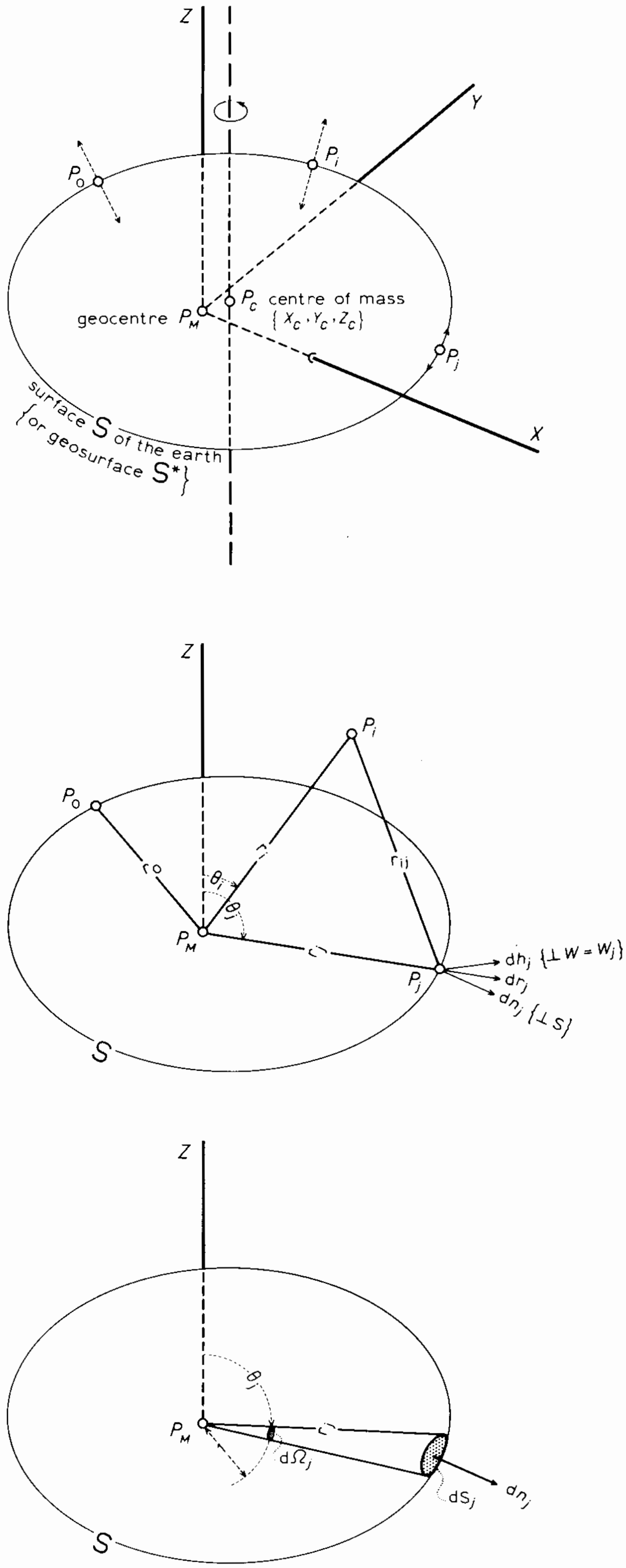
The series can be used in the coefficients of difference-integral equations; in doing so, one may put $r_j \simeq R$ {see section 3.1}.

The series for $\dfrac{r_j}{r_{ij}}$ do not converge any more if $r_i = r_j$; in applying (1.6.1) this limiting case must therefore be treated with caution.

1.7 Surface integrals and surfaces S and S*

With the approximation $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r} \cos(n,r)$ {see section 1.8} Gauss' integral becomes:

$$p = -\iint\limits_{S} \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) dS_j = -\iint\limits_{S} \frac{\partial}{\partial r_j} \left(\frac{1}{r_{ij}}\right) \cos(r_j,n_j) \, dS_j$$



Z
Y
X
P_o
P_i
P_j
P_c centre of mass
$\{X_c, Y_c, Z_c\}$
geocentre P_M
surface S of the earth
{or geosurface S*}
Z
P_o
P_i
P_j
P_M
r_o
r_i
r_j
r_{ij}
θ_i
θ_j
dh_j $\{\perp W = W_j\}$
dr_j
dn_j $\{\perp S\}$
S
Z
P_M
θ_j
r_j
$d\Omega_j$
dS_j
dn_j
S

Fig. 1.7-1

With (1.5.1), see figure 1.7-1, the solid angle being $d\Omega_j$:

$$p = \iint\limits_S \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} \frac{\cos(r_j, n_j)}{r_j^2} dS_j = \iint\limits_\Omega \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} d\Omega_j$$

With [HM, p. 12] or [PICK et al, 1973, p. 460]:

$$p = -\iint\limits_S \frac{\partial}{\partial n_j}\left(\frac{1}{r_{ij}}\right) dS_j = \iint\limits_\Omega \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} d\Omega_j \begin{cases} = 0, P_i \text{ outside S} \\ = 2\pi, P_i \text{ on S} \\ = 4\pi, P_i \text{ inside S} \end{cases} \quad (1.7.1)$$

In Gauss' integral (1.7.1) and other integrals the problem arises how S, being the surface of the earth and hence the boundary, must be defined.
In this connection, we first consider Green's {third} integral formula [HM, (1–29)] for a continuous and finite function V in the spatial region v, inside and on the surface S {p from (1.7.1)}:

$$\iiint\limits_{\text{earth}} \frac{1}{r_{ik}} (-\nabla^2 V_k)\, dv_k - pV_i = \iint\limits_S \left[V_j \frac{\partial}{\partial n_j}\left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial V_j}{\partial n_j} \right] dS_j \quad . \quad . \quad . \quad . \quad . (1.7.2')$$

If the gravitational potential is V, then with $\nabla^2 V_k \neq 0$ for P_k inside and on S:

$$(4\pi - p)V_i = \iint\limits_S \left[V_j \frac{\partial}{\partial n_j}\left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial V_j}{\partial n_j} \right] dS_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . (1.7.2'')$$

A comparison of (1.7.2) with [HM, (1–29′)] shows that (1.7.2) {with (1.7.1)} can also be interpreted as the {third} Green's integral formula for the harmonic function $V\{\nabla^2 V_k = 0\}$ in the spatial region v', outside and on S {P_k outside or on S}.
Both interpretations of (1.7.2) are only realistic if S in the first interpretation is considered as a part of the earth {a material boundary}, and in the second interpretation as the boundary between the atmosphere and the earth, hence as a part of the atmosphere {an immaterial boundary}. For the further derivations from (1.7.2) for the observational area outside and on the earth, the gravitational potential V can therefore be considered as a harmonic function $\{\nabla^2 V = 0\}$ . . . . (1.7.3)

In (1.7.2), p can be eliminated by means of (1.7.1), so that with $V_{ij} = V_j - V_i$:

$$\boxed{\begin{array}{l} V_i = \dfrac{1}{4\pi} \displaystyle\iint\limits_S \left[V_{ij} \frac{\partial}{\partial n_j}\left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial V_j}{\partial n_j} \right] dS_j \\ \\ P_i \text{ outside, on or inside S}; P_j \text{ on S} \end{array}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . (1.7.4)$$

This notation is introduced on the analogy of a derivation by M. S. MOLODENSKII {[MOLODENSKI, 1958], [MOLODENSKII et al, 1962], [PICK et al, 1973, pp. 467, 468]}.

(1.7.4) can be reduced in an analogous way as (1.7.1):

$$V_i = \frac{1}{4\pi} \iint \left[-V_{ij} \left\{ -r_j^2 \frac{\partial}{\partial r_j} \left(\frac{1}{r_{ij}} \right) \right\} + \frac{r_j}{r_{ij}} \left\{ -r_j \frac{\partial V_j}{\partial r_j} \right\} \right] \frac{\cos(r_j, n_j)}{r_j^2} \, dS_j$$

or, with (1.5.1), Ω the surface of a sphere of unit radius:

$$V_i = \frac{1}{4\pi} \iint_{\Omega} \left[-V_{ij} \frac{1 + \delta_{ij}}{2} + r_j \left(- \frac{\partial V_j}{\partial r_j} \right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j \qquad \ldots (1.7.5)$$

P_i outside, on or inside S; P_j on S

In this notation, the surface over which one has to integrate appears to cause no difficulties.

But several difficulties remain: $\frac{\partial^2 V}{\partial n^2}$ is undefined on the boundary between the earth and the atmosphere, and so it is also undefined on S, although one wishes to measure the second derivates of V. It is still more essential that in practice measurements are never executed on S itself, but at some distance outside S. Here, a comparison can be made with the spatial geometric networks, by which one determines coordinates of points usually situated at some distance from the earth, on towers, pillars, etc.*). For cartographic purposes these points are projected on a reference ellipsoid or on a plane, but this is not essential for spatial geometric geodesy. Similarly, the "reduction" of observations in gravimetric geodesy does not belong to the essence of the theory, so that in principle all reductions should be excluded.
The exclusion of reductions is attained by replacing the surface S by the geosurface S*, containing the observation points on, or near and connected with, the earth's surface. S* may locally coincide with S but it may also deviate from S. S* has to fulfil the same requirements as S; the surface may contain a finite number of singular points and a finite number of edges, which divide the surface into a finite number of pieces with continuously changing normal direction [STERNBERG and SMITH, 1952, chapter 3].

The equations (1.7.4) and (1.7.5), as well as (1.7.1) remain valid if S is replaced by the geosurface S*. Points P_j, connected with the earth are therefore always situated on S*. "P_i inside S*" now assumes a more realistic meaning. . . . (1.7.6)

*) This means that man-made structures are not considered as belonging to S. Probably this is something similar to the influence of the mass of the atmosphere.

Of course the introduction of S* does not solve all difficulties; the geosurface always remains a somewhat vague concept because observations will always be made at discrete points. In fact we are here faced with the problem of discrete observations in physical geodesy, first formulated by A. BJERHAMMAR [BJERHAMMAR 1963, 1964]. For solutions suggested by himself and other authors, see the survey paper [BJERHAMMAR, 1975].

Finally, a remark on the gravity potential W in points connected with the earth:
The centrifugal potential is [HM, (2–3)]:

$$C_i = \tfrac{1}{2}\omega^2 (X_i^2 + Y_i^2)$$
$$\nabla^2 C_i = 2\omega^2$$

Replacing V by W in (1.7.2′) gives:

$$\iiint\limits_{\text{earth}} \frac{1}{r_{ik}} (-\nabla^2 W_k)\, \mathrm{d}v_k - pW_i = \iint\limits_{S} \left[W_j \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial W_j}{\partial n_j} \right] \mathrm{dS}_j \qquad (1.7.7')$$

With: $W_i = V_i + C_i \quad , \quad \nabla^2 W_i = \nabla^2 V_i + \nabla^2 C_i$

one obtains:

$$\iiint \frac{1}{r_{ik}} (-\nabla^2 W_k)\, \mathrm{d}v_k = \iiint \frac{1}{r_{ik}} (-\nabla^2 V_k)\, \mathrm{d}v_k - 2\omega^2 \iiint \frac{1}{r_{ik}}\, \mathrm{d}v_k =$$

$$= 4\pi V_i - 2\omega^2 \iiint \frac{1}{r_{ik}}\, \mathrm{d}v_k =$$

$$= 4\pi W_i - 4\pi \frac{\omega^2}{2} \left\{ X_i^2 + Y_i^2 + \frac{1}{\pi} \iiint \frac{1}{r_{ik}}\, \mathrm{d}v_k \right\}$$

And hence, on the analogy of (1.7.2″), compare [HM, (8–16)]:

$$(4\pi - p)\, W_i - 4\pi \frac{\omega^2}{2} \left\{ X_i^2 + Y_i^2 + \frac{1}{\pi} \iiint\limits_{\text{earth}} \frac{1}{r_{ik}}\, \mathrm{d}v_k \right\} =$$

$$= \iint\limits_{S} \left[W_j \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial W_j}{\partial n_j} \right] \mathrm{dS}_j \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1.7.7'')$$

For points P_i outside S and not connected with the earth {i.e. points outside S*}, (1.7.7) has no meaning because the influence of the centrifugal potential is zero. For this situation one has to revert to (1.7.2) in the form {with $p = 0$}:

$$4\pi V_i = \iint\limits_{S} \left[\{W_j - C_j\} \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial \{W_j - C_j\}}{\partial n_j} \right] \mathrm{dS}_j \quad . \; . \; . \; . \; (1.7.8)$$

For a more automatic form of derivation it is better to contract (1.7.7) and (1.7.8). To do this, consider *temporarily* P_i as being always connected with the earth, then (1.7.7) is always valid. Substract from (1.7.7) Green's {third} integral formula for the centrifugal potential C:

$$-2\omega^2 \iiint \frac{1}{r_{ik}}\, \mathrm{d}v_k - pC_i = \iint \left[C_j \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial C_j}{\partial n_j} \right] \mathrm{dS}_j$$

or:

$$(4\pi - p)\, C_i - 4\pi \left\{ C_i + \frac{2\omega^2}{4\pi} \iiint_{\text{earth}} \frac{1}{r_{ik}}\, \mathrm{d}v_k \right\} =$$

$$= \iint_S \left[C_j \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial C_j}{\partial n_j} \right] \mathrm{dS}_j$$

This substraction gives:

$$(4\pi - p)\, \{W_i - C_i\} = \iint_S \left[\{W_j - C_j\} \frac{\partial}{\partial n_j} \left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}} \frac{\partial \{W_j - C_j\}}{\partial n_j} \right] \mathrm{dS}_j \qquad (1.7.9)$$

or (1.7.2″), with for $p = 0$ (1.7.8).

(1.7.9) therefore includes (1.7.7) and (1.7.8). Consequently it suffices to use the formulae (1.7.2) – (1.7.6) of the gravitational potential, provided one fills in $V = W - C$ when W has been observed.
The essence of this line of thought is that (1.7.9) and therefore (1.7.2) – (1.7.6) is a relation in non-linearized form between measurable quantities, which can in principle be verified by an experiment. In practice, one is compelled to use a linearized form; in chapter 4 this will be further discussed.

In the present sketch of the theory we will not go into possible refinements. Reference can be made to [MORITZ, 1974] for such considerations.

1.8 Appendix. Effects of approximations in partial derivatives

When this publication was being made ready for the press, the author's co-worker, the mathematician D.T. VAN DAALEN found an error in the oldest part of the theory, which had been considered as the part that had been most carefully checked. The finding of this error provides the answer to the question as to what degree of approximation is involved in the present theory. This question had been raised long ago but so far could not be answered.
It appears that the theory developed cannot be corrected in a simple straightforward way, so that the only ways to take account of the error are to make an appraisal of its effect, or to execute additional computations. The main lines of the theory appear to be unaffected, although its spherical approximation character is more pronounced. Because of this, and also in view of some interesting conclusions, the author has deemed it right to add, in a separate section, his provisional considerations on the degree of approximation of the theory.

In the following parts of this section, we will therefore consider some effects of the approximation formula used: $\frac{\partial}{\partial n} = \frac{\partial}{\partial l} \cos(u, l)$, mainly relating to the chapters 3 and 4. Apart from adding this section 1.8, only section 7.4 had to be re-written, all other sections have been left unchanged.

I. With a h-direction corresponding to the gravitational potential {and therefore differing slightly from the h-direction used in (3.4.2) which corresponds to the gravity potential} the following holds [MAGNIZKI et al, 1964, p. 23]:

$$\frac{\partial V_j}{\partial n_j} = \frac{\partial V_j}{\partial h_j} \cos(n_j, h_j) \quad \ldots \quad (1.8.1)$$

$$\frac{\partial V_j}{\partial h_j} = \left\{ \left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial V}{\partial Z}\right)^2 \right\}^{\frac{1}{2}}$$

Then one obtains, with $(r_j, h_j) << \frac{\pi}{2}$:

$$\frac{\partial V_j}{\partial r_j} = \frac{\partial V_j}{\partial h_j} \cos(r_j, h_j) \qquad \text{, hence:}$$

$$\frac{\partial V_j}{\partial n_j} = \frac{\partial V_j}{\partial r_j} \frac{\cos(n_j, h_j)}{\cos(r_j, h_j)}$$

Consider the points of intersection of the vectors $\vec{n}_j$, $\vec{h}_j$ and $\vec{r}_j$ with the unit sphere around P_j, and denote the angle between the planes through $\vec{r}_j, \vec{h}_j$ and $\vec{r}_j, \vec{n}_j$ by (n_j, r_j, h_j). Then the rule of cosines gives:

$$\cos(n_j, h_j) = \cos(r_j, n_j) \cos(r_j, h_j) + \sin(r_j, n_j) \sin(r_j, h_j) \cos(n_j, r_j, h_j)$$

By this we have:

$$\frac{\partial V_j}{\partial n_j} = \frac{\partial V_j}{\partial r_j} \cos(r_j, n_j) \{1 + \operatorname{tg}(r_j, n_j) \operatorname{tg}(r_j, h_j) \cos(n_j, r_j, h_j)\}$$

$$\underset{\text{say}}{=} \frac{\partial V_j}{\partial r_j} \cos(r_j, n_j)(1 + B_j) \quad \ldots \quad (1.8.2')$$

Using N for North component and E for East component one obtains:

$$B_j = \operatorname{tg}(r_j, n_j)_N \operatorname{tg}(r_j, h_j)_N + \operatorname{tg}(r_j, n_j)_E \operatorname{tg}(r_j, h_j)_E \quad \ldots \quad (1.8.2'')$$

i.e. a correction term comparable to [HM, (8–21)].

According to (4.6.3) we have: $\operatorname{tg}(r_j, h_j) \lesssim 3.10^{-3}$, hence:

$$B_j \lesssim 3.3 \cdot 10^{-3} \cdot \operatorname{tg}(r_j, n_j)$$

If one wishes to neglect B_j when establishing the difference equations, then one must have, according to (1.2.4):

$$B_j < 0.01$$

or:

$$\mathrm{tg}(r_j, n_j) < 3 \quad , \quad \boxed{(r_j, n_j) < 71^\circ} \qquad (1.8.3)$$

Therefore, the surface to be determined should not have any slopes greater than around 70°. This implies that the introduction of the geosurface S* in (1.7.6) is necessary for the formula system developed in this publication. In view of the choice of stations in practice, the restriction imposed on S* by (1.8.3) does not seem to cause difficulties.

II. In analogy with (1.8.1) one obtains:

$$\left.\begin{aligned} \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial n_j} &= \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_{ij}} \cos(n_j, r_{ij}) \\ \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} &= \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_{ij}} \cos(r_j, r_{ij}) \end{aligned}\right\} \qquad (1.8.4)$$

If, for brevity, we leave the limiting situation $(r_j, r_{ij}) = \frac{\pi}{2}$ out of consideration {this situation turns out to be harmless when the formula system is written out}, then the equations (1.8.4) result in:

$$\frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial n_j} = \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \frac{\cos(n_j, r_{ij})}{\cos(r_j, r_{ij})}$$

In analogy with (1.8.2′) one obtains:

$$\frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial n_j} = \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \cos(r_j, n_j) \{1 + \mathrm{tg}(r_j, n_j)\, \mathrm{tg}(r_j, r_{ij}) \cos(n_j, r_j, r_{ij})$$

$$\underset{\text{say}}{=} \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \cos(r_j, n_j)(1 + A_{ij}) \qquad (1.8.5')$$

From (1.8.5′) it follows that A_{ij} can always be neglected if $(r_j, n_j) \simeq 0$, i.e. in the spherical approximation for S. This is important for the derivation of the Poisson integrals in section 3.2, for this derivation is completely founded on the use of difference equations for which the spherical approximation of S is considered to be sufficient. Therefore:

$$\text{In Poisson integrals (3.2.5): } A_{ij} = 0 \quad \ldots \ldots \ldots \ldots \quad (1.8.6)$$

For Green integrals this is not immediately valid, because these are obtained by the linearization of an integral equation in which (1.8.5′) is completely valid. Therefore it has to be investigated for which situations one may neglect A_{ij} all the same. With this in mind, write (1.8.5′) in the form:

$$\frac{\partial \frac{1}{r_{ij}}}{\partial n_j} = \left\{ r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} - \left(\frac{r_j}{r_{ij}}\right)^2 \operatorname{tg}(r_j, n_j) \sin(r_j, r_{ij}) \cos(n_j, r_j, r_{ij}) \right\} \frac{\cos(r_j, n_j)}{r_j^2}$$

$$\underset{\text{say}}{=} \left\{ r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} + C_{ij} \right\} \frac{\cos(r_j, n_j)}{r_j^2} \quad \ldots \ldots \ldots \quad (1.8.5'')$$

The application of (1.8.2) – (1.8.5) to (1.7.4) gives:

$$V_i = \frac{1}{4\pi} \iint \left[V_{ij} r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} + \frac{r_j}{r_{ij}} \left(-r_j \frac{\partial V_j}{\partial r_j}\right) \right] d\Omega_j + \frac{1}{4\pi} \iint V_{ij} C_{ij} \, d\Omega_j \quad (1.8.7)$$

in which the first term in the right hand member corresponds to (1.7.5). The second term then represents the correction term to be investigated. It is conjectured that, given certain situations of P_i, always two points P_j, $P_{j'}$ can be found whose combined contribution to the correction term in (1.8.7) is approximately zero.
In order to elucidate this, consider points P_j and $P_{j'}$ situated at about the same "height" on opposite sides of a symmetric mountain ridge, so that one may put:

$$\left.\begin{array}{llll} V_{ij} \simeq V_{ij'} & ; & r_j \simeq r_{j'} & ; \\ (r_j, n_j) \simeq (r_{j'}, n_{j'}) & ; & \vec{n}_j, \vec{r}_j, \vec{n}_{j'}, \vec{r}_{j'} \text{ coplanar} & \end{array}\right\} \quad \ldots \ldots \quad (1.8.8)$$

For the purpose of making an appraisal for points P_i situated on S* beyond the direct neighbourhood of the mountain ridge, S* is approximated by a sphere with centre P_M and radius R {radial distances from P_M are $r_i \simeq r_j \simeq r_{j'} \simeq R$ and $r_{\bar{M}}$}. The section of this sphere with the plane through $P_i, P_j, P_{j'}$ is a circle with centre $P_{\bar{M}}$ and radius $\bar{R}$ {radial distances from $P_{\bar{M}}$ are $\bar{r}_i \simeq \bar{r}_j \simeq \bar{r}_{j'} \simeq \bar{R}$}. Finally, apply a plane through $P_M, P_{\bar{M}}, P_j$ and a plane through $P_M, P_{\bar{M}}, P_{j'}$; the angle between these planes is:

$$\nu = (\bar{r}_j, \bar{r}_{j'}) \simeq \frac{R}{\bar{R}} (r_j, r_{j'})$$

Further:

$$\cos(r_j, r_{ij}) \simeq \frac{r_{ij}}{2R}; \quad \cos(\overline{r}_j, r_{ij}) \simeq \frac{r_{ij}}{2\overline{R}}; \quad \operatorname{tg}(r_j, \overline{r}_j) \simeq \frac{r_{\overline{M}}}{\overline{R}}$$

$$(\overline{r}_{j'}, r_{ij'}) \simeq (\overline{r}_j, r_{ij}) - \tfrac{1}{2}\nu$$

Hence we obtain in (1.8.7), with (1.8.8):

$$C_{ij} + C_{ij'} \simeq -\left(\frac{r_j}{r_{ij}}\right)^2 \Bigg[-\sin(r_j, r_{ij}) \sin\left((n_j, r_j, r_{ij}) - \frac{\pi}{2} + \tfrac{1}{2}\nu\right) + \left(\frac{r_{ij}}{r_{ij'}}\right)^2 \sin(r_{j'}, r_{ij'}) \sin\left((n_{j'}, r_{j'}, r_{ij'} + \frac{\pi}{2} - \tfrac{1}{2}\nu\right) + \nu \operatorname{tg}(r_j, r_{ij}) \cos\left((n_j, r_j, r_{ij}) - \frac{\pi}{2} + \tfrac{1}{2}\nu\right) \cos(r_j, r_{ij}) \Bigg] \operatorname{tg}(r_j, n_j)$$

in which between the square brackets the difference between the first and second term is neglected in the third term.

The use of right-angled spherical triangles on a unit sphere with spherical representation of all directions {on the analogy of item 1} then leads to a reduction to the plane through P_i, P_j, $P_{j'}$:

$$\begin{aligned} C_{ij} + C_{ij'} &\simeq -\left(\frac{r_j}{r_{ij}}\right)^2 \left[-\sin(\overline{r}_j, r_{ij}) + \frac{\cos^2(\overline{r}_j, r_{ij})}{\cos^2(\overline{r}_{j'}, r_{ij'})} \sin(\overline{r}_{j'}, r_{ij'}) + \nu \cdot \operatorname{tg}(r_j, \overline{r}_j) \cos(r_j, r_{ij}) \right] \operatorname{tg}(r_j, n_j) \\ &\simeq -\left(\frac{r_j}{r_{ij}}\right)^2 \left[\left(-\frac{1}{\cos^2(\overline{r}_j, r_{ij})} + \tfrac{1}{2}\right) \cos(\overline{r}_j, r_{ij}) + \operatorname{tg}(r_j, \overline{r}_j) \cos(r_j, r_{ij}) \right] \nu \cdot \operatorname{tg}(r_j, n_j) \\ &\simeq -\left(\frac{r_j}{r_{ij}}\right)^2 \cos(r_j, r_{ij}) \left[\left(-4\left(\frac{\overline{R}}{r_{ij}}\right)^2 + \tfrac{1}{2}\right)\left(\frac{R}{\overline{R}}\right)^2 + \frac{r_{\overline{M}}}{\overline{R}} \frac{R}{\overline{R}} \right] (r_j, r_{j'}) \operatorname{tg}(r_j, n_j) \end{aligned}$$

Or:

$$\boxed{C_{ij} + C_{ij'} \simeq r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \left[-4\left(\frac{R}{r_{ij}}\right)^2 + \left(\tfrac{1}{2} + \frac{r_{\overline{M}}}{R}\right)\left(\frac{R}{\overline{R}}\right)^2 \right] (r_j, r_{j'}) \operatorname{tg}(r_j, n_j)}$$

$0 < r_{ij} \leqslant 2R$	$0 \leqslant r_{\overline{M}} < R$	$0 < \overline{R} \leqslant R$	. . . . (1.8.9′)

The most critical value is obtained when P_i lies on the great circle through P_j and $P_{j'}$, with $r_{\bar{M}} = 0$ and $\bar{R} = R$ one then obtains:

$$\left| C_{ij} + C_{ij'} \right| \lesssim \left| r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \left[-4\left(\frac{R}{r_{ij}}\right)^2 + \tfrac{1}{2} \right] (r_j, r_{j'}) \operatorname{tg}(r_j, n_j) \right| \qquad (1.8.9'')$$

In order to find a criterion, the combined correction term for the points P_j and $P_{j'}$ is joined to the main term for P_j in (1.8.7). This gives:

$$V_{ij} r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \left\{ 1 + \left[-4\left(\frac{R}{r_{ij}}\right)^2 + \left(\tfrac{1}{2} + \frac{r_{\bar{M}}}{R}\right)\left(\frac{R}{\bar{R}}\right)^2 \right] (r_j, r_{j'}) \operatorname{tg}(r_j, n_j) \right\}$$

$$\underset{\text{say}}{=}\; V_{ij} r_j^2 \frac{\partial\left(\frac{1}{r_{ij}}\right)}{\partial r_j} \left\{ 1 + A_{i,j,j'} \right\} \quad \dots\dots\dots\dots \qquad (1.8.10)$$

on the analogy of (1.8.5′).

Like in item I, $A_{i,j,j'}$ may now be neglected when establishing difference equations if {the appraisal concerns *two* points, P_j and $P_{j'}$}:

$$\left| A_{i,j,j'} \right| < 0.02$$

Hence, with the maximum value from (1.8.9″):

$$\left[4\left(\frac{R}{r_{ij}}\right)^2 - \tfrac{1}{2} \right] (r_j, r_{j'}) \operatorname{tg}(r_j, n_{j'}) < 0.02 \quad \dots\dots\dots\dots \qquad (1.8.11)$$

Now consider the following reasonably realistic mountain cross section:

$(r_j, r_{j'}) \operatorname{tg}(r_j, n_j) \simeq 2.10^{-4}$			
Slope (r_j, n_j)	$\operatorname{tg}(r_j, n_j)$	$(r_j, r_{j'})$	$\overline{P_jP_{j'}}$
70°	3	$0.7 \cdot 10^{-4}$	0.4 km
45°	1	$2 \cdot 10^{-4}$	1.3 km
27°	0.5	$4 \cdot 10^{-4}$	2.6 km
11°	0.2	$10 \cdot 10^{-4}$	6.4 km
6°	0.1	$20 \cdot 10^{-4}$	13.0 km

. . . . (1.8.12′)

Then one obtains from (1.8.11) and (1.8.12′):

$$4\left(\frac{R}{r_{ij}}\right)^2 < 100 \quad , \quad \boxed{r_{ij} > 1300 \text{ km}} \quad \ldots\ldots\ldots\ldots \quad (1.8.12'')$$

This means that points P_i must be more than a thousand kilometres away from a mountain mass if the quantities A_{ij} or C_{ij} from (1.8.5) can be neglected. This is a heavy requirement for Green integrals.

In chapter 4 the emphasis is on regional application, with the datum point P_0 within the region considered. Roughly speaking, one is then dealing with the difference of the equations (1.8.9) for P_i and P_0 if $V_{ij} \simeq V_{0j}$. A coarse spherical appraisal indicates that the requirement (1.8.12) is valid for the datum point P_0 if $\{r_{0i}\}_{max} \leqslant \frac{1}{3} r_{0j}$. This provides some more room for the application of the formulae in chapter 4.

III. Of course, the assumptions (1.8.12) can be considered as too unrestrictive. But in that case the limitations to the application of Green integrals are so strict that correction should be considered. (1.7.5) then should be replaced by (1.8.7).
In view of (1.8.6), the application of the train of thought of section 4.1 then gives, instead of (4.1.8):

$$\boxed{\begin{aligned} &\tfrac{1}{2}\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i} \quad \Delta\left(\frac{V_0}{\Phi_0}\right)\right\} = \\ &= \frac{1}{4\pi}\iint\left[-\tfrac{1}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right) d\Omega_j + \text{C-term}\end{aligned}}$$

(1.8.13′)

$$\boxed{\begin{aligned} &\text{C-term} = \\ &= \frac{1}{4\pi}\iint\left[-\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right)\right\} C_{ij} - \frac{r_0}{r_i}\,\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) C_{0j}\right] d\Omega_j\end{aligned}}$$

(1.8.13″)

The solution of (1.8.13) might be found by means of an iteration process proposed by MOLODENSKII, as described e.g. in [HM, (8–45) – (8–49)]. To achieve this, rewrite (1.8.13) in a form analogous to (6.3.2) and solve these equations without C-term. Substitute the solution found into the C-term and use this term with its proper sign to replace the term $\overline{\Delta X}_{0i}$ in (6.3.2). Then this process is repeated once or more times.
The convergence of this procedure has not been investigated; the same applies to the establishment of practical computing rules making Molodenskii's method so attractive.

So far, it is unknown how important the influence of the C-term is. In section 6.1, the formulae without C-term have been compared with formulae in [HM, section 8–8], in which an interpretation is given to the first order correction term of Stokes' formula. A clear correspondence is found, in particular with the formulae [HM, (8–68)] and hence with [HM, (8–52)], in which this correction term is clearly recognized. Perhaps this indicates that the C-term only exerts an influence of the second order.

The greatest influence of the C-term is to be expected for points P_i situated in a mountain area. But closer investigation of this problem leads to a surprising conclusion. To see this, consider the solution of (1.8.13) without C-term in section 4.2, which led to two types of integral formulae, (4.2.9) and (4.2.10). In section 4.3, the possible applications are said to be regions at sea and regions on land respectively.
For the "land"-solution one needs a sufficiently dense network of points where gravimeter *and* levelling observations have been made. But in very rough mountain areas this is *not* possible because levelling is only feasible along a limited number of valley lines. Gravimeter observations too will be less dense than in plane areas.
In fact, the mountain situation is therefore comparable to the sea situation, be it understood that in the mountain situation data on the vertical dimension have to be derived from geometric data {trigonometric levelling, satellite measurements}. In addition, there is a lower precision of measurements in comparison with spirit levelling; the same may apply to gravity observations in very rough terrain. In this respect, too, the mountain situation is comparable to the sea situation. This implies that a small ill-effect of the C-term in the mountain situation can be accepted more readily.
The conclusion is therefore that in very rough mountain areas one should apply (4.2.9) instead of (4.2.10). The sparse levelling data can then be used for an adjustment procedure, as indicated with (4.3.3). After this, potential differences are computed by (4.3.4). The applicability of (4.2.10) {and, consequently, the classical Stokes formula} is therefore even considerably more restricted than was assumed in section 4.3.

IV. On reviewing the different conclusions in the preceding items, it seems possible that the formula system of this publication has a greater significance for practice than was thought immediately after the discovery of the shortcoming mentioned at the beginning of this section. This is one of the reasons why a further study of possible effects on the matter treated in chapter 5, was relegated to the future.

Doubt generates doubt. So it seems that the uniqueness of the linearization procedure by (4.1.6) in chapter 4 needs reconsideration. And other questions will have to be reconsidered as well. Time will tell!

2. ESTIMABLE QUANTITIES AND S-TRANSFORMATIONS IN PHYSICAL GEODESY

2.1 Dimensionless quantities

By studying some examples it is tried in this chapter to find out which quantities are "estimable" and to what extent they are so. This is done by investigating the connection between observations and theory.

As an introduction we assume in this section that the following are measurable quantities in a "universal" system (a) not to be further specified: rays r {section 1.2}, gravity potential W, gravity $g = -\dfrac{\partial W}{\partial h}$ and gravity gradient $q = \dfrac{\partial^2 W}{\partial h^2}$.

On the analogy of (3.1.1), the dimensions of length, mass and time are eliminated by forming the quantities $\{P_i, P_k$ denote points$\}$:

$$\frac{r_i W_i}{r_k^2 g_k}\,,\ \frac{r_i^2 g_i}{r_k^2 g_k}\,,\ \frac{r_i^3 q_i}{r_k^2 g_k} \qquad \ldots\ldots\ldots\ldots \quad (2.1.1)$$

Now the difference quantities of (2.1.1) are relevant, and under the assumptions of section 1.2 we have according to chapter 3:

$$\left.\begin{aligned}
\left\{\frac{r_i W_i}{r_k^2 g_k}\right\}^{\text{appr}} &\simeq \left\{\frac{r_i V_i}{r_k \Phi_k}\right\}^{\text{appr}} \simeq 1 \\
\left\{\frac{r_i^2 g_i}{r_k^2 g_k}\right\}^{\text{appr}} &\simeq \left\{\frac{r_i \Phi_i}{r_k \Phi_k}\right\}^{\text{appr}} \simeq 1 \\
\left\{\frac{r_i^3 q_i}{r_k^2 g_k}\right\}^{\text{appr}} &\simeq \left\{\frac{r_i \Psi_i}{r_k \Phi_k}\right\}^{\text{appr}} \simeq 2
\end{aligned}\right\} \qquad \ldots\ldots\ldots \quad (2.1.2)$$

From (2.1.1) follows then, for example:

$$\begin{aligned}
\Delta\left(\frac{r_i W_i}{r_k^2 g_k}\right) &= \left\{\frac{r_i}{r_k}\right\}^{\text{appr}} \Delta\left(\frac{W_i}{r_k g_k}\right) + \left\{\frac{W_i}{r_k g_k}\right\}^{\text{appr}} \Delta\left(\frac{r_i}{r_k}\right) = \\
&= \left\{\frac{r_i}{r_k}\right\}^{\text{appr}} \Delta\left(\frac{W_i}{r_k g_k}\right) + \left\{\frac{r_i W_i}{r_k^2 g_k}\right\}^{\text{appr}} \Delta\left(\ln\frac{r_i}{r_k}\right) = \\
&= \left\{\frac{r_i}{r_k}\right\}^{\text{appr}} \left[\Delta\left(\frac{W_i}{r_k g_k}\right) + \left\{\frac{r_k}{r_i}\right\}^{\text{appr}} \Delta\left(\ln\frac{r_i}{r_k}\right)\right]
\end{aligned}$$

If in the coefficients of difference quantities we further omit the suffix "appr" we have consequently:

$$\frac{r_k}{r_i}\,\Delta\left(\frac{r_i W_i}{r_k^2 g_k}\right) = \Delta\left(\frac{W_i}{r_k g_k}\right) + \frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)$$

and hence, with

$$W_{ki} = W_i - W_k :$$

$$\boxed{\frac{r_k}{r_i}\,\Delta\left(\frac{r_i W_i}{r_k^2 g_k} - \frac{W_k}{r_k g_k}\right) = \Delta\left(\frac{W_{ki}}{r_k g_k}\right) - \frac{r_k - r_i}{r_i}\,\Delta\left(\frac{W_k}{r_k g_k}\right) + \frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)} \qquad (2.1.3)$$

In analogy:

$$\boxed{\begin{aligned}\frac{r_k}{r_i}\,\Delta\left(\frac{r_i^2 g_i}{r_k^2 g_k}\right) &= \Delta\left(\frac{r_i g_i}{r_k g_k}\right) + \frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)\\ &= \frac{r_k}{r_i}\left\{\Delta\left(\ln\frac{g_i}{g_k}\right) + 2\Delta\left(\ln\frac{r_i}{r_k}\right)\right\}\end{aligned}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.1.4)$$

$$\begin{aligned}\frac{r_k}{r_i}\,\Delta\left(\frac{r_i^3 q_i}{r_k^2 g_k}\right) &= \Delta\left(\frac{r_i^2 q_i}{r_k g_k}\right) + 2\,\frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)\\ &= 2\,\frac{r_k}{r_i}\left\{\Delta\left(\ln\frac{r_k q_i}{g_k}\right) + 3\Delta\left(\ln\frac{r_i}{r_k}\right)\right\}\end{aligned}$$

and therefore also, with

$$\frac{r_k^3 q_k}{r_k^2 g_k} = \frac{r_k^2 q_k}{r_k g_k} = \frac{r_k q_k}{g_k} :$$

$$\boxed{\begin{aligned}\frac{r_k}{r_i}\,\Delta\left(\frac{r_i^3 q_i}{r_k^2 g_k} - \frac{r_k q_k}{g_k}\right) &= \Delta\left(\frac{r_i^2 q_i}{r_k g_k} - \frac{r_k q_k}{g_k}\right) - \frac{r_k - r_i}{r_i}\,\Delta\left(\frac{r_k q_k}{g_k}\right) + 2\,\frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)\\ &= 2\,\frac{r_k}{r_i}\,\Delta\left(\ln\frac{q_i}{q_k}\right) + 6\,\frac{r_k}{r_i}\,\Delta\left(\ln\frac{r_i}{r_k}\right)\end{aligned}}$$

(2.1.5)

If P_i denotes a set of points, and P_k one point, then it is evident from (2.1.3) – (2.1.5) that P_k has the function of "datum point".

In these formulae we have:

$$\Delta\left(\frac{W_k}{r_k g_k}\right) \neq 0 \quad , \quad \text{see (4.4.6) or (4.4.7)}$$

$$\Delta\left(\frac{r_k q_k}{g_k}\right) \neq 0 \quad , \quad \text{see (5.4.4)}$$

The introduction of quantities $(W_i - W_k)$ and $(r_i^2 q_i - r_k^2 q_k)$ is determined by the analysis of the measuring procedure in sections 2.6 and 2.7.

2.2 S-transformations

For the (a)-system mentioned in section 2.1 we have:

$$\left.\begin{aligned}
\Delta\left(\frac{W_{ki}}{r_k g_k}\right) &= \frac{\Delta W_{ki}^{(a)}}{r_k g_k} - \frac{W_{ki}}{r_k g_k}\,\Delta\,(\ln r_k g_k)^{(a)} \\
\Delta\left(\frac{r_i g_i}{r_k g_k}\right) &= \frac{\Delta(r_i g_i)^{(a)}}{r_k g_k} - \frac{r_i g_i}{r_k g_k}\,\Delta\,(\ln r_k g_k)^{(a)} \\
\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right) &= \frac{\Delta(r_i^2 q_i - r_k^2 q_k)^{(a)}}{r_k g_k} - \frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\,\Delta\,(\ln r_k g_k)^{(a)}
\end{aligned}\right\} \quad (2.2.1)$$

If P_k is a station of measurement which acts as datum point, then define the following quantities in the $\underset{k}{\text{S}}$-system {compare the definition of $\Delta z_i^{(r,s)}$ in the $\underset{r,s}{\text{S}}$-system in [BAARDA, 1973]}:

$$\left.\begin{aligned}
\Delta W_i^{(k)} &\underset{\text{def}}{=} r_k g_k\,\Delta\left(\frac{W_{ki}}{r_k g_k}\right) \\
\Delta(r_i g_i)^{(k)} &\underset{\text{def}}{=} r_k g_k\,\Delta\left(\frac{r_i g_i}{r_k g_k}\right) \\
\Delta(r_i^2 q_i)^{(k)} &\underset{\text{def}}{=} r_k g_k\,\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right)
\end{aligned}\right\} \quad \dots\dots\dots\dots (2.2.2')$$

and hence:

$$\left.\begin{aligned}
\Delta(\ln r_i g_i)^{(k)} &= \Delta\left(\ln \frac{r_i g_i}{r_k g_k}\right) \\
\Delta(\ln r_i)^{(k)} &= \Delta\left(\ln \frac{r_i}{r_k}\right) \\
\Delta(\ln g_i)^{(k)} &= \Delta\left(\ln \frac{g_i}{g_k}\right)
\end{aligned}\right\} \quad \dots\dots\dots\dots (2.2.2'')$$

Now assume that the quantities (2.2.2) have already been computed in an $\underset{l}{\text{S}}$-system, with P_l as datum point, hence:

$$\Delta W_i^{(l)} = r_l g_l \,\Delta \left(\frac{W_{li}}{r_l g_l}\right) \quad , \quad \text{etc.}$$

If now one wishes to transform the computed results, obtained in the $\underset{l}{\text{S}}$-system, to the $\underset{k}{\text{S}}$-system, then (2.2.1) can be applied, (a) being replaced by l and taking account of the definitions (2.2.2). This transformation is indicated as S-transformation:

$$\begin{aligned}
\Delta W_i^{(k)} &= \Delta W_i^{(l)} - \Delta W_k^{(l)} - W_{ki}\Delta(\ln r_k g_k)^{(l)}\\
\Delta(r_i g_i)^{(k)} &= \Delta(r_i g_i)^{(l)} - r_i g_i \Delta(\ln r_k g_k)^{(l)}\\
\Delta(r_i^2 q_i)^{(k)} &= \Delta(r_i^2 q_i)^{(l)} - \Delta(r_k^2 q_k)^{(l)} - (r_i^2 q_i - r_k^2 q_k)\,\Delta\,(\ln r_k g_k)^{(l)}
\end{aligned}$$

S-transformation {approximate values unchanged} (2.2.3)

$$\begin{aligned}
\Delta(\ln r_i g_i)^{(k)} &= \Delta(\ln r_i g_i)^{(l)} - \Delta(\ln r_k g_k)^{(l)}\\
\Delta(\ln r_i)^{(k)} &= \Delta(\ln r_i)^{(l)} - \Delta(\ln r_k)^{(l)}\\
\Delta(\ln g_i)^{(k)} &= \Delta(\ln g_i)^{(l)} - \Delta(\ln g_k)^{(l)}
\end{aligned}$$

This S-transformation makes it possible to connect regional S-systems to form one global S-system. Such regional systems arise from the possibilities, offered by the following chapters, to restrict computations with Poisson- and Green integrals to regional areas.

However, the elegant system shown by the first three equations of (2.2.3) is only appearance, as will be demonstrated in the next sections. The cause of this is the relative smallness of W_{ki} and, among other things, the relatively low precision of q as compared with r_k and g_k. This implies that in practice the terms with $\Delta\,(\ln r_k g_k)^{(l)}$ in the first and third equation of (2.2.3) may be neglected. Indeed an analysis of (2.1.5) shows that the third equation in (2.2.2′) and in (2.2.3) can be replaced by the more manageable equation:

$$\Delta\,(\ln q_i)^{(k)} \underset{\text{def}}{=} \Delta \left(\ln \frac{q_i}{q_k}\right) \quad \ldots \ldots \quad (2.2.4')$$

$$\boxed{\Delta\,(\ln q_i)^{(k)} = \Delta\,(\ln q_i)^{(l)} - \Delta\,(\ln q_k)^{(l)}} \qquad \ldots\ldots\ldots\ldots \quad (2.2.4'')$$

Conversely, one can find again the dimensionless quantities from the results of S-transformations:

$$\left.\begin{aligned}
&\Delta\left(\frac{W_{ki}}{r_k g_k}\right) = \frac{\Delta\,W_i^{(k)}}{r_k g_k} \\
&\Delta\left(\frac{r_i g_i}{r_k g_k}\right) = \frac{\Delta\,(r_i g_i)^{(k)}}{r_k g_k} \underset{r_i \simeq r_k}{=} \Delta\,(\ln g_i)^{(k)} + \Delta\,(\ln r_i)^{(k)} \\
&\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right) = \frac{\Delta\,(r_i^2 q_i)^{(k)}}{r_k g_k} \underset{r_i \simeq r_k}{=} 2\,\Delta\,(\ln q_i)^{(k)} + 4\,\Delta\,(\ln r_i)^{(k)}
\end{aligned}\right\} \quad (2.2.5)$$

2.3 Transition from gravity potential to gravitational potential

This transition makes only sense for points P_i, P_k, P_l connected with the earth.
For such points one can, in the first place, extend the invariant quantities in (2.2.1) with ω^2-terms from (3.4.7''):

$$\left.\begin{aligned}
&\Delta\left(\frac{W_{ki}}{r_k g_k}\right) + \frac{\omega^2 R^2}{r_k g_k}\,(\overline{Y}_i^{(1)} - \overline{Y}_k^{(1)}) \\
&\Delta\left(\frac{r_i g_i}{r_k g_k}\right) - \frac{\omega^2 R^2}{r_k g_k}\,(\overline{Y}_i^{(1)} - \overline{Y}_k^{(1)}) \\
&\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right) + 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2.3.1)$$

to which the definitions of (2.2.2') apply, with analogous extension. If necessary, the quantities g and q in all formulae of this chapter up to (2.3.1) can be replaced by the quantities g' and q' from (3.4.2). See the text referring to (3.4.7').

In the second place, (2.3.1) can be extended with a multiple of the term, see (2.1.2):

$$\frac{r_i g_i}{r_k g_k}\,\Delta\left(\ln\frac{r_i}{r_k}\right) \quad , \quad \left\{\frac{r_i g_i}{r_k g_k}\right\}^{\text{appr}} \simeq \frac{r_k}{r_i} \qquad \ldots\ldots\ldots\ldots \quad (2.3.2)$$

to which the definitions of (2.2.2') apply, with analogous extension.

But in the situation of this section, (3.1.9) and (3.1.10) are valid, so that (2.3.1) with (2.3.2) generate the quantities in (3.4.7):

$r_i \simeq r_k \simeq r_l \simeq R$	$g_i \simeq g_k \simeq g_l \simeq G$

$$\Delta\left(\frac{W_{ki}}{r_k g_k}\right) + \Delta\left(\ln\frac{r_i}{r_k}\right) + \frac{\omega^2 R}{G}(\overline{Y}_i^{(1)} - \overline{Y}_k^{(1)}) = \Delta\left(\frac{V_{ki}}{\Phi_k} - \frac{r_k}{r_i}\right)$$

$$\Delta\left(\frac{r_i g_i}{r_k g_k}\right) + \Delta\left(\ln\frac{r_i}{r_k}\right) - \frac{\omega^2 R}{G}(\overline{Y}_i^{(1)} - \overline{Y}_k^{(1)}) = \Delta\left(\frac{\Phi_i}{\Phi_k} - \frac{r_k}{r_i}\right)$$

$$\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right) + 2\,\Delta\left(\ln\frac{r_i}{r_k}\right) = \Delta\left(\frac{\Psi_{ki}}{\Phi_k} - 2\,\frac{r_k}{r_i}\right)$$

$$\Delta\left(\frac{r_i g_i}{r_k g_k}\right) = \Delta\left(\ln\frac{g_i}{g_k}\right) + \Delta\left(\ln\frac{r_i}{r_k}\right)$$

$$\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right) = 2\,\Delta\left(\ln\frac{q_i}{q_k}\right) + 4\,\Delta\left(\ln\frac{r_i}{r_k}\right)$$

(2.3.3′)

or, expressed in quantities in the $\underset{k}{S}$-system {see (2.2.5)}, with:

$$\left(\overline{Y}_i^{(1)}\right)^{(k)} = \overline{Y}_i^{(1)} - \overline{Y}_k^{(1)} \qquad \text{and:}$$

$$\Delta\left(\frac{W_{ki}}{r_k g_k}\right) \simeq \frac{r_k g_k}{RG}\;\frac{\Delta W_i^{(k)}}{r_k g_k} = \frac{\Delta W_i^{(k)}}{RG} \qquad \text{\{see also (2.6.7)\}}$$

$$\frac{\Delta W_i^{(k)}}{RG} + \Delta(\ln r_i)^{(k)} + \frac{\omega^2 R}{G}(\overline{Y}_i^{(1)})^{(k)} = \Delta\left(\frac{V_{ki}}{\Phi_k} - \frac{r_k}{r_i}\right)$$

$$\Delta(\ln g_i)^{(k)} + 2\,\Delta(\ln r_i)^{(k)} - \frac{\omega^2 R}{G}(\overline{Y}_i^{(1)})^{(k)} = \Delta\left(\frac{\Phi_i}{\Phi_k} - \frac{r_k}{r_i}\right)$$

$$2\,\Delta(\ln q_i)^{(k)} + 6\,\Delta(\ln r_i)^{(k)} = \Delta\left(\frac{\Psi_{ki}}{\Phi_k} - 2\,\frac{r_k}{r_i}\right)$$

(2.3.3″)

$\frac{W_{ki}}{G}$ can be considered as a kind of "dynamic height". Then, with (1.2.3) we have for points on S*:

$$\frac{W_{ki}}{RG} < 10^{-3}\text{, hence: } \frac{W_{ki}}{RG}\,\Delta\,(\ln r_k g_k)^{(l)} < 10^{-9}$$

and consequently small with respect to $\frac{\Delta\, W_i^{(l)}}{RG}$ and $\frac{\Delta\, W_k^{(l)}}{RG}$. Then in (2.2.3) one can put:

$$W_{ki}\,\Delta\,(\ln r_k g_k)^{(l)} \simeq 0 \qquad (2.3.4)$$

With this result it follows from (2.2.3) and (2.2.4) in connection with (2.3.3) that:

S-transformation	$r_i \simeq r_k \simeq r_l \simeq R$
$\Delta\, W_i^{(k)}$	$= \Delta\, W_i^{(l)} - \Delta\, W_k^{(l)}$
$\Delta\,(\ln g_i)^{(k)}$	$= \Delta\,(\ln g_i)^{(l)} - \Delta\,(\ln g_k)^{(l)}$
$\Delta\,(\ln q_i)^{(k)}$	$= \Delta\,(\ln q_i)^{(l)} - \Delta\,(\ln q_k)^{(l)}$
$\Delta\,(\ln r_i)^{(k)}$	$= \Delta\,(\ln r_i)^{(l)} - \Delta\,(\ln r_k)^{(l)}$
$\Delta\,(\overline{Y}_i^{(1)})^{(k)}$	$= \Delta\,(\overline{Y}_i^{(1)})^{(l)} - \Delta\,(\overline{Y}_k^{(1)})^{(l)}$

(2.3.5)

This means that one obtains the simplest type of S-transformations, as studied in [BAARDA, 1973, section 11].
Then it also follows from (2.3.3) and (2.3.5) that:

$r_i \simeq r_k \simeq r_l \simeq R$
$\Delta\left(\frac{V_{ki}}{\Phi_k} - \frac{r_k}{r_i}\right) = \Delta\left(\frac{V_{li}}{\Phi_l} - \frac{r_l}{r_i}\right) - \Delta\left(\frac{V_{lk}}{\Phi_l} - \frac{r_l}{r_k}\right)$
$\Delta\left(\frac{\Phi_i}{\Phi_k} - \frac{r_k}{r_i}\right) = \Delta\left(\frac{\Phi_i}{\Phi_l} - \frac{r_l}{r_i}\right) - \Delta\left(\frac{\Phi_k}{\Phi_l} - \frac{r_l}{r_k}\right)$
$\Delta\left(\frac{\Psi_{ki}}{\Phi_k} - 2\frac{r_k}{r_i}\right) = \Delta\left(\frac{\Psi_{li}}{\Phi_l} - 2\frac{r_l}{r_i}\right) - \Delta\left(\frac{\Psi_{lk}}{\Phi_l} - 2\frac{r_l}{r_k}\right)$

(2.3.6)

in accordance with the expansions (3.1.7).

2.4 Absolute gravity measurement on S*

Results of measurements with some instruments are expressed in the instrumental unit belonging to this instrument. I am firmly convinced that this unit varies from instrument to instrument, and for each instrument from period to period. "Instrument" is here to be taken in the sense of "hardware" plus circumstances of measurement {including the observer and the climatic conditions} and "reductions" and "corrections" to be applied. If the measurements are to be expressed in the unit of a "universal" (a)-system in order to combine the results of different instruments and different periods, then the results of the measurements per instrument and per period will have to be multiplied by a scale factor $\lambda^{(a)}$, and in general these scale factors can rarely be determined by calibration with a sufficient precision and reliability. One of the reasons for this is that the "universal" (a)-system itself cannot be defined with sufficient exactness. In other words: the definition of such a universal system is not operational, i.e. the system cannot be constructed from measurable quantities which {with respect to the precision and reliability of measurements} are sufficiently free from uncertainties of scale, orientation, etc. Possibly, an interpretation of this can be that a universal system is not constructed from estimable quantities and therefore is not estimable itself, be it that the concept "estimable" is then disconnected from the further numerical computing process.

Now consider absolute gravity measurements with one instrument, over a period in which the instrumental unit can be deemed constant.
Expressing the measurements g_i {in points P_i} in the (a)-system with the {unknown} g-scale factor $\lambda_g^{(a)}$ then results in:

$$g_i^{(a)} = \lambda_g^{(a)} g_i \qquad (2.4.1)$$

Elimination of $\lambda_g^{(a)}$ can be effected by measurement with the same instrument and in the same period in a "datum point" P_k:

$$g_k^{(a)} = \lambda_g^{(a)} g_k$$

$$\text{or:} \quad \frac{g_i^{(a)}}{g_k^{(a)}} = \frac{g_i}{g_k}$$

are estimable quantities in the sense indicated above. Not only the dimension, but *also* the instrumental unit is eliminated {provided it is constant!}.

More important for this theory are difference quantities:

$$\Delta(\ln g_i^{(a)}) = \Delta(\ln \lambda_g^{(a)}) + \Delta(\ln g_i)$$

$$\Delta(\ln g_k^{(a)}) = \Delta(\ln \lambda_g^{(a)}) + \Delta(\ln g_k)$$

$$\text{Consequently:} \quad \Delta\left(\ln \frac{g_i}{g_k}\right)^{(a)} = \Delta\left(\ln \frac{g_i}{g_k}\right) = \Delta\,(\ln g_i)^{(k)} \qquad (2.4.2)$$

are estimable quantities.

Rays r are derived from one geometric geodetic system, therefore we have, with the {also unknown} r-scale factor $\lambda_r^{(a)}$:

$$r_i^{(a)} = \lambda_r^{(a)} r_i \quad \ldots \quad (2.4.3)$$

On the analogy of (2.4.2) we then have:

$$\Delta \left(\ln \frac{r_i}{r_k}\right)^{(a)} = \Delta \left(\ln \frac{r_i}{r_k}\right) = \Delta (\ln r_i)^{(k)} \quad \ldots \quad (2.4.4)$$

are estimable quantities. But this was already one of the assumptions on which the geometric quaternion theory was built.

Because approximate values can be chosen in an operational way, they are "estimable". The combination with (2.4.2) and (2.4.4) then shows that quantities $\Delta \left(\frac{\Phi_i}{\Phi_k} - \frac{r_k}{r_i}\right)$ in (2.3.3) can be estimable quantities.

2.5 Relative gravity measurement on S*

Assume a trajectory between the points P_k and P_i to be divided into partial trajectories with a small g-difference between their terminals.
Then we have, with $g_{j,j+1} = g_{j+1} - g_j$, for trajectory points

$$P_j, P_{j+1} \; \{j = k, \ldots, i\}:$$

$$\frac{g_i}{g_k} = \frac{g_{k+1}}{g_k} \cdot \frac{g_{k+2}}{g_{k+1}} \cdots \frac{g_i}{g_{i-1}} = \prod_j \frac{g_{j+1}}{g_j} = \prod_j \left(1 + \frac{g_{j,j+1}}{g_j}\right)$$

Relative gravity measurement of $\frac{g_{j,j+1}}{g_j}$ requires the introduction of a dg-scale factor λ_{dg} per instrument and period in order to make possible a comparison with $\frac{g_i}{g_k}$ from absolute gravity measurement.
Assume that the instrument is so well calibrated that:

$$\{\lambda_{dg}\}^{\text{appr}} = 1 \quad \ldots \quad (2.5.1')$$

Then:

$$\ln \frac{g_i}{g_k} = \sum_j \ln \left(1 + \lambda_{dg} \frac{g_{j,j+1}}{g_j}\right) = \lambda_{dg} \sum_j \frac{g_{j,j+1}}{g_j}$$

$$\Delta \left(\ln \frac{g_i}{g_k}\right) = \left\{\ln \frac{g_i}{g_k}\right\}^{\text{appr}} \Delta (\ln \lambda_{dg}) + \sum_j \Delta \left(\frac{g_{j,j+1}}{g_j}\right) \quad \ldots \quad (2.5.1'')$$

According to [TORGE, 1975, pp. 109, 244] the uncertainty in the scale determination after calibration is:

$$\Delta\,(\ln \lambda_{dg}) \simeq 10^{-4} \text{ à } 10^{-5} \quad \ldots \quad (2.5.2)$$

The influence of this on (2.5.1) amounts to:

$$\left| \left\{ \ln \frac{g_i}{g_k} \right\}^{\text{appr}} \Delta\,(\ln \lambda_{dg}) \right| \simeq \left| \frac{g_{ki}}{g_k}\, \Delta\,(\ln \lambda_{dg}) \right| \lessapprox (|\, g_{ki} \,|\ \text{mgal})\ 10^{-10} \qquad (2.5.3)$$

In view of the value 10^{-8} mentioned in (1.2.3), (2.5.3) can be neglected if:

$$(|g_{ki}|\ \text{mgal})\ 10^{-10} \lesssim 10^{-9}, \text{ or } |g_{ki}| \lesssim 10 \text{ mgal}$$

which seems an unrealistic requirement.
If, more realistically, on puts:

$$(|g_{ki}|\ \text{mgal})\ 10^{-10} \lesssim 10^{-8}, \text{ or } |g_{ki}| \lesssim 100 \text{ mgal} \quad \ldots \quad (2.5.4)$$

then one is obliged to apply the law of propagation of variances to (2.5.1″), by which the differences $\Delta \left(\ln \dfrac{g_i}{g_k} \right)$ become correlated variates.

Whichever the case may be, the conclusion seems right that $\Delta \left(\ln \dfrac{g_i}{g_k} \right)$ from relative gravity measurements is only estimable to a limited extent.

2.6 Spirit levelling on S*

Assume that spirit levelling has been executed on the trajectory from P_k to P_i. Partial trajectories are formed by the individual instrument set-ups with backsight and foresight; the terminals P_j and P_{j+1} of these partial trajectories are the successive staff positions, with height differences

$$h_{j,j+1} = h_{j+1} - h_j \quad ; \quad j = k, \ldots, i$$

For each partial trajectory a gravity measurement is available, e.g. the average value of gravity measurements in the terminals P_j and P_{j+1}:

$$g_{j,j+1} = \tfrac{1}{2}(g_j + g_{j+1})$$

With the g-scale and h-scale factors $\lambda_g^{(a)}$ and $\lambda_h^{(a)}$ the potential difference between the terminals of the trajectory in the (a)-system can be computed from:

$$W_{ki}^{(a)} = - \sum_j \lambda_g^{(a)}\, g_{j,j+1}\, \lambda_h^{(a)}\, h_{j,j+1} \quad \ldots \quad (2.6.1)$$

$h_{j,j+1}$ is there a metric quantity, which can be compared with the rays r previously mentioned, although it will never be possible to measure a ratio $\dfrac{h}{r}$. One can at most establish a connection between h-quantities and distance-quantities of geometric networks by calibration via a standard metre.

For the g_j's this is in principle feasible via the measuring procedure of section 2.4.
In this way one can try to eliminate $\lambda_g^{(a)}$ and $\lambda_h^{(a)}$ entire or partially via the data in a datum point P_h:

$$r_k^{(a)}\, g_k^{(a)} = \lambda_r^{(a)}\, r_k\, \lambda_g^{(a)}\, g_k$$

Or:
$$\frac{W_{ki}^{(a)}}{r_k^{(a)}\, g_k^{(a)}} = -\frac{\lambda_h^{(a)}}{\lambda_r^{(a)}} \sum_j \frac{g_{j.j+1}\, h_{j,j+1}}{r_k g_k}$$

are estimable quantities if $\dfrac{\lambda_h^{(a)}}{\lambda_r^{(a)}}$ may be put equal to 1.

Assume:
$$\left\{\frac{\lambda_h^{(a)}}{\lambda_r^{(a)}}\right\}^{\text{appr}} = 1 \qquad (2.6.2')$$

then difference quantities are:

$$\Delta \left(\frac{W_{ki}}{r_k g_k}\right)^{(a)} = \frac{W_{ki}}{r_k g_k}\, \Delta \left(\ln \frac{\lambda_h^{(a)}}{\lambda_r^{(a)}}\right) - \sum_j \Delta \left(\frac{g_{j.j+1}\, h_{j,j+1}}{r_k g_k}\right). \qquad (2.6.2'')$$

According to [Torge, 1975, p. 133] the uncertainty in the scale determination of levelling staves after calibration is:

$$|\, \Delta (\ln \lambda_h)\, | \simeq 10^{-5} \qquad (2.6.3')$$

Because $|\, \Delta (\ln \lambda_r)\, |$ with respect to this calibration system will certainly not be greater, it seems perfectly reasonable to assume:

$$\left|\, \Delta \left(\ln \frac{\lambda_h^{(a)}}{\lambda_r^{(a)}}\right) \right| \simeq 10^{-5} \qquad (2.6.3'')$$

Now put:

$$\frac{W_{ki}}{g_k} = H_{ki} \quad , \quad \frac{W_{ki}}{r_k g_k} \simeq \frac{H_{ki}}{R} \qquad (2.6.4')$$

with, like in (2.3.4), H_{ki} a dynamic height.
If, like in section 2.5, the limit for the order of magnitude of neglected terms is put at 10^{-9}, we have, with (2.6.3):

$$\left|\, \frac{W_{ki}}{r_k g_k}\, \Delta \left(\ln \frac{\lambda_h^{(a)}}{\lambda_r^{(a)}}\right) \right| \simeq \frac{|H_{ki}|}{R}\, 10^{-5} \lesssim 10^{-9} \qquad (2.6.4'')$$

Or:
$$|H_{ki}| \lesssim 640 \text{ metres} \qquad (2.6.4''')$$

From (2.6.2) and (2.6.4) it follows that the following are almost always estimable quantities:

$$\Delta \left(\frac{W_{ki}}{r_k g_k}\right)^{(a)} = \Delta \left(\frac{W_{ki}}{r_k g_k}\right) = -\sum_j \Delta \left(\frac{g_{j.j+1}\, h_{j,j+1}}{r_k g_k}\right) \quad . \; . \; . \; . \; . \; (2.6.5)$$

Because approximate values are always determined from previous measurements, one will in view of the requirement (1.2.3) have, beside (2.6.3):

$$\left| \Delta \left(\ln \frac{\lambda_h}{\lambda_{\{r\}}^{\text{appr}}}\right) \right| \simeq 10^{-5} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2.6.6)$$

Then one may write:

$$\Delta \left(\frac{g_{j.j+1}\, h_{j,j+1}}{r_k g_k}\right) = \frac{g_{j.j+1}\, h_{j,j+1}}{r_k g_k} \left\{ \Delta \left(\ln \frac{g_{j.j+1}}{g_k}\right) - \Delta (\ln r_k) \right\} +$$

$$+ \frac{g_{j.j+1}}{g_k} \; \frac{\Delta h_{j,j+1}}{r_k}$$

The first term in the right hand member will be much smaller than the order of magnitude 10^{-9}, so that it can be safely neglected. Then from (2.6.5) follows:

$$\Delta \left(\frac{W_{ki}}{r_k g_k}\right) = -\frac{\sum_j g_{j.j+1}\, \Delta h_{j,j+1}}{r_k g_k} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2.6.7')$$

Therefore, with (2.2.2), the following is almost always an estimable quantity too:

$$\Delta W_i^{(k)} = r_k g_k \, \Delta \left(\frac{W_{ki}}{r_k g_k}\right) = -\sum_j g_{j.j+1}\, \Delta h_{j,j+1} \, . \quad . \; . \; . \; . \; . \; . \; . \; . \; (2.6.8')$$

This explains the first equation in (2.3.5).

In (2.6.7′) one may without hesitation replace $r_k g_k$ in the right hand member by RG. Introducing for brevity:

$$\Delta W_{ki} \underset{\text{def}}{=} -\sum_j g_{j.j+1}\, \Delta h_{j,j+1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2.6.7'')$$

then:

$$\Delta \left(\frac{W_{ki}}{r_k g_k}\right) = -\frac{\sum_j g_{j.j+1}\, \Delta h_{j,j+1}}{RG} = \frac{\Delta W_{ki}}{RG} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2.6.7''')$$

$$\Delta W_i^{(k)} = RG \, \Delta \left(\frac{W_{ki}}{r_k g_k}\right) = \Delta W_{ki} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2.6.8'')$$

From (2.6.7) it consequently follows that for the computation of a potential difference from spirit levelling a {good} approximate value for $g_{j,j+1}$ is sufficient, or:

$$W_{ki} \underset{\text{def}}{=} -\sum_j \{g_{j,j+1}\}^{\text{appr}} h_{j,j+1} \quad \ldots \ldots \quad (2.6.9)$$

In view of the high demands on approximate values in the present theory, it seems safer for practical applications to derive $g_{j,j+1}$ from measured values.

2.7 Gravity gradient measurement on S*

We only consider the measurement of the vertical gradient:

$$q = \frac{\partial^2 W}{\partial h^2}$$

For a summary of existing theories, see [GROTEN, 1975] and [HEIN, 1977].

As an introduction, we assume q_i to be computed from absolute gravity measurements in two points P_i and $P_{i'}$, close to each other, situated on the same vertical, and from measurement of the distance between these points. In the (a)-system we then have:

$$q_i^{(a)} = -\frac{\lambda_g^{(a)} g_{ii'}}{\lambda_h^{(a)} h_{ii'}} \quad \ldots \ldots \quad (2.7.1)$$

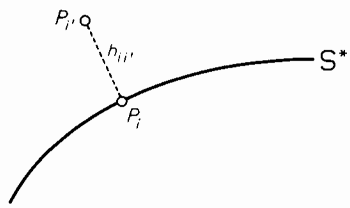


Fig. 2.7-1

It is again tried to eliminate $\dfrac{\lambda_g^{(a)}}{\lambda_h^{(a)}}$ via data in a datum point P_k:

$$\frac{g_k^{(a)}}{r_k^{(a)}} = \frac{\lambda_g^{(a)} g_k}{\lambda_r^{(a)} r_k}$$

Or:

$$\left(\frac{r_k q_i}{g_k}\right)^{(a)} = -\frac{\lambda_r^{(a)}}{\lambda_h^{(a)}} \frac{r_k g_{ii'}}{g_k h_{ii'}}$$

Assume:

$$\left\{\frac{\lambda_r^{(a)}}{\lambda_h^{(a)}}\right\}^{\text{appr}} = 1 \quad \ldots\ldots \quad (2.7.2')$$

then difference quantities are:

$$\Delta\left(\frac{r_k\, q_i}{g_k}\right)^{(a)} = \frac{r_k\, q_i}{g_k}\,\Delta\left(\ln\frac{\lambda_r^{(a)}}{\lambda_h^{(a)}}\right) - \Delta\left(\frac{r_k\, g_{ii'}}{g_k\, h_{ii'}}\right) \quad \ldots\ldots \quad (2.7.2'')$$

But evidently this does *not* produce estimable quantities because according to (2.1.2), and assuming (2.6.3) to be valid here, we have:

$$\left|\frac{r_k\, q_i}{g_k}\,\Delta\left(\ln\frac{\lambda_r^{(a)}}{\lambda_h^{(a)}}\right)\right| \simeq 2\cdot 10^{-5} \gg 10^{-9}$$

Elimination of this term may be attempted now by working with differences of quantities having the same order of magnitude, like in section 2.6:

$$\Delta\left(\frac{r_i^2}{r_k^2}\;\frac{r_k\, q_i}{g_k}\right)^{(a)} - \Delta\left(\frac{r_k\, q_k}{g_k}\right)^{(a)}$$

Or:

$$\Delta\left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right)^{(a)} = \frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\,\Delta\left(\ln\frac{\lambda_r^{(a)}}{\lambda_h^{(a)}}\right) - \Delta\left(\frac{r_i^2\, g_{ii'}}{r_k g_k\, h_{ii'}} - \frac{r_k\, g_{kk'}}{g_k\, h_{kk'}}\right) \quad (2.7.3)$$

However, it is difficult to appraise the order of magnitude of:

$$\frac{r_i^2\, q_i - r_k^2\, q_k}{r_k g_k}$$

so it does not seem possible to get certainty in this way {which also applies to the term $\Delta(\ln r_k g_k)^{(l)}$ in (2.2.3)}

However, from (2.1.5) it appears that for points on S* we have, with sufficient approximation:

$$\Delta\left(\frac{r_i^2\, q_i - r_k^2\, q_k}{r_k g_k}\right)^{(a)} = 2\,\Delta\left(\ln\frac{r_i^2\, q_i}{r_k^2\, q_k}\right)^{(a)} \quad \ldots\ldots \quad (2.7.4)$$

A tentative approach might be to introduce, for one arbitrary measuring process for q, a q-scale factor $\lambda_q^{(a)}$:

$$q_i^{(a)} = \lambda_q^{(a)} q_i \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (2.7.5)$$

and consequently also:

$$q_k^{(a)} = \lambda_q^{(a)} q_k$$

so that one obtains as estimable quantities:

$$\frac{q_i^{(a)}}{q_k^{(a)}} = \frac{q_i}{q_k}$$

Or, in difference quantities as estimable quantities:

$$\Delta \left(\ln \frac{q_i}{q_k}\right)^{(a)} = \Delta \left(\ln \frac{q_i}{q_k}\right) = \Delta (\ln q_i)^{(k)} \quad \ldots \ldots \ldots \ldots \quad (2.7.6)$$

If these are coupled with the estimable quantities from (2.4.4) $\Delta \left(\ln \frac{r_i}{r_k}\right)$ this results with (2.7.4) in the estimable quantities:

$$2\,\Delta \left(\ln \frac{r_i^2\, q_i}{r_k^2\, q_k}\right)^{(a)} = \Delta \left(\frac{r_i^2 q_i - r_k^2 q_k}{r_k g_k}\right)^{(a)} = 2\,\Delta \left(\ln \frac{q_i}{q_k}\right) + 4\,\Delta \left(\ln \frac{r_i}{r_k}\right) \quad (2.7.7)$$

to be used in (2.3.3).

According to [GROTEN, 1975] a standard deviation $\sigma_{\ln q} \gtrsim 10^{-3}$, or:

$$\sigma_{\ln \frac{q_i}{q_k}} \gtrsim 10^{-3} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (2.7.8)$$

This is at least a thousand times greater than assumed for $\ln \frac{r_i}{r_k}$ and $\ln \frac{g_i}{g_k}$ in (1.2.3), so that the combination of $\ln \frac{q_i}{q_k}$ and $\ln \frac{r_i}{r_k}$ in (2.7.7) and consequently in (2.3.3) and chapter 5, has little practical significance for the time being.

2.8 Some remarks

In sections 2.4–2.7 it has been tried to demonstrate that measuring procedures are possible which lead to estimable quantities (2.3.3):

$$\Delta \left(\frac{V_{ki}}{\Phi_k} - \frac{r_k}{r_i}\right) \quad , \quad \Delta \left(\frac{\Phi_i}{\Phi_k} - \frac{r_k}{r_i}\right) \quad , \quad \Delta \left(\frac{\Psi_{ki}}{\Phi_k} - \frac{r_k}{r_i}\right)$$

The concept "estimable" has here been coupled with the elimination of uncertainties in scale factors remaining after calibration, and therefore it is limited to the measuring processes. But

in the opinion of the present author this is sufficient; the computing process linked to the measurements does not permit any extension.
For, a computing model contains only mathematical quantities, to which operational names can only be given *after* establishing the link with measuring processes. In the first instance, the linking of measuring- and computing process should therefore be done by means of nameless {dimensionless} quantities, as introduced in section 2.1. The choice of a computing model can in principle only be made by the execution of an experiment, hence with "repetition" of measurements. Relationships between means in this repeated process then should agree {within a previously agreed margin} with relationships between quantities in the mathematical computing model to be chosen {or between means of stochastic variables in this model}. But "repetition" implies constancy of the measuring instrument {in the extended sense, as described in section 2.4}. Any instrumental unit that is possibly variable will therefore have to be eliminated. It follows that the concept "estimable" can only refer to the sufficient elimination of dimensions *and* instrumental units.
This line of thought was published earlier in [BAARDA, 1967, chapter 4] A further elaboration was given in [BAARDA, 1973]. The present theory should also be seen as a further elaboration and application. It should be noted that the contents of this chapter only give a first exploration of the possibilities. It has been tried to explain how the author arrived at the formation of estimable quantities.
It is evident that the line of thought developed in this chapter has been influenced by elements of the theory of the following chapters that where found at an early stage. It may be remarked that the whole theory originates from an intuitive belief in dimensionless and unitless quantities.

3. SERIES OF SPHERICAL HARMONICS. POISSON INTEGRALS

3.1 Harmonic functions V, Φ, Ψ

Henceforth, the datum point will be denoted by P_0.
Spirit levelling can only produce differences in gravity potential: W_{0i}. Within the computing model we may, however, split up this difference:

$$W_{0i} = W_i - W_0$$

Since we consider points P_i situated outside or on the surface S of the earth, the formulae are developed for the gravitational potential V, in accordance with chapter 1.7. If necessary, V will later be replaced by the difference of the gravity potential and the centrifugal potential.

Now, in connection with chapter 2, consider the following quantities, the integration being over the mass distribution of the earth, and under the condition: $\nabla^2 V_i = 0$ outside and on S {in the sense of: down to S; see the contents of (1.7.3) – (1.7.6)}:

According to Newton's model theory:	Eliminating the units of length and mass:
$V_i = \iiint \frac{1}{r_{ik}} \, \mathrm{d}M_k$	$\frac{V_i}{\Phi_0} \frac{r_i}{r_0} = \iiint \frac{r_i}{r_{ik}} \, \mathrm{d}\left(\frac{M_k}{\Phi_0 r_0}\right)$
$-\frac{\partial V_i}{\partial r_i} = \iiint \frac{1}{r_{ik}^2} \cos(r_i, r_{ik}) \, \mathrm{d}M_k$	$\frac{\Phi_i}{\Phi_0} \frac{r_i}{r_0} = \iiint \left(\frac{r_i}{r_{ik}}\right)^2 \cos(r_i, r_{ik}) \, \mathrm{d}\left(\frac{M_k}{\Phi_0 r_0}\right)$
$-r_i \frac{\partial V_i}{\partial r_i} \equiv \Phi_i$	$1 = \iiint \left(\frac{r_0}{r_{0k}}\right) \cos(r_0, r_{0j}) \, \mathrm{d}\left(\frac{M_k}{\Phi_0 r_0}\right)$
$\frac{\partial^2 V_i}{\partial r_i^2} =$	$\frac{\Psi_i}{\Phi_0} \frac{r_i}{r_0} = \iiint \left(\frac{r_i}{r_{ik}}\right)^3 \{3\cos^2(r_i, r_{ik}) - 1\} \, \mathrm{d}\left(\frac{M_k}{\Phi_0 r_0}\right)$
$= \iiint \frac{1}{r_{ik}^3} \{3\cos^2(r_i, r_{ik}) - 1\} \, \mathrm{d}M_k$	
$r_i^2 \frac{\partial^2 V_i}{\partial r_i^2} \equiv \Psi_i \quad \{P_i \text{ outside S}\}$	

(3.1.1)

In analogy [HOPFNER, 1933, p. 389] we have if $\nabla^2 V = 0$:

$$\nabla^2 V = 0$$

$$-\Phi = r \frac{\partial V}{\partial r} \quad ; \quad \frac{\partial \Phi}{\partial r} = -\frac{\partial V}{\partial r} - r \frac{\partial^2 V}{\partial r^2}$$

$$-\nabla^2 \Phi = 2 \cdot \nabla^2 V + X \cdot \nabla^2 \frac{\partial V}{\partial X} + Y \cdot \nabla^2 \frac{\partial V}{\partial Y} + Z \cdot \nabla^2 \frac{\partial V}{\partial Z} =$$

$$= 2 \cdot \nabla^2 V + \sum_{X, Y, Z} X \cdot \nabla^2 \frac{\partial V}{\partial X} = 2 \cdot \nabla^2 V + \sum_{X, Y, Z} X \cdot \frac{\partial}{\partial X} (\nabla^2 V) = 0$$

$$\Psi = r^2 \frac{\partial^2 V}{\partial r^2} \quad ; \quad \text{from } \frac{\partial \Phi}{\partial r} \quad : \quad -\Psi = r \frac{\partial \Phi}{\partial r} - \Phi \quad ; \quad P \text{ outside S}$$

$$-\nabla^2 \psi = 2 \cdot \nabla^2 \Phi + \sum_{X, Y, Z} X \cdot \frac{\partial}{\partial X} (\nabla^2 \Phi) - \nabla^2 \Phi =$$

$$= \nabla^2 \Phi + \sum_{X, Y, Z} X \cdot \frac{\partial}{\partial X} (\nabla^2 \Phi) = 0$$

Hence: V, Φ, Ψ {Ψ outside S} are harmonic functions . . . . (3.1.2)

With r_0, $r_i > r_k$ we have according to (1.4.3):

$$\frac{r_i}{r_{ik}} = \sum_{n=0}^{\infty} \left(\frac{r_k}{r_i}\right)^n Y_i^{\prime(n)} Y_k^{\prime(n)} \quad , \qquad Y^{\prime(0)} = 1$$

Hence with (3.1.1):

$$\frac{V_i}{\Phi_0} \frac{r_i}{r_0} = \sum_{n=0}^{\infty} \left(\frac{1}{r_i}\right)^n Y_i^{\prime(n)} \iiint (r_k)^n Y_k^{\prime(n)} \, \mathrm{d} \left(\frac{M_k}{\Phi_0 r_0}\right)$$

With the introduction of M {mass of the earth, see section 1.4}, R {radius of a sphere with centre P_M, the geocentre} and the spherical harmonics $Y^{(n)}$ this can be written as:

$$\frac{V_i}{\Phi_0} \frac{r_i}{r_0} = \frac{M}{\Phi_0 r_0} \sum_{n=0}^{\infty} \left(\frac{R}{r_i}\right)^n Y_i^{\prime(n)} \iiint \left(\frac{r_k}{R}\right)^n Y_k^{\prime(n)} \, \mathrm{d}\left(\frac{M_k}{M}\right)$$

$$= \frac{M}{\Phi_0 r_0} \sum_{n=0}^{\infty} \left(\frac{R}{r_i}\right)^n Y_i^{(n)} \quad . . . . (3.1.3)$$

<table>
<tr><td colspan="2">$\frac{V_i}{\Phi_0} = \frac{M}{\Phi_0 r_0} \cdot \frac{r_0}{r_i} \sum_{n=0}^{\infty} \left(\frac{R}{r_i}\right)^n Y_i^{(n)}$</td><td colspan="2">$r_i \geq R \geq \{r_k\}_{\max}$</td></tr>
<tr><td>$Y_i^{(n)} = C^{(n)} Y_i'^{(n)}$</td><td colspan="2">$C^{(n)} = \iiint \left(\frac{r_k}{R}\right)^n Y_k'^{(n)} \, \mathrm{d}\left(\frac{M_k}{M}\right)$</td><td>$Y_i^{(0)} = Y_i'^{(0)} = 1$</td></tr>
</table>

(3.1.4)

For considerations concerning the convergence of these and following series, see [HOPFNER, 1933, chapter 10] and [HM, p. 60]*). Differentiation with respect to r_i gives the following series:

$$\left.\begin{aligned}
\frac{\Phi_i}{\Phi_0} &= \frac{M}{\Phi_0 r_0} \frac{r_0}{r_i} \sum_{n=0}^{\infty} (n+1) \left(\frac{R}{r_i}\right)^n Y_i^{(n)} \\
\frac{\Phi_0}{\Phi_0} &= 1 = \frac{M}{\Phi_0 r_0} \sum_{n=0}^{\infty} (n+1) \left(\frac{R}{r_0}\right)^n Y_0^{(n)} \quad , \quad \text{hence: } \frac{M}{\Phi_0 r_0} \simeq 1 \\
\frac{\Psi_i}{\Phi_0} &= \frac{M}{\Phi_0 r_0} \frac{r_0}{r_i} \sum_{n=0}^{\infty} (n+2)(n+1) \left(\frac{R}{r_i}\right)^n Y_i^{(n)} \quad , \quad P_i \text{ outside S}
\end{aligned}\right\} \quad (3.1.5)$$

Linearization using approximate values {the suffix "appr" will further be omitted in the coefficients of Δ-quantities} gives:

$$\left.\begin{aligned}
\Delta\left(\frac{V_i}{\Phi_0} - \frac{r_0}{r_i}\right) &= \frac{r_0}{r_i}\left\{ \Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} \left(\frac{R}{r_i}\right)^n \cdot \Delta Y_i^{(n)} \right\} \\
\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) &= \frac{r_0}{r_i}\left\{ \Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} (n+1) \left(\frac{R}{r_i}\right)^n \cdot Y_i^{(n)} \right\} \\
\Delta\left(\frac{\Psi_i}{\Phi_0} - 2\frac{r_0}{r_i}\right) &= \frac{r_0}{r_i}\left\{ 2\,\Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} (n+2)(n+1) \left(\frac{R}{r_i}\right)^n \cdot \Delta Y_i^{(n)} \right\} \\
\Delta\left(\frac{V_0}{\Phi_0}\right) &= \Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} \left(\frac{R}{r_0}\right)^n \cdot \Delta Y_0^{(n)} \\
\Delta\left(\frac{\Phi_0}{\Phi_0}\right) = 0 &= \Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} (n+1) \left(\frac{R}{r_0}\right)^n \cdot \Delta Y_0^{(n)} \\
\Delta\left(\frac{\Psi_0}{\Phi_0}\right) &= 2\,\Delta\left(\frac{M}{\Phi_0 r_0}\right) + \sum_{n=1}^{\infty} (n+2)(n+1) \left(\frac{R}{r_0}\right)^n \cdot \Delta Y_0^{(n)}
\end{aligned}\right\} \quad (3.1.6)$$

*) But see: Arnold, K. Beweis der gleichmässigen Konvergenz der Kugelfunctionsentwicklung der Erde im Aussenraum. – Vermessungstechnik 1978, Heft 7.

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)=\frac{r_0}{r_i}\sum_{n=1}^{\infty}\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} \quad (3.1.7')$$

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{r_0}{r_i}\sum_{n=1}^{\infty}(n+1)\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} \quad (3.1.7'')$$

$$\Delta\left(\frac{\Psi_{0i}}{\Phi_0}-2\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{\Psi_0}{\Phi_0}\right)=\frac{r_0}{r_i}\sum_{n=1}^{\infty}(n+2)(n+1)\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} \quad (3.1.7''')$$

$$\Delta Y_i^{(n)}=c^{(n)}\cdot Y_i'^{(n)}$$

If now the approximate values are chosen in accordance with section 1.2, then the Δ-quantities in (3.1.6) are small. A comparison of (3.1.6) with [HM, p. 34] then shows the possibility to interpret the left hand members as harmonic functions on and outside a sphere $\overline{S}$ with radius R, in such a way that for points outside $\overline{S}$ the series expansions coincide with the linearized series (3.1.4) and (3.1.5). Then, contrary to (3.1.4) and taking account of (1.7.6) we have:

$$\left.\begin{array}{l}\text{Points } P_j \text{ on } S^*\text{: } \{r_j\}_{\min}\lesssim R\lesssim\{r_j\}_{\max}\\ \qquad R \text{ in (3.1.6) and (3.1.7)}\end{array}\right\}\quad\ldots(3.1.8)^{*)}$$

The spherical surface $\overline{S}$ should be seen only as an interpretative mathematical approximation of the geo-surface S*. $\overline{S}$ can never lead a life of its own, in particular one can never speak of a mass transfer from S to $\overline{S}$. We have, e.g. for points P_i and P_0:

$$\left.\begin{array}{l}\text{If: } \{r_j\}_{\min}\lesssim r_i,r_0\lesssim\{r_j\}_{\max}\\ \text{then in (3.1.6) and (3.1.7): } r_i\simeq r_0\simeq r_j\simeq R\end{array}\right\}\quad\ldots(3.1.9)$$

(3.1.9) is important for the datum point P_0, because for practical reasons this will always be chosen on S*.

In situations analogous to the ones for which (3.1.9) is applicable it can be useful to use an average value for g:

$$\left.\begin{array}{l}\text{If: } \{g_j\}_{\min}\lesssim g_i,g_0\lesssim\{g_j\}_{\max}\\ \text{then in such situations } g_i\simeq g_0\simeq g_j\simeq G\end{array}\right\}\quad\ldots\ldots(3.1.10)$$

*) For R one may e.g. choose the equatorial radius of the reference ellipsoid, as is done in satellite dynamics [HM, p. 59].

3.2 Poisson integrals

Combine the series (3.1.7) into:

$$\Delta X_{0i} = \frac{r_0}{r_i} \sum_{n=1}^{\infty} A_n \left\{ \left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\} ; \Delta Y_i^{(n)} = c^{(n)} Y_i'^{(n)} \tag{3.2.1}$$

and with $\dfrac{r_0}{r_j} - \dfrac{r_0}{r_i} = \dfrac{r_0}{r_j} \dfrac{r_i - r_j}{r_i}$

$$-\Delta X_{0i} + \Delta X_{0j} = -\frac{r_0}{r_i} \sum_{n=1}^{\infty} A_n \left\{ \left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_j}\right)^n c^{(n)} Y_j'^{(n)} \right\} +$$

$$+ \frac{r_0}{r_j} \frac{r_i - r_j}{r_i} \sum_{n=1}^{\infty} A_n \left\{ \left(\frac{R}{r_j}\right)^n c^{(n)} Y_j'^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\}$$

From (1.6.1): $(-\delta_{ij}) \dfrac{r_j}{r_{ij}} = \sum_{\bar{n}=0}^{\infty} (2\bar{n} + 1) \left(\dfrac{r_j}{r_i}\right)^{\bar{n}+1} Y_i'^{(\bar{n})} Y_j'^{(\bar{n})} \quad , \quad r_i > r_j$

Choose P_j on S*, then with (3.1.8), see (1.4.4), we have:

$$\frac{1}{4\pi} \iint \left(\frac{R}{r_j}\right)^{n+1} Y_j'^{(n)} (2\bar{n} + 1) \left(\frac{r_j}{r_i}\right)^{\bar{n}+1} Y_j'^{(\bar{n})} \, d\Omega_j \begin{cases} = \left(\dfrac{R}{r_i}\right)^{n+1} & , \bar{n} = n \\ \simeq 0 & , \bar{n} \neq n \end{cases} \tag{3.2.2}$$

The influence of the replacement of r_j by R is small according to (3.2.2), therefore we have with a very good approximation:

$$I_1 \equiv \frac{1}{4\pi} \iint (-\Delta X_{0i} + \Delta X_{0j}) (-\delta_{ij}) \frac{r_j}{r_{ij}} \, d\Omega_j =$$

$$= \frac{r_0}{r_i} \frac{r_i - R}{r_i} \sum_{n=1}^{\infty} A_n \left\{ \left(\frac{R}{r_i}\right)^n c^{(n)} Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\}$$

or, with (3.2.1):

$$I_1 = \frac{r_i - R}{r_i} \Delta X_{0i} \quad , \quad r_i > \{r_j\}_{\max}$$

The indetermination of the radius R of the surface $\bar{S}$, which is expressed by (3.1.8), together with the remarks with (1.7.6), makes it desirable to substitute $(-\delta_{ij}) \dfrac{r_j}{r_{ij}}$ for $r_i < r_j$ in I_1. In this situation we always have $r_i \simeq r_j$, so that:

$$I_1 \simeq 0 \quad , \quad r_i \lesssim \{r_j\}_{\min}$$

The following integral is formed with {see (1.6.1)}:

$$\frac{r_j}{r_{ij}} = \sum_{n=0}^{\infty} \left(\frac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)} Y_j'^{(n)} \quad , r_i > r_j$$

$$= \sum_{n=0}^{\infty} \left(\frac{r_i}{r_j}\right)^{n} Y_i'^{(n)} Y_j'^{(n)} \quad , r_i < r_j$$

With, probably, a similar degree of approximation we then have:

$$I_2 \equiv \Delta X_{0i} \cdot \frac{1}{4\pi} \iint \frac{r_j}{r_{ij}} \,\mathrm{d}\Omega_j = \frac{R}{r_i} \Delta X_{0i} \quad , \quad r_i > r_j$$

$$= 1 \cdot \Delta X_{0i} \quad , \quad r_i < r_j$$

The way of derivation makes it permissible to pass to the limit $r_i \simeq r_j$, so that for the whole region where r_i is greater than a value somewhat smaller than $\{r_j\}_{\min}$:

$$I_1 + I_2 = \Delta X_{0i} \quad \ldots \quad (3.2.3)$$

in which the uncertainty in the choice of R is highly reduced.

Now write $I_1 + I_2$ in the following form:

$$\tfrac{1}{2}(I_1 + I_2) = \Delta X_{0i} \cdot \frac{1}{4\pi} \iint \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} \,\mathrm{d}\Omega_j + \frac{1}{4\pi} \iint \Delta X_{0j} \cdot \frac{-\delta_{ij}}{2} \frac{r_j}{r_{ij}} \,\mathrm{d}\Omega_j$$

With (1.7.1) and (3.2.3) this gives:

$$\frac{1}{4\pi} \iint \Delta X_{0j} \cdot (-\delta_{ij}) \frac{r_j}{r_{ij}} \,\mathrm{d}\Omega_j \quad \begin{cases} = \Delta X_{0i} & , P_i \text{ outside } S^* \\ = 0^{*)} & , P_i \text{ on } S^* \\ = -\Delta X_{0i} & , P_i \text{ inside } S^* \; \{r_i \lesssim r_j\} \end{cases} \qquad (3.2.4)$$

(3.2.4) with (3.2.1) and (3.1.7) then result in the Poisson integrals, with datumpoint P_0 on S*, hence $\dfrac{r_0 - r_j}{r_j} \simeq 0$:

---

*) In accordance with [PICK et al, 1973, p. 485].

P_0, P_j on S*	P_i outside S*	P_i on S* and $P_i \to P_0$	P_i inside S* $\{r_i \lesseqgtr r_j\}$	
$\frac{1}{4\pi}\iint \Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_j}\right)(-\delta_{ij})\frac{r_j}{r_{ij}}\,d\Omega_j =$	$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right) - \frac{r_0-r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)$	0	$-\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)$	(3.2.5′)
$\frac{1}{4\pi}\iint \Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)(-\delta_{ij})\frac{r_j}{r_{ij}}\,d\Omega_j =$	$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)$	0	$-\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)$	(3.2.5″)
$\frac{1}{4\pi}\iint \Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)(-\delta_{ij})\frac{r_j}{r_{ij}}\,d\Omega_j =$	$\Delta\left(\frac{\Psi_{0i}}{\Phi_0}-2\frac{r_0}{r_i}\right) - \frac{r_0-r_i}{r_i}\,\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$	0	$-\Delta\left(\frac{\Psi_{0i}}{\Phi_0}-2\frac{r_0}{r_i}\right)$	(3.2.5‴)

These Poisson integrals follow directly from the series expansion for difference quantities (3.1.7), which in turn follow from (3.1.1), i.e. Newton's gravitation theory.
The approximation of S* by the spherical surface $\overline{S}$, with the indetermination of the radius R formulated by (3.1.8), is not directly expressed in (3.2.5). This makes it difficult to understand fully the significance of the Bjerhammar sphere, the more so because numerical computations seem to suggest an optimal value for the radius. See [KRARUP, 1969] and subsequent connected publications by several authors.
The question arises what is the essential significance of (3.2.5). The Poisson integrals will be used in section 4 for the derivation of Green integrals, consequently the degree of approximation is the same. It follows that the relations (3.2.5) may be introduced as difference equations in the computational model of adjustment theory; their possible dependence on difference equations obtained from the Green integrals will have to be investigated. A handicap is the occurrence of difference quentities $\Delta\left(\frac{V_0}{\Phi_0}\right)$ and $\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$; therefore it is to be expected that the second relation in (3.2.5) will be the most useful one.

As far as I can see, it follows from the derivation that:

$$(3.2.5) \text{ is dependent on } (3.1.7) \quad \ldots\ldots \quad (3.2.6)$$

3.3 First degree terms

By means of geometric techniques the direction of the Z-axis can be chosen so as to be parallel to the axis of rotation of the earth to a sufficient degree. We have, however, no similar means to let the position of the geocentre coincide with the gravity centre of the earth. Here one must expect a deviation, see figure 3.3-1, with an order of magnitude:

$$\frac{r_C}{R} \lesssim 10^{-5} \quad \ldots\ldots \quad (3.3.1)$$

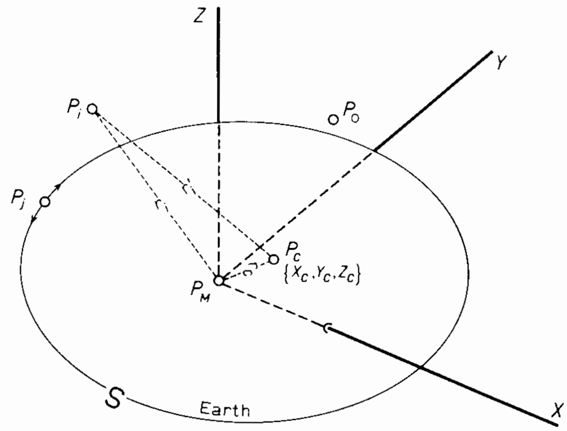


Fig. 3.3-1

Use the shortened notation:

$$X_iX_k + Y_iY_k + Z_iZ_k = \Sigma\, X_iX_k$$

then the following is valid [HOPFNER 1933, p. 296]:

$$\frac{1}{r_{ik}} = \frac{1}{r_i} + \frac{1}{r_i^3}\ \Sigma\, X_iX_k + \ldots$$

$$V_i = \iiint\limits_{\text{earth}} \frac{1}{r_{ik}}\ \Sigma\, \mathrm{d}M_k$$

$$X_k = X_C + (X_k - X_C) \quad , \quad \iiint (X_k - X_C)\, \mathrm{d}M_k = 0$$

or:
$$V_i = \frac{M}{r_i} \left\{ 1 + \Sigma\, \frac{X_i}{r_i}\, \frac{X_C}{r_i} + \ldots \right\}$$

and in accordance with (3.1.4):

$$\frac{V_i}{\Phi_0} = \frac{M}{\Phi_0 r_0}\, \frac{r_0}{r_i} \left\{ 1 + \frac{R}{r_i}\ \Sigma\, \frac{X_i}{r_i}\, \frac{X_C}{R} + \ldots \right\}$$

or:
$$\Sigma\, \frac{X_i}{r_i}\, \frac{X_C}{R} = \frac{X_i}{r_i}\, \frac{X_C}{R} + \frac{Y_i}{r_i}\, \frac{Y_C}{R} + \frac{Z_i}{r_i}\, \frac{Z_C}{R} = Y_i^{(1)} \quad \ldots\ldots\ldots \quad (3.3.2)$$

With (3.3.1) we have, see figure 3.3-1:

$$\frac{r_0}{r_i} - \frac{r_0'}{r_i'} = \frac{r_0}{r_i} \left\{ 1 - \frac{1 - \frac{r_0 - r_0'}{r_0}}{1 - \frac{r_i - r_i'}{r_i}} \right\} \simeq - \frac{r_0}{r_i} \left\{ \frac{r_i - r_i'}{r_i} - \frac{r_0 - r_0'}{r_0} \right\}$$

$$2\, r_i(r_i - r_i') \simeq r_i^2 - r_i'^2 = \Sigma\, X_i^2 - \Sigma\, (X_i - X_C)^2 = 2 \left\{ \Sigma\, X_iX_C - \tfrac{1}{2}\, \Sigma\, X_C^2 \right\}$$

Hence with (3.3.2):

$$\frac{r_0}{r_i} - \frac{r_0'}{r_i'} \simeq - \frac{r_0}{r_i} \left[\left\{ \frac{R}{r_i} Y_i^{(1)} - \frac{R}{r_0} Y_0^{(1)} \right\} - \frac{1}{2} \left\{ \left(\frac{R}{r_i}\right)^2 - \left(\frac{R}{r_0}\right)^2 \right\} \left(\frac{r_C}{R}\right)^2 \right] \qquad (3.3.3)$$

in which, in view of (3.3.1), the second term in the right hand member can be neglected when establishing the difference equation.
In accordance with the considerations in the final part of section 1.3, it is assumed that by geometric methods the direction of $\overrightarrow{P_M P_i}$ is determined with a higher precision than the modulus r_i. Then, with (X_C, Y_C, Z_C) unknown to start with, it follows from (3.3.2) that:

$$\Delta Y_i^{(1)} = Y_i^{(1)}$$

so that we have, from (3.3.3):

$\Delta \left(\frac{r_0}{r_i}\right) - \Delta \left(\frac{r_0'}{r_i'}\right) = - \frac{r_0}{r_i} \left\{ \frac{R}{r_i} \Delta Y_i^{(1)} - \frac{R}{r_0} \Delta Y_0^{(1)} \right\}$	(3.3.4)
$\Delta Y_i^{(1)} = Y_i^{(1)}$ from (3.3.2)	

From (3.3.4) it follows that the first-degree terms in (3.1.7) are exclusively due to the excentric position of P_M with respect to P_C. Its influence on further terms in (3.1.4), together with the influence of a small remaining influence in the orientation of the X,Y,Z-system, has been investigated in [AARDOOM, 1969]. With (3.3.1) this influence turned out to be of the order 10^{-8} or smaller.
This implies that the influence of (X_C, Y_C, Z_C) on (3.1.6) and (3.1.7) is practically confined to the first-degree terms. Thus it follows for (3.1.7) with respect to P_C that:

$$\boxed{\begin{aligned}
&\Delta \left(\frac{V_{0i}}{\Phi_0'} - \frac{r_0'}{r_i'} \right) - \frac{r_0 - r_i}{r_i} \Delta \left(\frac{V_0}{\Phi_0'} \right) = \frac{r_0}{r_i} \sum_{n=2}^{\infty} \left\{ \left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\} \\
&\Delta \left(\frac{\Phi_i'}{\Phi_0'} - \frac{r_0'}{r_i'} \right) = \frac{r_0}{r_i} \sum_{n=2}^{\infty} (n+1) \left\{ \left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\} \\
&\Delta \left(\frac{\Psi_{0i}'}{\Phi_0'} - 2\frac{r_0'}{r_i'} \right) - \frac{r_0 - r_i}{r_i} \Delta \left(\frac{\Psi_0'}{\Phi_0'} \right) = \frac{r_0}{r_i} \sum_{n=2}^{\infty} (n+2)(n+1) \left\{ \left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\}
\end{aligned}} \qquad (3.3.5)$$

This matter is relevant for formulae like (4.2.9) and (4.2.10).

3.4 Transition from gravitational to gravity potential

From section 1.7 it follows that for points connected with the earth we shall have to consider the transition from V to W.
Since the Z-axis and the axis of rotation of the earth are (to a sufficient degree) parallel but do not coincide one has to write:

$$V_i = W_i - C_i \quad , \quad C_i = \tfrac{1}{2}\,\omega^2 \left\{ (X_i - X_C)^2 + (Y_i - Y_C)^2 \right\} \qquad (3.4.1)$$

hence:

$$V_i = W_i - \omega^2 \left\{ \tfrac{1}{2}(X_i^2 + Y_i^2) - (X_i X_C + Y_i Y_C) + \tfrac{1}{2}(X_C^2 + Y_C^2) \right\}$$

$$\Phi_i = -\, r_i \frac{\partial V_i}{\partial r_i} = -\, r_i \frac{\partial W_i}{\partial r_i} + \omega^2 \left\{ (X_i^2 + Y_i^2) - (X_i X_C + Y_i Y_C) \right\}$$

$$\Psi_i = r_i^2 \frac{\partial^2 V_i}{\partial r_i^2} = r_i^2 \frac{\partial^2 W_i}{\partial r_i^2} - \omega^2 (X_i^2 + Y_i^2)$$

and with {compare (3.3.2)}:

$$\left.\begin{aligned}
&-\frac{\partial W_i}{\partial r_i} = -\frac{\partial W_i}{\partial h_i} \cos(h_i, r_i) = g_i \cos(h_i, r_i) \underset{\text{say}}{=} g_i' \\
&+\frac{\partial^2 W_i}{\partial r_i^2} = \frac{\partial^2 W_i}{\partial h_i^2} \cos^2(h_i, r_i) = q_i \cos^2(h_i, r_i) \underset{\text{say}}{=} q_i' \\
&\frac{X_i}{r_i}\frac{X_C}{R} + \frac{Y_i}{r_i}\frac{Y_C}{R} = \overline{Y}_i^{(1)} \\
&V_i = W_i - \omega^2 R^2 \left[\tfrac{1}{2} \frac{X_i^2 + Y_i^2}{R^2} - \frac{r_i}{R} \overline{Y}_i^{(1)} + \tfrac{1}{2} \frac{X_C^2 + Y_C^2}{R^2} \right] \\
&\Phi_i = r_i g_i' + \omega^2 R^2 \left[\frac{X_i^2 + Y_i^2}{R^2} - \frac{r_i}{R} \overline{Y}_i^{(1)} \right] \\
&\Phi_0^{-1} \simeq \frac{1}{r_0 g_0'} \left[1 - \frac{\omega^2 R^2}{r_0 g_0'} \left\{ \frac{X_0^2 + Y_0^2}{R^2} - \frac{r_0}{R} \overline{Y}_0^{(1)} \right\} \right] \\
&\Psi_i = r_i^2 q_i' - \omega^2 R^2 \frac{X_i^2 + Y_i^2}{R^2}
\end{aligned}\right\} \qquad (3.4.2)$$

With a slight simplification in coefficients of ω^2, such as:

$$\frac{W_{0i}}{r_0 g_0'} \doteq \frac{V_{0i}}{\Phi_0} \simeq \frac{r_0 - r_i}{r_i}$$

one obtains:

$$\frac{V_{0i}}{\Phi_0} - \left\{\frac{r_0 - r_i}{r_i}\right\}^{\text{appr}} \frac{V_0}{\Phi_0} = \frac{W_{0i}}{r_0 g_0'} - \left\{\frac{r_0 - r_i}{r_i}\right\}^{\text{appr}} \frac{W_0}{r_0 g_0'} +$$

$$- \frac{\omega^2 R^2}{r_0 g_0'} \left\{ \tfrac{1}{2} \left(\frac{X_i^2 + Y_i^2}{R^2} - \frac{r_0}{r_i} \frac{X_0^2 + Y_0^2}{R^2} \right) - \left(\frac{r_i}{R} \overline{Y}_i^{(1)} - \frac{r_0}{r_i} \frac{r_0}{R} \overline{Y}_0^{(1)} \right) \right\} \tag{3.4.3'}$$

$$\frac{\Phi_i}{\Phi_0} = \frac{r_i g_i'}{r_0 g_0'} +$$

$$- \frac{\omega^2 R^2}{r_0 g_0'} \left\{ \left(\frac{X_i^2 + Y_i^2}{R^2} - \frac{r_0}{r_i} \frac{X_0^2 + Y_0^2}{R^2} \right) - \left(\frac{r_i}{R} \overline{Y}_i^{(1)} - \frac{r_0}{r_i} \frac{r_0}{R} \overline{Y}_0^{(1)} \right) \right\} \tag{3.4.3''}$$

$$\frac{\Psi_{0i}}{\Phi_0} - \left\{\frac{r_0 - r_i}{r_i}\right\}^{\text{appr}} \frac{\Psi_0}{\Phi_0} = \left\{ \left(\frac{r_i}{r_0}\right)^2 \frac{r_0 q_i'}{g_0'} - \frac{r_0 q_0'}{g_0'} \right\} - \left\{\frac{r_0 - r_i}{r_i}\right\}^{\text{appr}} \frac{r_0 q_0'}{g_0'} +$$

$$- \frac{\omega^2 R^2}{r_0 g_0'} \left\{ \frac{X_i^2 + Y_i^2}{R^2} - \frac{r_0}{r_i} \frac{X_0^2 + Y_0^2}{R^2} \right\} . \quad \ldots\ldots\ldots\ldots \tag{3.4.3'''}$$

The angle (h,r) is the small angle δ from section 4.6:

$\delta \simeq 3 \cdot 10^{-3} \sin 2\varphi$, φ geographic latitude
$\delta^2 \simeq 10^{-5} \sin 2\varphi$, hence:
$1 \geq \cos(h,r) \geq 1 - 10^{-5} \sin 2\varphi$ . . . . (3.4.4')

Using this, one obtains { compare the quantity m in [HM, (2–100)]}:

$$\frac{\omega^2 R^2}{r_0 g_0'} \simeq 3 \cdot 10^{-3} \quad \ldots\ldots\ldots\ldots \tag{3.4.4''}$$

so that, in view of (1.2.3), the use of approximate values in the coefficients of $\frac{\omega^2 R^2}{r_0 g_0'}$ is justified.

With:

$$-\Delta \left(\frac{r_0}{r_i}\right) = - \frac{r_0}{r_i} \Delta \left(\ln \frac{r_0}{r_i}\right) = \frac{r_0}{r_i} \Delta \left(\ln \frac{r_i}{r_0}\right)$$

and similar slight simplifications as used with (3.4.3) one obtains:

$$\left.\begin{aligned}
-\Delta\left(\frac{r_0}{r_i}\right) &= \frac{r_0}{r_i}\,\Delta\left(\ln\frac{r_i}{r_0}\right)\\
\Delta\left(\frac{r_i}{r_0}\,\frac{g_i'}{g_0'}\right) &= \frac{r_0}{r_i}\left\{\Delta\left(\ln\frac{r_i}{r_0}\right)+\Delta\left(\ln\frac{g_i'}{g_0'}\right)\right\}\\
\Delta\left(\frac{r_i^2}{r_0^2}\,\frac{r_0q_i'}{g_0'}\right) &= 2\,\frac{r_0}{r_i}\left\{2\,\Delta\left(\ln\frac{r_i}{r_0}\right)+\Delta\left(\ln\frac{r_0q_i'}{g_0'}\right)\right\}
\end{aligned}\right\}\quad(3.4.5)$$

(X_C, Y_C, Z_C) remains unknown for the time being, with an order of magnitude (3.3.1). In view of (3.4.4), the terms with $\bar{Y}^{(1)}$ in (3.4.3) can therefore reach the same order of magnitude as (1.2.3), so that these terms have to be included in the difference equations. Then from (3.4.3) and 3.4.5) {omitting the suffix "appr" in the coefficients of difference quantities} one obtains:

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)=\Delta\left(\frac{W_{0i}}{r_0g_0'}\right)-\frac{r_0-r_i}{r_i}\,\Delta\left(\frac{W_0}{r_0g_0'}\right)+\frac{r_0}{r_i}\,\Delta\left(\ln\frac{r_i}{r_0}\right)+$$

$$+\frac{\omega^2R^2}{r_0g_0'}\left\{\frac{r_i}{R}\,\bar{Y}_i^{(1)}-\frac{r_0}{R}\left(1+\frac{r_0-r_i}{r_i}\right)\bar{Y}_0^{(1)}\right\}$$

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{r_0}{r_i}\left\{\Delta\left(\ln\frac{g_i'}{g_0'}\right)+2\,\Delta\left(\ln\frac{r_i}{r_0}\right)\right\}+$$

$$-\frac{\omega^2R^2}{r_0g_0'}\left\{\frac{r_i}{R}\,\bar{Y}_i^{(1)}-\frac{r_0}{R}\left(1+\frac{r_0-r_i}{r_i}\right)\bar{Y}_0^{(1)}\right\}$$

$$\Delta\left(\frac{\Psi_{0i}}{\Phi_0}-2\,\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\,\Delta\left(\frac{\Psi_0}{\Phi_0}\right)=2\,\frac{r_0}{r_i}\,\Delta\left(\ln\frac{r_0q_i'}{g_0'}.-\ln\frac{r_0q_0'}{g_0'}\right)+$$

$$-2\,\frac{r_0-r_i}{r_i}\,\Delta\left(\ln\frac{r_0q_0'}{g_0'}\right)+6\,\frac{r_0}{r_i}\,\Delta\left(\ln\frac{r_i}{r_0}\right)$$

and in view of (4.4.6) and (5.4.4):

$$\Delta\left(\frac{V_0}{\Phi_0}\right)=\Delta\left(\frac{W_0}{r_0g_0'}\right)+\frac{\omega^2R^2}{r_0g_0'}\,2\,\frac{r_0}{R}\,\bar{Y}_0^{(1)}$$

$$\Delta\left(\frac{\Psi_0}{\Phi_0}\right)=2\,\Delta\left(\ln\frac{r_0q_0'}{g_0'}\right)+\frac{\omega^2R^2}{r_0g_0'}\,2\,\frac{r_0}{R}\,\bar{Y}_0^{(1)}$$

(3.4.6′)

The replacement of V by W, and hence (3.4.6), is only relevant for points connected with the earth. In this case, (3.1.9) is valid for the coefficients of difference quantities. Therefore in (3.4.6) one may put:

$$\frac{r_0 - r_i}{r_i} = 0 \quad , \quad \frac{r_0}{r_i} = \frac{r_i}{R} = \frac{r_0}{R} = 1 \quad \ldots\ldots\ldots\ldots \quad (3.4.6'')$$

Further we have from (3.4.2) and (3.4.4′), with:

$\cos\alpha \; \mathrm{d}\,(\cos^{-1}\alpha) = -\mathrm{d}\,(\ln\cos\alpha) = \mathrm{tg}\,\alpha\, \mathrm{d}\alpha = \alpha\, \mathrm{d}(\alpha)$

$$\left.\begin{aligned}
&\Delta\left(\frac{W_{0i}}{r_0 g_0'}\right) = \sec(h_0,r_0)\,\Delta\left(\frac{W_{0i}}{r_0 g_0}\right) + \frac{W_{0i}}{r_0 g_0'}\,(h_0,r_0)\,\Delta\,(h_0,r_0)\\
&\Delta\left(\ln\frac{g_i'}{g_0'}\right) = \Delta\left(\ln\frac{g_i}{g_0}\right) - \{(h_i,r_i)\,\Delta\,(h_i,r_i) - (h_0,r_0)\,\Delta\,(h_0,r_0)\}\\
&\Delta\left(\ln\frac{r_0 q_i'}{g_0'} - \ln\frac{r_0 q_0'}{g_0'}\right) =\\
&= \Delta\left(\ln\frac{q_i'}{q_0'}\right) = \Delta\left(\ln\frac{q_i}{q_0}\right) - 2\,\{(h_i,r_i)\,\Delta\,(h_i,r_i) - (h_0,r_0)\,\Delta\,(h_0,r_0)\}\\
&\frac{\omega^2 R^2}{r_0 g_0'} = \frac{\omega^2 R^2}{r_0 g_0}\,\sec(h_0,r_0)
\end{aligned}\right\} \quad \ldots \quad (3.4.6''')$$

The angle (h,r) is small, and its effect on the above equations too, but it cannot be simply neglected. Its computation belongs to geometric geodesy, as sketched in section 1.3, but the computation of the approximate value belongs partly to gravimetric theory, depending on the potential model chosen. The angle (h,r) replaces the "deflection of the vertical", a concept which according to section 5.2 has no place in the theory developed here. To determine (h,r), one has to revert to astronomical measurements of latitude and longitude. For, the determination of the h-direction with respect to the geometric network by vertical angle measurement gives a $\sigma_{(h,r)} \simeq 1'' = 5\cdot10^{-6}$ rad. {assuming a standard deviation of estimable coordinate functions of $\lesssim 10^{-6}$}; the influence on (3.4.6‴), being $|\,(h,r)\,\sigma_{(h,r)}\,| \lesssim 1.5\cdot10^{-8}$, cannot be neglected. According to [HUSTI, 1978] astronomical measurements can attain $\sigma_{(h,r)} \simeq 0''.2 = 10^{-6}$ rad*) so that $|\,(h,r)\sigma_{(h,r)}\,| \lesssim 3\cdot10^{-9}$, and in view of (1.2.3) this can be neglected.
This does not mean that $|\,(h,r)\;\Delta\;(h,r)\,|$ can be neglected. For the h-direction is strongly influenced by irregularities in the mass distribution in the earth's crust around the station. The computation of the model of approximated values will have to be of such quality that

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*) In a discussion with Prof. W. TORGE and Prof. G. SEEBER in October 1978 in Hanover they pointed out that this estimate was too optimistic.
The author is in particular Prof. TORGE very thankful for his comments on the theory developed but for the time being it is preferred to make no changes in the manuscript.

$$\Delta\,(h,r) \lesssim 10^{-6} \text{ rad} \quad \ldots \quad (3.4.7')$$

to assure that the neglect of $|\ (h,r)\ \Delta\ (h,r)\ |$ is permissible. Here, too, it is evident, that the greatest difficulties for the application of this theory are presented by the computation of approximate values. In any case, the influence of (h_0,r_0) on $\Delta\left(\frac{W_{0i}}{r_0 g_0}\right)$ and on $\frac{\omega^2 R^2}{r_0 g_0}$ can be neglected.

Then one obtains from (3.4.6):

$r_i \simeq r_0 \simeq R$
$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) = \Delta\left(\frac{W_{0i}}{r_0 g_0}\right) + \Delta\left(\ln\frac{r_i}{r_0}\right) + \frac{\omega^2 R^2}{r_0 g_0}\left(\bar{Y}_i^{(1)} - \bar{Y}_0^{(1)}\right)$
$\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) = \Delta\left(\ln\frac{g_i}{g_0}\right) + 2\,\Delta\left(\ln\frac{r_i}{r_0}\right) - \frac{\omega^2 R^2}{r_0 g_0}\left(\bar{Y}_i^{(1)} - \bar{Y}_0^{(1)}\right)$
$\Delta\left(\frac{\Psi_{0i}}{\Phi_0} - 2\frac{r_0}{r_i}\right) = 2\,\Delta\left(\ln\frac{q_i}{q_0}\right) + 6\,\Delta\left(\ln\frac{r_i}{r_0}\right)$

(3.4.7″)

with G from (3.1.10) one can write without any practical objection:

$$\frac{\omega^2 R^2}{r_0 g_0} = \frac{\omega^2 R}{G} \quad \ldots \quad (3.4.7''')$$

Because V_{0i} is small compared to Φ_0, the following derivation can be useful:

$$\Delta\left(\frac{V_i}{\Phi_0}\right) = \frac{\Delta V_i}{\Phi_0} - \frac{V_i}{\Phi_0}\,\Delta(\ln\Phi_0) \simeq \frac{\Delta V_i}{\Phi_0} - \frac{r_0}{r_i}\,\Delta(\ln\Phi_0)$$

$$\frac{r_0}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{r_0}{r_i}\,\frac{\Delta V_0}{\Phi_0} - \frac{r_0}{r_i}\,\Delta(\ln\Phi_0)\ , \quad \text{hence:}$$

$$\Delta\left(\frac{V_{0i}}{\Phi_0}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{\Delta V_{0i}}{\Phi_0} - \frac{r_0 - r_i}{r_i}\,\frac{\Delta V_0}{\Phi_0} \quad \ldots \quad (3.4.8)$$

For an analogous derivation pertaining to $\Delta\left(\frac{W_{0i}}{r_0 g_0}\right)$, reference is made to (2.6.7).

4. GREEN INTEGRALS

4.1 Linearization

We start from the application of Green's third integral formula for the gravitational potential V, as elaborated in section 1.7. Points P_j are part of the geosurface S*, P_0 is the datum point, like in chapter 3, and is also part of S*.
Like in section 1.3, one has to take account of the fact that measurements can only provide differences of V. Provisionally, however, we shall operate with V itself within the model.
(1.7.5) with (1.7.6) gives:

$$V_i = \frac{1}{4\pi} \iint \left[-V_{ij} \frac{1+\delta_{ij}}{2} + r_j \left(-\frac{\partial V_j}{\partial r_j} \right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j$$

or, with the introduction of the quantity Φ and elimination of the units of length and mass according to (3.1.1), see also (3.1.4) and (3.1.5):

$$\frac{V_i}{\Phi_0} = \frac{1}{4\pi} \iint \left[-\frac{V_{ij}}{\Phi_0} \frac{1+\delta_{ij}}{2} + \frac{\Phi_j}{\Phi_0} \right] \frac{r_j}{r_{ij}} \, d\Omega_j \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (4.1.1)$$

With (1.7.1) and $V_{ij} = V_{0j} - V_{0i}$ this results in:

$$\frac{V_i}{\Phi_0} - \frac{p}{4\pi} \frac{V_{0i}}{\Phi_0} = \frac{1}{4\pi} \iint \left[-\frac{V_{0j}}{\Phi_0} \frac{1+\delta_{ij}}{2} + \frac{\Phi_j}{\Phi_0} \right] \frac{r_j}{r_{ij}} \, d\Omega_j$$

After the introduction of a consistent model of approximate values according to section 1.2, this equation can be linearized. Here it can be noted that (1.7.1) is also valid for the model of approximate values, hence:

$$\Delta p = 0 \quad . \; (4.1.2)$$

Again omitting the suffix "appr" for functions of approximate values in the coefficients of Δ-quantities, one obtains:

$$\Delta \left(\frac{V_i}{\Phi_0} \right) - \frac{p}{4\pi} \cdot \Delta \left(\frac{V_{0i}}{\Phi_0} \right) = \frac{1}{4\pi} \iint \left[-\frac{1+\delta_{ij}}{2} \Delta \left(\frac{V_{0j}}{\Phi_0} \right) + \Delta \left(\frac{\Phi_j}{\Phi_0} \right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j +$$
$$+ \frac{1}{4\pi} \iint \left[-\frac{V_{0j}}{\Phi_0} \Delta \left(\frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} \right) + \frac{\Phi_j}{\Phi_0} \Delta \left(\frac{r_j}{r_{ij}} \right) \right] d\Omega_j \; . \; . \; . \; (4.1.3)$$

If the approximate values are close, the coefficients $\frac{V_{0j}}{\Phi_0}$ and $\frac{\Phi_j}{\Phi_0}$ can be computed from the zero-order terms of (3.1.4) and (3.1.5). This means that the special harmonic function [HM, p. 37] is used for the approximation of these coefficients:

$$V_i = \frac{M}{r_i} \quad \ldots \quad (4.1.4)$$

This gives with (3.1.8):

$$\left.\begin{aligned} \frac{V_{0i}}{\Phi_0} &= \frac{r_0}{r_i} - \frac{r_0}{r_0} = \frac{r_0 - r_i}{r_i} \quad , \quad \frac{V_{0j}}{\Phi_0} \simeq 0 \\ \frac{\Phi_i}{\Phi_0} &= \frac{r_0}{r_i} \quad , \quad \frac{\Phi_j}{\Phi_0} \simeq 1 \end{aligned}\right\} \quad \ldots \quad (4.1.5)$$

Now in the second integral in the right hand member of (4.1.3) one has only to calculate the influence of $\Delta\left(\frac{r_j}{r_{ij}}\right)$. But a more convenient device is to apply (4.1.3) to (4.1.4). One then obtains an identity because only rays r occur:

$$\Delta\left(\frac{r_0}{r_i}\right) - \frac{p}{4\pi} \cdot \Delta\left(\frac{r_0}{r_i} - 1\right) \equiv \frac{1}{4\pi} \iint \left[- \frac{1 + \delta_{ij}}{2} \Delta\left(\frac{r_0}{r_j} - 1\right) + \Delta\left(\frac{r_0}{r_j}\right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j +$$

$$+ \frac{1}{4\pi} \iint \left[- \frac{r_0 - r_j}{r_j} \Delta\left(\frac{1 + \delta_{ij}}{2} \frac{r_j}{r_{ij}}\right) + \frac{r_0}{r_j} \Delta\left(\frac{r_j}{r_{ij}}\right) \right] d\Omega_j \quad (4.1.6)$$

Subtracting (4.1.6) from (4.1.3) results then with (4.1.5) in the elimination of the second integral in the right hand member of (4.1.3) $\{ \Delta\left(\frac{r_0}{r_i} - 1\right) = \Delta\left(\frac{r_0}{r_i}\right) \}$:

$$\Delta\left(\frac{V_i}{\Phi_0} - \frac{r_0}{r_j}\right) - \frac{p}{4\pi} \cdot \Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) = \frac{1}{4\pi} \iint \left[- \frac{1 + \delta_{ij}}{2} \Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \right.$$

$$\left. + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j$$

The application of the Poisson integral (3.2.5′) makes it possible to eliminate p, in view of (1.7.1) and (3.1.9):

$$\Delta\left(\frac{V_i}{\Phi_0} - \frac{r_0}{r_j}\right) - \frac{1}{2} \left\{ \Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i} \Delta\left(\frac{V_0}{\Phi_0}\right) \right\} =$$

$$= \frac{1}{4\pi} \iint \left[- \frac{1}{2} \Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) \right] \frac{r_j}{r_{ij}} \, d\Omega_j$$

and with $i \to 0$:

$$\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{1}{4\pi}\iint\left[-\tfrac{1}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{0j}}\,d\Omega_j \tag{4.1.7}$$

Subtraction results in:

$$\tfrac{1}{2}\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) + \frac{r_0 - r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)\right\} =$$

$$= \frac{1}{4\pi}\iint\left[-\tfrac{1}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\left(\frac{r_j}{r_{ij}} - \frac{r_j}{r_{0j}}\right)d\Omega_j$$

Subtracting $\dfrac{r_0 - r_i}{r_i}\Delta\left(\dfrac{V_0}{\Phi_0}\right)$, one obtains with (4.1.7):

$$\boxed{\begin{aligned}&\tfrac{1}{2}\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)\right\} =\\ &= \frac{1}{4\pi}\iint\left[-\tfrac{1}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right)d\Omega_j\\ &P_i \text{ outside, on or inside } S^*;\ P_0,\ P_j \text{ on } S^*\end{aligned}} \tag{4.1.8}$$

The integral equation (4.1.8) is the result of many years of trying; frankly, the author is not quite sure that the derivation is correct for the cases that P_i is on or inside S*. The further elaboration will therefore only apply to the case that P_i is outside S*, and the other cases will here for brevity be considered as a limiting situation.

Nevertheless the linearization executed in this manner is more complete than the linearization of Molodenskii's problem [HM, section 8-5], although there the derivations are mathematically more reliable. The most remarkable thing is that (4.1.8) does not give any indication of an "ill-posed problem".

With the aid of (1.4.3), the kernel of (4.1.8) can be expanded in the following series $\left\{Y'^{(0)} = 1\right\}$:

$$\left.\begin{aligned}&\left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right) = \frac{r_j}{r_i}\sum_{n=1}^{\infty}\left\{\left(\frac{r_j}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{r_j}{r_0}\right)^n Y_0'^{(n)}\right\}Y_j'^{(n)};\\ &\qquad r_i > r_j \quad , \quad r_0 > r_j\ (?)\end{aligned}\right\} \tag{4.1.9}$$

This kernel has a great practical significance. For, if P_0 is chosen in a regional area of points P_i

on S* {the above-mentioned limiting situation}, then the influence of points P_j situated further from P_0 almost vanishes {"the influence of distant zones", [HM, section 7-4]}.

(4.1.8) is a basic relation, which can be used as difference equation in the adjustment- or computing model. This relation may therefore be rearranged or transformed.

For brevity, introduce: $-\dfrac{\partial V_i}{\partial r_i} \equiv \gamma_i$.

Then:

$$\Delta\left(\frac{\Phi_i}{\Phi_0}\right) = \Delta\left(\frac{r_i}{r_0}\cdot\frac{\gamma_i}{\gamma_0}\right) = \frac{r_i}{r_0}\cdot\Delta\left(\frac{\gamma_i}{\gamma_0}\right) + \frac{\gamma_i}{\gamma_0}\cdot\Delta\left(\frac{r_i}{r_0}\right)$$

and with:

$$\Delta\left(\frac{r_i}{r_0}\right) = \frac{r_i}{r_0}\,\Delta\left(\ln\frac{r_i}{r_0}\right) = -\frac{r_i}{r_0}\,\Delta\left(\ln\frac{r_0}{r_i}\right) = -\left(\frac{r_i}{r_0}\right)^2\Delta\left(\frac{r_0}{r_i}\right)$$

and, in view of (3.1.5): $$\frac{\gamma_i}{\gamma_0} = \frac{r_0}{r_i}\,\frac{\Phi_i}{\Phi_0} \simeq \left(\frac{r_0}{r_i}\right)^2$$

$$\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) = \frac{r_i}{r_0}\,\Delta\left(\frac{\gamma_i}{\gamma_0}\right) - 2\,\Delta\left(\frac{r_0}{r_i}\right) \quad \text{. . . . .(4.1.10)}$$

If, consequently, (4.1.8) is written in the form:

$$\frac{1}{2}\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)\right\} =$$

$$= \frac{3}{4\pi}\iint\left[\frac{1}{2}\,\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \frac{1}{3}\left\{-2\,\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\}\right]\left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\,\frac{r_j}{r_{0j}}\right) d\Omega_j$$

P_i outside, on or inside S*; P_0, P_j on S*

(4.1.11)

then one obtains with (4.1.10) that $\Delta\left(\dfrac{r_0}{r_j}\right)$ is eliminated in {for the series see (3.1.7)}:

$$-2\,\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) =$$

$$= \frac{r_0}{r_j}\sum_{n=2}^{\infty}(n-1)\left\{\left(\frac{R}{r_j}\right)^n \Delta Y_j^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)}\right\} \quad \text{. . . (4.1.12)}$$

It must be emphasized that (4.1.8) and (4.1.11) are two ways of expressing the same difference equation. The so-called "solutions" of these integral equations can therefore only result in dependent solutions.
Furthermore, (4.1.8) emerged from (4.1.3) and (3.2.5′) and only one relation (4.1.8) is to be used further.
Therefore:

$$\left.\begin{array}{l}(4.1.8)\ \text{independent of}\ (3.2.5')\\ (4.1.11)\ \text{dependent on}\ \ (4.1.8)\end{array}\right\} \quad \ldots\ldots \quad (4.1.13)$$

4.2 Solution of integral equations

The approach used with (3.1.8) is again applied:

$$r_j \simeq R \quad \ldots\ldots \quad (4.2.1)$$

In order to avoid difficulties with divergent series we provisionally assume:

$$r_i\,, r_0 > \{r_j\}_{\max} \quad \ldots\ldots \quad (4.2.2)$$

However, in that case $\Delta \left\{\dfrac{V_{0j}}{\Phi_0} - \dfrac{r_0}{r_j}\right\}$ in (4.1.8) must be supplemented with:

$$-\frac{r_0 - r_j}{r_j}\,\Delta\left(\frac{V_0}{\Phi_0}\right) \simeq -\frac{r_0 - R}{R}\,\Delta\left(\frac{V_0}{\Phi_0}\right)$$

but this constant may always be added because there is no zero-degree term in (4.1.9).
Put:

$$\left.\begin{array}{ll}\Delta\left(\dfrac{V_{0i}}{\Phi_0} - \dfrac{r_0}{r_i}\right) - \dfrac{r_0 - r_i}{r_i}\,\Delta\left(\dfrac{V_0}{\Phi_0}\right) & = \Delta X_{0i}\\[2ex] \Delta\left(\dfrac{\Phi_j}{\Phi_0} - \dfrac{r_0}{r_j}\right) & = \Delta\bar{X}_{0j}\end{array}\right\} \quad \ldots\ldots \quad (4.2.3)$$

Then (4.1.8) becomes with (4.1.9), on account of (4.2.1):

$$\tfrac{1}{2}\,\Delta X_{0i} = \tfrac{1}{4}\,\frac{R}{r_i}\sum_{n=1}^{\infty}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}(K_n + I_n) \quad \ldots \quad (4.2.4)$$

$$K_n = \iint Y_j'^{(n)}\left(-\tfrac{1}{2}\Delta X_{0j}\right) d\Omega_j \quad , \quad I_n = \iint Y_j'^{(n)}\,\Delta\bar{X}_{0j}\, d\Omega_j$$

Substitution of (4.2.4) in the equation for K_n gives:

$$K_n = -\frac{1}{4\pi}\iint \sum_{\bar{n}=1}^{\infty} \left\{ Y_j'^{(\bar{n})} - \left(\frac{R}{r_0}\right)^{\bar{n}} Y_0'^{(\bar{n})} \right\} (K_{\bar{n}} + I_{\bar{n}})\, Y_j'^{(n)}\, d\Omega_j$$

$$K_n = -\frac{1}{2n+1}(K_n + I_n)$$

$$K_n = -\frac{1}{2(n+1)} I_n \quad , \quad K_n + I_n = \frac{2n+1}{2(n+1)} I_n \quad , \quad n > 0$$

Substitution of $K_n + I_n$ and of I_n in (4.2.4) gives:

$$\Delta X_{0i} = \frac{1}{4\pi}\iint \frac{R}{r_i} \sum_{n=1}^{\infty} \frac{2n+1}{n+1} \left\{ \left(\frac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)} \right\} Y_j'^{(n)}\, \Delta\overline{X}_{0j}\, d\Omega_j \qquad (4.2.5)$$

With:

$$-2\Delta X_{0j} + \Delta\overline{X}_{0j} = \Delta\overline{\overline{X}}_{0j} \quad \ldots\ldots\ldots\ldots \quad (4.2.6)$$

one can similarly write for (4.1.11):

$$\tfrac{1}{2}\Delta X_{0i} = \frac{3}{4\pi}\frac{R}{r_i}\sum_{n=1}^{\infty} \left\{ \left(\frac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)} \right\} (\overline{K}_n + \tfrac{1}{3}\overline{I}_n)$$

$$\overline{K}_n = \iint Y_j'^{(n)} (\tfrac{1}{2}\Delta X_{0j})\, d\Omega_j \quad , \quad \overline{I}_n = \iint Y_j'^{(n)}\, \Delta\overline{\overline{X}}_{0j}\, d\Omega_j \quad , \quad n > 0$$

$$\overline{K}_n = \frac{1}{2(n-1)}\overline{I}_n \quad , \quad \overline{K}_n + \tfrac{1}{3}\overline{I}_n = \frac{2n+1}{6(n-1)}\overline{I}_n \quad , \quad n > 1$$

$\overline{K}_1$ proves to be indeterminate, which according to (4.1.12) corresponds with $\overline{I}_1 = 0$. The solution is therefore determinate up to a $Y^{(1)}$-term:

$$\Delta X_{0i} = Y^{(1)}\text{-term} + \frac{1}{4\pi}\iint \frac{R}{r_i}\sum_{n=2}^{\infty}\frac{2n+1}{n-1} \left\{ \left(\frac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)} \right\} Y_j'^{(n)}\, \Delta\overline{\overline{X}}_{0j}\, d\Omega_j \qquad (4.2.7)$$

(4.2.5) and (4.2.7) can be checked by substitution of the series for $\Delta\overline{X}_{0j}$ and $\Delta\overline{\overline{X}}_{0j}$ from (3.1.7) and (4.1.12). Then it appears from a comparison with the series for ΔX_{0i} from (3.1.7) that:

$$Y^{(1)}\text{-term in (4.2.7)} = \frac{r_0}{r_i}\left\{ \frac{R}{r_i}\Delta Y_i^{(1)} - \frac{R}{r_0}\Delta Y_0^{(1)} \right\} \quad \ldots\ldots \quad (4.2.8)$$

The series in (4.2.5) and (4.2.7) present no difficulties in the limiting situation: $r_i, r_0 \simeq R$.
Now we replace the ΔX-notation by (4.2.3) and (4.2.6). We also use the first difference equation in (3.1.7), by which according to (3.2.6) the Poisson integral (3.2.5′) ceases to be an independent difference equation {compare (4.1.13)}. From (4.2.5) and (4.2.7) with (4.2.8) then follows:

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right) =$$

$$\text{or: } \frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}+\frac{r_0}{r_i}\sum_{n=2}^{\infty}\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\}=$$

$$=\frac{1}{4\pi}\iint S_{0i;j}^{(n+1)}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)d\Omega_j \qquad (4.2.9)$$

$$S_{0i;j}^{(n+1)}=\frac{R}{r_i}\sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}Y_j'^{(n)}$$

P_i outside, on or inside S*; P_0, P_j on S*; $r_0, r_j \simeq R$

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)-\frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}=$$

$$\text{or: } \frac{r_0}{r_i}\sum_{n=2}^{\infty}\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} =$$

$$=\frac{1}{4\pi}\iint S_{0i;j}^{(n-1)}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}d\Omega_j \qquad (4.2.10)$$

$$S_{0i;j}^{(n-1)}=\frac{R}{r_i}\sum_{n=2}^{\infty}\frac{2n+1}{n-1}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}Y_j'^{(n)}$$

Compare [HOTINE, 1969, (29.17) and (29.14) respectively].
In the right hand member of (4.2.10) one recognizes {apart from a small correction} the difference of two Stokes' integrals [HM, sections 2-16, 17]. There is no doubt about $Y^{(0)}$- and $Y^{(1)}$-terms [HM, sections 2-18, 19]; the connection of the situation P_i outside S* with points P_0 and P_j on S* does not present any difficulty [HM, section 2-20].

4.3 Interpretation, at sea and on land

To avoid interruption of the argumentation, some complementary derivations have been deferred to section 4.4. Now we shall first give an interpretation of the difference equations (4.2.9) and (4.2.10). Two main situations are to be distinguished: first, measurements above and on the surface of the sea, and second, measurements above and on the continents.

I. In recent years it has become clear that the surface of the sea does not coincide with an equipotential surface, the differences may be several metres. This means that potential differences between points on the surface of the sea, far removed from the land, cannot be measured. Therefore (4.2.10) cannot be applied, so that (4.2.9) has to be further considered. Both equations are transformations of the same basic equation (4.1.8), and because of the form of the kernel they have possibilities for regional application, see (4.1.9). The two equations have in common that use can be made of $\Delta Y^{(n)}$ from satellite dynamics [HM, section 2–20].
"At" sea level, gravity can be measured, whereas the rays to these points can be derived from satellite altimetry. From the right hand member of (4.2.9) and from (3.4.6) it follows that standard deviations of $\Delta \ln g$ and $\Delta \ln r$ should have the same order of magnitude. At present this seems to be realized on the level of 10^{-6} {from $\sigma_g \simeq 1$ mgal, $\sigma_r \simeq \frac{1}{2} R\, 10^{-6} \simeq 3$ m}, although for practical application one will need 10^{-7} {or possibly 10^{-8}}.
A remark on satellite altimetry:
From the orbit data of the satellite, values $\bar{r}_j$ are computed, see figure 4.3-1. Assuming good calibration with respect to orbit data, the satellite altimeter produces values h_j with an instrumental length scale factor $(1 + \Delta\lambda_h)$.
Then, if $\Delta\lambda_h$ is small*):

$$\ln r_j = \ln \left\{ \bar{r}_j - (1 + \Delta\lambda_h)\, h_j \right\} = \ln (\bar{r}_j - h_j) - \left(\frac{h_j}{r_j}\right)^{\text{appr}} \Delta\lambda_h$$

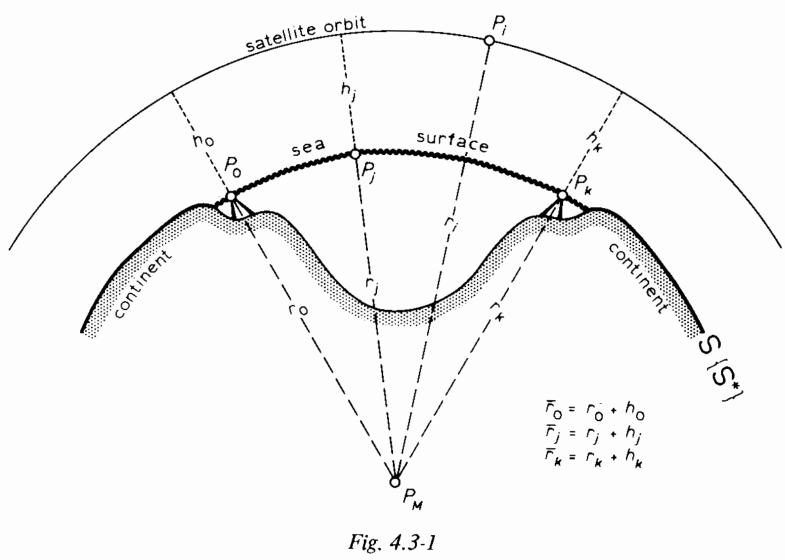

Fig. 4.3-1

*) For the definition of "corrected" heights h, see section 4.6

If now P_0 is chosen on or near the coast of the continent {and in any case connectible by spirit levelling}, then:

$$\ln \frac{r_j}{r_0} = \ln \frac{\bar{r}_j - h_j}{\bar{r}_0 - h_0} - \left(\frac{h_j}{r_j} - \frac{h_0}{r_0}\right)^{\text{appr}} \Delta\lambda_h \quad \ldots \ldots \quad (4.3.1')$$

If h does not vary much, the second term in the right hand member of (4.3.1′) can perhaps be considered as non-stochastic, or it can be neglected, so that:

$$\Delta \left(\ln \frac{r_j}{r_0}\right) = \Delta \left(\ln \frac{\bar{r}_j - h_j}{\bar{r}_0 - h_0}\right) \quad \ldots \ldots \quad (4.3.1'')$$

Similar to P_0, one can choose one or more points P_k on or near continental parts or islands, so that seas or oceans are bridged:

$$\ln \frac{r_k}{r_0} = \ln \frac{\bar{r}_k - h_k}{\bar{r}_0 - h_0} - \left(\frac{h_k}{r_k} - \frac{h_0}{r_0}\right)^{\text{appr}} \Delta\lambda_h \quad \ldots \ldots \quad (4.3.2')$$

$$\Delta \left(\ln \frac{r_k}{r_0}\right) = \Delta \left(\ln \frac{\bar{r}_k - h_k}{\bar{r}_0 - h_0}\right) \quad \ldots \ldots \quad (4.3.2'')$$

Now $\dfrac{r_k}{r_0}$ can also be determined {directly or indirectly} by e.g. geometric satellite methods, so that (4.3.2′) could also be used, if necessary, for the determination of a remaining small value $\Delta\lambda_h$.

If now we start from observations of $\Delta Y_i^{(n)} \{, \Delta Y_0^{(n)}\}$, $\dfrac{g_j}{g_0}$, $\dfrac{r_j}{r_0}$, and eliminate the $\Delta Y^{(1)}$-term via (4.4.12″), then (4.2.9) gives the contribution to the condition model of an adjustment procedure:

$\dfrac{r_0}{r_i} \sum\limits_{n=2}^{\infty} \left\{ \left(\dfrac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\dfrac{R}{r_0}\right)^n \Delta Y_0^{(n)} \right\} - \dfrac{1}{4\pi} \iint S'^{(n+1)}_{0i;j} \Delta \left(\dfrac{\Phi_j}{\Phi_0} - \dfrac{r_0}{r_j}\right) d\Omega_j = 0$	
$S'^{(n+1)}_{0i;j} = \dfrac{R}{r_i} \sum\limits_{n=2}^{\infty} \dfrac{2n+1}{n+1} \left\{ \left(\dfrac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\dfrac{R}{r_0}\right)^n Y_0'^{(n)} \right\} Y_j'^{(n)}$	(4.3.3)
P_i in satellite orbit; P_0, P_j on S^*; $r_0 \simeq r_j \simeq R$	

Because of the elimination of the $\Delta Y^{(1)}$- and $Y'^{(1)}$-terms, P_M could be replaced by P_C in figure 4.3-1, which might simplify the application of (4.3.1) and (4.3.2) {orbit computations are made with respect to P_C}.
By choosing P_i as a point of the satellite orbit {e.g. over the middle of the regional area considered} one meets the difficulty that dynamical satellite methods can only determine a limited number of $\Delta Y^{(n)}$-terms. But this interpretation of (4.3.3) is only given with reserve; a deeper analysis of methods and results of satellite techniques still has to be made.

If an adjustment has been made, one can afterwards again follow the course to compute via (4.2.9) the potential differences between points on the surface of the sea, by which one now also obtains $\frac{r_j}{r_0}$ from satellite altimetry {compare [MORITZ, 1974, (5-13)]}:

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint S_{0i;j}^{(n+1)}\,\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\,d\Omega_j$$

$$\Delta\left(\frac{V_{0i}}{\Phi_0}\right)=\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)+\Delta\left(\frac{r_0}{r_i}\right)$$

$$S_{0i;j}^{(n+1)}=\frac{R}{r_i}\sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}Y_j'^{(n)} \qquad \ldots\ldots\ (4.3.4')$$

P_i, P_0, P_j on S^*; $r_i \simeq r_j \simeq r_0 \simeq R$

one can also proceed like in (4.3.3) by taking the $Y^{(1)}$-term outside the integral sign via (4.4.12″). One obtains:

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0}{r_i}\left\{\frac{R}{r_i}\,\Delta Y_i^{(1)}-\frac{R}{r_0}\,\Delta Y_0^{(1)}\right\}=$$

$$=\frac{1}{4\pi}\iint S_{0i;j}'^{(n+1)}\,\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_i}\right)\,d\Omega_j \qquad \ldots\ (4.3.4'')$$

P_i, P_0, P_j on S^*; $r_i \simeq r_j \simeq r_0 \simeq R$

Then (3.3.5) is applied, so that one actually computes:

$$\Delta\left(\frac{V_{0i}}{\Phi_0'}-\frac{r_0'}{r_i'}\right)\quad \text{with respect to } P_C.$$

A situation similar to the one in (4.3.6) is then obtained. How this should be elaborated for practical application is a matter for further research – but the same applies to many other aspects of the present theory.
It seems to be that the connection with satellite altimetry, as sketched, has several points of contact with the interesting studies by R. S. MATHER, although at first sight these contacts are not clearly seen. See e.g. [MATHER, 1978] and the references there given.
An interesting situation is when for P_i the special points P_k are chosen, e.g. in (4.3.2). This means the transfer of potential from continent to continent {or island}, i.e. the now still missing possibility to connect continental systems of spirit levelling. But perhaps the controversy between geodetic and oceanic levelling [FISCHER, 1975] might be clarified too. See also [BRENNECKE and GROTEN, 1977, p. 50].

II. For points on land one has already the disposal of results from spirit levelling and gravity measurement guaranteeing standard deviations of 10^{-8}. This certainly does not apply to $\frac{r_i}{r_0}$, so that it is important to determine $\Delta\left(\frac{r_0}{r_i}\right)$. This means that (4.2.10) has to be applied, where $\Delta\left(\frac{r_0}{r_j}\right)$ has been eliminated from the right hand member.
On the analogy of (4.3.3) one obtains for the contribution to the condition model:

$$\frac{r_0}{r_i}\sum_{n=2}^{\infty}\left\{\left(\frac{R}{r_i}\right)^n \Delta Y_i^{(n)} - \left(\frac{R}{r_0}\right)^n \Delta Y_0^{(n)}\right\} + - \frac{1}{4\pi}\iint S_{0i;j}^{(n-1)}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\} d\Omega_j = 0 \qquad (4.3.5)$$

P_i in satellite orbit; P_0, P_j on S^*; $r_0 \simeq r_j \simeq R$

Then, with $\frac{V_{0i}}{\Phi_0}$ from spirit levelling, {"STOKES"}:

$$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) = \frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)} - \frac{R}{r_0}\Delta Y_0^{(1)}\right\} + + \frac{1}{4\pi}\iint S_{0i;j}^{(n-1)}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\} d\Omega_j \qquad (4.3.6)$$

$$\Delta\left(\frac{r_0}{r_i}\right) = -\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) + \Delta\left(\frac{V_{0i}}{\Phi_0}\right)$$

P_i, P_0, P_j on S^*; $r_i \simeq r_0 \simeq r_j \simeq R$

Practical application of (4.3.6) is only possible when (X_C, Y_C, Z_C) are known or have been computed, e.g. via (4.4.10) or (4.4.11).
Of course, one has to substitute the formulae (3.4.6) or (3.4.7) into (4.3.3.) – (4.3.6).

III. A problem is the sequence of application of (4.3.4) and (4.3.6), because the index j runs over the whole surface of the earth. Perhaps the solution of this problem follows from the following rough estimates of standard deviations*):

Formula		variate	standard deviation	
			at sea	on land
(4.3.4)	if	$\Delta\left(\ln \frac{g_j}{g_0}\right)$	10^{-6}	10^{-8}
		$\Delta\left(\ln \frac{r_j}{r_0}\right)$	10^{-7}	10^{-6}
	then	$\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)$	10^{-6}	10^{-6}
		$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right)$	10^{-6}	
		$\Delta\left(\frac{V_{0i}}{\Phi_0}\right)$	10^{-6}	
(4.3.6)	if	$\Delta\left(\frac{W_{0j}}{g_0 r_0}\right)$		10^{-8}
		$\Delta\left(\ln \frac{g_j}{g_0}\right)$		10^{-8}
	then	$-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)$	10^{-6}	10^{-8}
		$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right)$		$\lesssim 10^{-8}$
		$\Delta\left(\frac{r_0}{r_i}\right)$		$\lesssim 10^{-8}$

*) One or two decimal places may be lost by deriving block means taking means over sub-domains, some decimal places may be gained by the smoothing properties of the integral formulae [MEISSL, 1971].

This indicates that *first* (4.3.4) should be {regionally} applied to a sea area with an adjacent strip of land, and *afterwards* (4.3.6) should be {regionally} applied to a land area with an adjacent strip of sea.

4.4 Constants

I. Start from Green's second integral formula [MAGNIZKI et al, 1964, p. 52]:

$$\iiint_{\text{earth}} (U_k \nabla^2 V_k - V_k \nabla^2 U_k)\, \mathrm{d}V_k = \iint \left(U_j \frac{\partial V_j}{\partial n_j} - V_j \frac{\partial U_j}{\partial n_j} \right) \mathrm{d}S_j \quad . \; . \; . \; (4.4.1)$$

Choose: $U = 1$, $V =$ gravitational potential,

hence: $\nabla^2 U = 0, \nabla^2 V = -4\pi \dfrac{\mathrm{d}M}{\mathrm{d}v}$.

Then, from (4.4.1) {compare the derivation of (1.7.1)}:

$$M = \frac{1}{4\pi} \iint \left(- \frac{\partial V_j}{\partial n_j} \right) \mathrm{d}S_j = \frac{1}{4\pi} \iint r_j^2 \left(- \frac{\partial V_j}{\partial r_j} \right) \mathrm{d}\Omega_j =$$

$$= \frac{1}{4\pi} \iint r_j \, \Phi_j \, \mathrm{d}\Omega_j$$

$$\frac{M}{\Phi_0 r_0} = \frac{1}{4\pi} \iint \frac{r_j}{r_0} \frac{\Phi_j}{\Phi_0} \mathrm{d}\Omega_j \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4.4.2)$$

For linearization, apply the same device as in (4.1.3) – (4.1.8) {P_0, P_j on S or S*}:

$$\boxed{\Delta \left(\frac{M}{\Phi_0 r_0} \right) = \frac{1}{4\pi} \iint \frac{r_j}{r_0} \, \Delta \left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j} \right) \mathrm{d}\Omega_j} \quad . \; . \; . \; . \; . \; . \; . \; . \; (4.4.3)$$

which agrees with the series in (3.1.6) and (3.1.7). For the computation the formula concerned still has to be substituted into (3.4.6) or (3.4.7). One needs gravity- and length ratios, and (X_C, Y_C). A practical inconvenience is that integration extends over the whole earth.

But (4.4.3) can also be written in the following form:

$$\Delta \left(\frac{M}{\Phi_0 r_0} \right) = \frac{1}{4\pi} \iint \frac{r_j}{r_0} \left[2\Delta \left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j} \right) + \right.$$

$$\left. + \left\{ -2\Delta \left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j} \right) + \Delta \left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j} \right) \right\} \right] \mathrm{d}\Omega_j$$

and (4.2.10) can be substituted into this. This gives {compare [HM, (2-188)]}:

$$\Delta\left(\frac{M}{\Phi_0 r_0}\right) = -2\,\frac{R}{r_0}\left[\Delta Y_0^{(1)} + \frac{1}{4\pi}\iint \left\{S_{0;j}^{(n-1)} - \tfrac{1}{2}\right\}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\} d\Omega_j\right]$$

$$S_{0;j}^{(n-1)} = \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\, Y_j'^{(n)} \quad ; \quad r_0 \simeq r_j \simeq R \tag{4.4.4}$$

One now needs levelling- and gravity measurements and also (X_C, Y_C, Z_C).

With $\dfrac{M}{\Phi_0 r_0} \simeq 1$ one obtains:

$$\Delta\left(\frac{M}{\Phi_0 r_0}\right) = \Delta\left(\ln\frac{M}{\Phi_0 r_0}\right) = \Delta(\ln M^*) - \Delta(\ln \Phi_0^* r_0^*)$$

in which the $*$-sign serves as a reminder that these quantities have to be defined in some system {choice of man-made units}.

The application of section 3.4 gives {compare [HM, p. 107]}:

$$\Delta(\ln M^*) = \Delta\left(\frac{M}{\Phi_0 r_0}\right) + \Delta(\ln g_0^* \cos\{h_0, r_0\}) + 2\Delta(\ln r_0^*) + {} - \frac{\omega^2 R}{g_0 \cos\{h_0, r_0\}}\,\overline{Y}_0^{(1)} \tag{4.4.5'}$$

If so desired, one can write M as $k\overline{M}$, k being the gravitational constant:

$$\Delta(\ln \overline{M}^*) = \Delta(\ln M^*) - \Delta(\ln k^*) \tag{4.4.5''}$$

II. Start from (4.1.7):

$$\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{1}{4\pi}\iint\left[-\tfrac{1}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{0j}}\, d\Omega_j$$

and substitute (4.2.9) and (1.4.3) into this. This gives a formula of the same type as (4.4.3):

$$\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{R}{r_0}\cdot\frac{1}{4\pi}\iint\left\{S_{0;j}^{(n+1)} + 1\right\}\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) d\Omega_j$$

$$S_{0;j}^{(n+1)} = \sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\, Y_j'^{(n)} \quad ; \quad r_0 \simeq r_j \simeq R \tag{4.4.6}$$

Write (4.1.7) as:

$$\Delta\left(\frac{V_0}{\Phi_0}\right) = \frac{1}{4\pi}\iint\left[\tfrac{3}{2}\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \right.$$
$$\left. + \left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\}\right]\frac{r_j}{r_{0j}}\,d\Omega_j$$

and substitute (4.2.10) and (1.4.3) into this. This gives a formula of the same type as (4.4.4) {compare [HM, (2, 189)]}:

$$\boxed{\Delta\left(\frac{V_0}{\Phi_0}\right) = -\frac{R}{r_0}\left[\Delta Y_0^{(1)} + \frac{1}{4\pi}\iint\left\{S_{0;j}^{(n-1)} - 1\right\}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right\}d\Omega_j\right]} \tag{4.4.7}$$

With $\dfrac{V_0}{\Phi_0} \simeq 1$ one obtains:

$$\Delta\left(\frac{V_0}{\Phi_0}\right) = \Delta\left(\ln\frac{V_0}{\Phi_0}\right) = \Delta(\ln V_0^*) - \Delta(\ln \Phi_0^*)$$

Application of section 3.4 gives {compare [HM, p. 107]}:

$$\Delta(\ln V_0^*) = \Delta\left(\frac{V_0}{\Phi_0}\right) + \Delta(\ln g_0^* \cos\{h_0,r_0\}) + \Delta(\ln r_0^*) - \frac{\omega^2 R}{g_0\cos\{h_0,r_0\}}\,\bar{Y}_0^{(1)} \tag{4.4.8}$$

III. Start again from (4.4.1), but now choose for U successive point coordinates X,Y,Z:

$$\iint\left(X_j\frac{\partial V_j}{\partial n_j} - V_j\frac{\partial X_j}{\partial n_j}\right)dS_j = -\iiint_{\text{earth}} X_k\,dM_k = -MX_C$$

or, with $\dfrac{\partial X_j}{\partial r_j} = \dfrac{X_j}{r_j}$, and for $V = 1$: $\displaystyle\iint\frac{\partial X_j}{\partial n_j}\,dS_j = 0$, one obtains:

$$MX_C = \frac{1}{4\pi}\iint r_j^2\,\frac{X_j}{r_j}\left[V_{0j} + \Phi_j\right]d\Omega_j$$

$$\frac{M}{\Phi_0 r_0}\cdot\frac{X_C}{R} = \frac{1}{4\pi}\iint\frac{r_j}{r_0}\,\frac{r_j}{R}\,\frac{X_j}{r_j}\left[\frac{V_{0j}}{\Phi_0} + \frac{\Phi_j}{\Phi_0}\right]d\Omega_j \quad \ldots\ldots\ldots \tag{4.4.9}$$

Now $\dfrac{M}{\Phi_0 r_0} \simeq 1$, and X_C unknown, hence $X_C^{\text{appr}} = 0$ and $\Delta X_C = X_C$.

Then it follows from (4.4.9) {analogous for $\frac{Y_C}{R}$ and $\frac{Z_C}{R}$}:

$$\boxed{\frac{X_C}{R} = \frac{1}{4\pi}\iint \frac{r_j}{r_0}\,\frac{r_j}{R}\,\frac{X_j}{r_j}\left[\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right] d\Omega_j} \qquad (4.4.10)$$

According to (3.4.6), no terms with ω^2 occur in (4.4.10).

Now substitute (4.2.9) in (4.4.10). It should be noted that $\frac{X_j}{r_j}$ is one of the first degree terms. Hence, in our notation, we have for $n = 1$:

$$\frac{1}{4\pi}\iint \frac{3}{2}\,\frac{X_j}{r_j}\,Y_j'^{(1)}\,Y_k'^{(1)}\,d\Omega_j = \frac{1}{2}\,\frac{X_k}{r_k}$$

and a formula of the same type as (4.4.3) and (4.4.6) results:

$$\boxed{\frac{X_C}{R} = \frac{R}{r_0}\,\frac{1}{4\pi}\iint \frac{3}{2}\,\frac{X_j}{r_j}\,\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) d\Omega_j} \quad \ldots\ldots\ldots (4.4.11)$$

It is remarkable in this formula that the $Y^{(1)}$-term is used in $\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)$, whereas in current theory it is, in contrast, tried to suppress this term.

Substitution of (4.2.10) into (4.4.10) does not make sense because in doing so one introduces a $Y^{(1)}$-term, which has to be determined via (X_C, Y_C, Z_C).

With (3.3.2) and (3.3.4) one obtains from (4.4.11), with

$$\frac{X_i}{r_i}\,\frac{X_j}{r_j} + \frac{Y_i}{r_i}\,\frac{Y_j}{r_j} + \frac{Z_i}{r_i}\,\frac{Z_j}{r_j} = Y_i'^{(1)}\,Y_j'^{(1)}$$

$$Y_i^{(1)} = \Delta Y_i^{(1)} = \frac{R}{r_0}\,\frac{1}{4\pi}\iint \frac{3}{2}\,Y_i'^{(1)}\,Y_j'^{(1)}\,\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) d\Omega_j \quad \ldots \quad (4.4.12')$$

and hence:

$$\boxed{\frac{r_0}{r_i}\left\{\frac{R}{r_i}\,\Delta Y_i^{(1)} - \frac{R}{r_0}\,\Delta Y_0^{(1)}\right\} = \frac{1}{4\pi}\iint \frac{R}{r_i}\,\frac{3}{2}\left\{\frac{R}{r_i}\,Y_i'^{(1)} - \frac{R}{r_0}\,Y_0'^{(1)}\right\}\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) d\Omega_j}$$

, an important formula, in view of (4.2.9). (4.4.12″)

It remains an open question, to what extent the formulae derived in this section have practical significance.
From the previous sections it is evident that M {or $\overline{M}$} and V_0 are derived quantities which do not have to occur in the adjustment model. The same does not apply to (X_C, Y_C, Z_C): they will have to be computed in some way.

4.5 Combination of Poisson- and Green integrals

I. Write (3.2.5) as:

$$\Delta X_{0i} = \frac{1}{4\pi} \iint \Delta X_{0j} \, (-\delta_{ij}) \, \frac{r_j}{r_{ij}} \, \mathrm{d}\Omega_j \quad , \quad P_i \text{ outside } S^* \; . \; . \; . \; . \; . \; . \; . \qquad (4.5.1)$$

$$\text{For } i \to 0: \quad \frac{1}{4\pi} \iint \Delta X_{0j} \, (-\delta_{0j}) \, \frac{r_j}{r_{0j}} \, \mathrm{d}\Omega_j = 0$$

Therefore, on the analogy of the kernel (4.1.9), (4.5.1) can also be written as {series from (1.6.1)}:

$$\left.\begin{array}{l} \Delta \left(\dfrac{V_{0i}}{\Phi_0} - \dfrac{r_0}{r_i} \right) - \dfrac{r_0 - r_i}{r_i} \, \Delta \left(\dfrac{V_0}{\Phi_0} \right) \\[2ex] \Delta \left(\dfrac{\Phi_i}{\Phi_0} - \dfrac{r_0}{r_i} \right) \\[2ex] \Delta \left(\dfrac{\Psi_{0i}}{\Phi_0} - 2 \dfrac{r_0}{r_i} \right) - \dfrac{r_0 - r_i}{r_i} \, \Delta \left(\dfrac{\Psi_0}{\Phi_0} \right) \end{array} \right| \to \Delta X_{0i}$$

$$\Delta X_{0i} = \frac{1}{4\pi} \iint \Delta X_{0j} \, P_{0i;j} \, \mathrm{d}\Omega_j \qquad (4.5.2)$$

$$P_{0i;j} = (-\delta_{ij}) \frac{r_j}{r_{ij}} - \frac{r_0}{r_i} (-\delta_{0j}) \frac{r_j}{r_{0j}} =$$

$$= \frac{r_j}{r_i} \sum_{n=1}^{\infty} (2n+1) \left\{ \left(\frac{r_j}{r_i} \right)^n Y_i'^{(n)} - \left(\frac{r_j}{r_0} \right)^n Y_0'^{(n)} \right\} Y_j'^{(n)}$$

P_i outside S^*; $r_j \simeq R$

II. Substitute the difference equation (4.2.10) into the first relation of (4.5.2). This gives:

$$\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)=\frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}+$$

$$+\frac{1}{4\pi}\iint\left[\left\{-2\Delta\left(\frac{V_{0k}}{\Phi_0}-\frac{r_0}{r_k}\right)+\Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)\right\}\cdot\frac{1}{4\pi}\iint S_{0j;k}^{(n-1)}P_{0i;j}\,d\Omega_j\right]d\Omega_k=$$

$$=\frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}+\frac{1}{4\pi}\iint S_{0i;k}^{(n-1)}\left\{-2\Delta\left(\frac{V_{0k}}{\Phi_0}-\frac{r_0}{r_k}\right)+\Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)\right\}d\Omega_k$$

or again (4.2.10). This does not offer any new point of view.

III. Now substitute (4.2.10) into the second relation of (4.5.2). This gives, in view of the preceding:

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint 2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)P_{0i;j}\,d\Omega_j+$$

$$+\frac{1}{4\pi}\iint\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}P_{0i;j}\,d\Omega_j=$$

$$=2\frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}+$$

$$+\frac{1}{4\pi}\iint\left\{2S_{0i;j}^{(n-1)}+P_{0i;j}\right\}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}d\Omega_j$$

or:

$$\boxed{\begin{aligned}&\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)-2\frac{r_0}{r_i}\left\{\frac{R}{r_i}\Delta Y_i^{(1)}-\frac{R}{r_0}\Delta Y_0^{(1)}\right\}=\\&=\frac{1}{4\pi}\iint S_{0i;j}^{\left(\frac{n-1}{n+1}\right)}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}d\Omega_j\\&S_{0i;j}^{\left(\frac{n-1}{n+1}\right)}=\frac{R}{r_i}\sum_{n=2}^{\infty}\frac{(n+1)(2n+1)}{n-1}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}Y_j'^{(n)}\end{aligned}}\qquad(4.5.3)$$

(4.5.3) is only meaningful for P_i outside S*, because only in that case (4.5.2) is valid.

The Poisson integral we started from is used in the form (3.2.5″) for "the downward continuation of gravity anomalies" [HM, section 8-10], among other things for the reduction of gravity measured on board an aircraft. After this, Stokes' integral is applied.

Perhaps (4.5.3) can be applied to this situation, according to (3.4.6) one has to substitute into (4.5.3):

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right) \simeq \frac{r_0}{r_i}\left\{\Delta\left(\ln\frac{g_i}{g_0}\right)+2\Delta\left(\ln\frac{r_i}{r_0}\right)\right\} + \quad . \ . \ . \ . \quad (4.5.4)$$

Similar to what was done with (4.3.1), we substitute in (4.5.4), with $h_i \ll \overline{r}_i$ *)

$$\ln\frac{r_i}{r_0} = \ln\frac{\overline{r}_i+(1+\Delta\lambda_h)h_i}{r_0} = \ln\frac{\overline{r}_i}{r_0}+\frac{h_i}{\overline{r}_i}+\left(\frac{h_i}{\overline{r}_i}\right)^{\text{appr}}\Delta\lambda_h \quad . \ . \ . \ . \quad (4.5.5')$$

The last term is small and can again be taken to be non-stochastic {or zero}. Then:

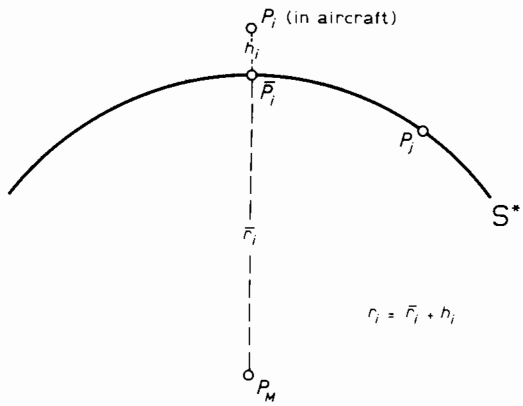

Fig. 4.5-1

$$\Delta\left(\ln\frac{r_i}{r_0}\right) = \Delta\left(\ln\frac{\overline{r}_i}{r_0}\right) + \frac{\Delta h_i}{(\overline{r}_i)^{\text{appr}}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4.5.5'')$$

If in the aircraft h_i is measured as well as g_i then it follows from (4.5.4) and (4.5.5) that with respect to the precision one must have:

$$\text{If: } \sigma\left(\ln\frac{g_i}{g_0}\right) = 10^{-p}, \text{ then: } \sigma_h \simeq \tfrac{1}{2}R\cdot 10^{-p} \simeq 3\cdot 10^{6-p} \text{ m}$$

At present one only attains $p = 5$ or 6, too low a precision, so that these considerations have a theoretical character for the time being.

If one now assumes a limited number of levelling- and gravity measurements on S* in and around the area considered, together with a great number of measurements of g_i and h_i on board of the aircraft, then (4.5.3), with substitution of (4.5.4) and (4.5.5), makes it possible to determine $\Delta\ln\left(\frac{\overline{r}_i}{r_0}\right)$ for points $\overline{P}_i$ on S*.

Because (3.2.5″) is linearly independent of (3.2.5′), (4.5.3) is linearly independent of (4.2.10). Also here, the form of the kernel makes regional application possible.

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*) The relatively small values for h imply that the corrections of section 4.6 can be neglected.

4.6 Appendix. Corrections in satellite altimetry

The content of the text written in relation with (4.3.1) and (4.3.2) has to be formulated a bit sharper. See [GOPALAPILLAI, 1974] and [BLAHA, 1977]. The reason is that in figure 4.3-1 $h_j \perp$ S, so that h_j is not exactly in line with r_j.

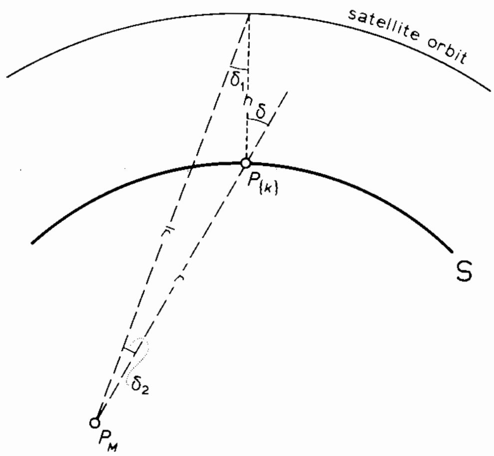

Fig. 4.6-1

I. Consider the situation in figure 4.6.-1. The angles δ are small, hence

$$\frac{\delta_2}{\delta_1} \simeq \frac{\sin\delta_2}{\sin\delta_1} = \frac{h}{r}, \ \delta_2 \simeq \frac{h}{r}\,\delta_1$$

$$\delta = \delta_1 + \delta_2 \simeq \left(1 + \frac{h}{r}\right)\,\delta_1$$

$$\boxed{\delta_1 \simeq \frac{r}{r+h}\,\delta \quad \Big| \quad \delta_2 \simeq \frac{h}{r+h}\,\delta} \quad \ldots\ldots\ldots\ldots \quad (4.6.1)$$

$$\bar{r} = h\cos\delta_1 + r\cos\delta_2$$

$$\simeq r + h - \tfrac{1}{2}\left(h\{\delta_1\}^2 + r\{\delta_2\}^2\right)$$

$$\simeq r + h - \tfrac{1}{2}h\,\frac{r}{r+h}\,\delta^2$$

or:

$$\boxed{r \simeq \bar{r} - h\left(1 - \tfrac{1}{2}\,\frac{r}{r+h}\,\delta^2\right)} \quad \ldots\ldots\ldots\ldots \quad (4.6.2)$$

According to the above-mentioned publications, the difference between geographic and geocentric latitudes gives a sufficient estimate for δ. Or {with φ one of these types of latitude}:

$$\delta \simeq \tfrac{1}{2}\, e^2 \sin 2\varphi \quad \ldots \quad (4.6.3')$$

or for the Geodetic Reference System 1967:

$$\delta \simeq 0.00335 \sin 2\varphi \lesssim 12' \quad \ldots \quad (4.6.3'')$$

or with $r \simeq 6.4 \cdot 10^6$ m and $h \simeq 10^6$ m [BLAHA, 1977]:

$$r - (\bar{r} - h) \simeq h \tfrac{1}{2}\, \frac{r}{r+h}\, \delta^2 \simeq (4.8 \sin^2 2\varphi)\ \mathrm{m} \quad \ldots \quad (4.6.4)$$

The influence of deviations of the vertical up to $1'$ is consequently at most $(\tfrac{1}{12})^2$ of this value, and therefore negligible with respect to the accuracy of measurement {> 0.1 m}.
The correction (4.6.4) can therefore be computed completely from approximate values, so that from (4.6.2) follows:

$$\Delta r = \Delta \bar{r} - \Delta h \quad \ldots \quad (4.6.5)$$

to be used in (4.3.1″) or (4.3.2″).

The situation in figure 4.6-1 is met at points on or near offshore constructions, whose coordinates have been determined and which are connected to the shore by levelling, (4.3.2).

II. A possibly different situation is met at points P_j in the open sea, where the direction of $\vec{r}$ is not clearly fixed. Here, one can perhaps define P_j better as the intercept of S with the straight line P_M-satellite.

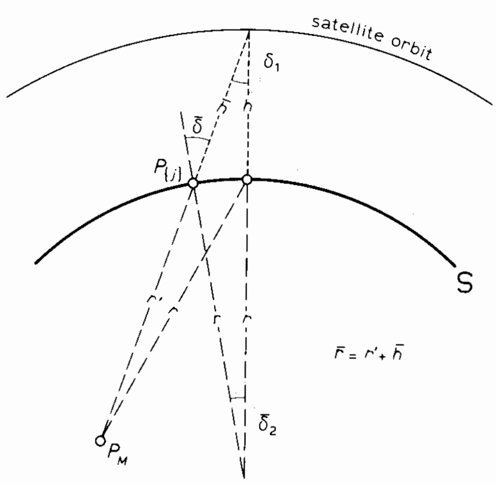

Fig. 4.6-2

Consider S locally as a sphere with radius r, see figure 4.6-2. Then we have, with (4.6.1):

$$\frac{\bar{\delta}}{\delta_1} \simeq \frac{\sin \bar{\delta}}{\sin \delta_1} = \frac{r+h}{r}$$

$$\left.\begin{aligned} &\bar{\delta} \simeq \frac{r+h}{r}\,\delta_1 \simeq \delta \\ &\bar{\delta}_2 = \bar{\delta} - \delta_1 \simeq \delta_2 \end{aligned}\right\} \qquad (4.6.6)$$

$$\bar{h}^2 = r^2 + (r+h)^2 - 2r\,(r+h)\cos\delta_2$$

$$\frac{\bar{h}^2 - h^2}{r^2} \simeq 2\left(1+\frac{h}{r}\right) - 2\left(1+\frac{h}{r}\right)\left(1-\tfrac{1}{2}\{\delta_2\}^2\right)$$

$$\frac{\bar{h}-h}{r}\;2\,\frac{h}{r} \simeq \left(1+\frac{h}{r}\right)\left(\frac{h}{r+h}\right)^2 \delta^2$$

$$\bar{h} - h \simeq h\,\tfrac{1}{2}\,\frac{r}{r+h}\,\delta^2 \qquad (4.6.7)$$

which is a correction term similar to (4.6.4).
Or with, see figure 4.6-2:

$$r' = \bar{r} - \bar{h}$$

we have:

$$\boxed{r' \simeq \bar{r} - h\left(1+\tfrac{1}{2}\,\frac{r}{r+h}\,\delta^2\right)} \qquad (4.6.8)$$

again with:

$$\Delta r' = \Delta\bar{r} - \Delta h$$

to be used in (4.3.1″); the prime in r' may be omitted.

5. TENTATIVE CONSIDERATIONS CONCERNING GREEN INTEGRALS

5.1 Elimination of discontinuities

When differentiating (1.7.4) with respect to r_i, one is confronted with the differentiation of the potentials of a double layer and a single layer. K. R. KOCH has studied this problem thoroughly [KOCH, 1967[a], 1967[b]]; his results can be used for a test.
The question then arises whether the derivations of the difference equations in chapter 4 are sufficiently rigorous, because in fact there one is faced with the same problem.
The elegant way in which M. S. MOLODENSKII has approached this problem for a spherical body [MOLODENSKI, 1958, p. 18], [MOLODENSKII et al, 1962, pp. 45-48] raises the question whether this line of thought cannot be generalized for the earth as a non-spherical body*). For this, the form of the kernel (4.1.9) may indicate the direction to be taken, but in the first instance one has to abstain from expansions with spherical harmonics if the formulae are to remain valid outside and on S {or S*}.

Starting from (1.7.5) with (1.7.6), one obtains:

$$V_i = \frac{1}{4\pi} \iint \left[-V_{ij} \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} + \Phi_j \frac{r_j}{r_{ij}} \right] d\Omega_j$$

$-\dfrac{1+\delta_{ij}}{2} \dfrac{r_j}{r_{ij}} = r_j^2 \dfrac{\partial}{\partial r_j}\left(\dfrac{1}{r_{ij}}\right)$	$\Phi_j = -r_j \dfrac{\partial V_j}{\partial r_j}$	$d\Omega_j = \dfrac{\cos(r_j,n_j)}{r_j^2} dS_j$

(5.1.1)

In (5.1.1.), the discontinuity of the double layer for P_i from "outside S" to "on S" has already been eliminated by the coefficient V_{ij}. It will now be tried to eliminate in an analogous way the discontinuity of the derivative of the single layer with respect to r_i for the same situation.

For this purpose, consider the potential of the single layer with density $\dfrac{\cos(r_j,n_j)}{r_j}$ for P_i outside or on S:

$$A_i = \frac{1}{4\pi} \iint \frac{r_j}{r_{ij}} d\Omega_j = \frac{1}{4\pi} \iint \frac{1}{r_{ij}} \frac{\cos(r_j,n_j)}{r_j} dS_j \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5.1.2)$$

Then the following is valid:

$$A_i - \frac{A_i}{A_0} A_0 = \frac{1}{4\pi} \iint \left(\frac{r_j}{r_{ij}} - \frac{A_i}{A_0} \frac{r_j}{r_{0j}} \right) d\Omega_j = 0 \, . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5.1.3)$$

*) But see section 1.8

For a spherical surface with radius R we have according to (1.6.1):

$$\left.\begin{aligned} & r_j = R \quad ; \quad A_i = \frac{R}{r_i} \quad ; \quad \frac{A_i}{A_0} = \frac{r_0}{r_i} \\ & \frac{1}{4\pi} \iint\limits_{\text{sphere}} \left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i} \frac{r_j}{r_{0j}} \right) \mathrm{d}\Omega_j = 0 \end{aligned}\right\} \qquad (5.1.4)$$

which demonstrates the meaning of the kernel (4.1.9).

Taking account of (5.1.3), which permits the replacement of Φ_j by Φ_{ij} as a coefficient, it follows from (5.1.1) for P_i outside or on S that:

$$\boxed{\begin{aligned} V_i - \frac{A_i}{A_0} V_0 = \frac{1}{4\pi} \iint \Bigg[& - V_{ij} \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} + \frac{A_i}{A_0} V_{0j} \frac{1+\delta_{0j}}{2} \frac{r_j}{r_{0j}} + \\ & + \Phi_{ij} \left(\frac{r_j}{r_{ij}} - \frac{A_i}{A_0} \frac{r_j}{r_{0j}} \right) \Bigg] \mathrm{d}\Omega_j \end{aligned}} \qquad (5.1.5)$$

by which the objective has been reached.

It is interesting to note the connection with section 4.1. With $V_{ij} = V_{0j} - V_{0i}$, (5.1.5) becomes, compare (4.1.8):

$$\begin{aligned} & V_{0i} - \left(\frac{A_i}{A_0} - 1 \right) V_0 = \\ & = \frac{1}{4\pi} \iint \left[V_{0i} \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} + \tfrac{1}{2} V_{0j} \left\{ (-\delta_{ij}) \frac{r_j}{r_{ij}} - \frac{A_i}{A_0} (-\delta_{0j}) \frac{r_j}{r_{0j}} \right\} \right] \mathrm{d}\Omega_j + \\ & \qquad + \frac{1}{4\pi} \iint \left[-\tfrac{1}{2} V_{0j} + \Phi_j \right] \left(\frac{r_j}{r_{ij}} - \frac{A_i}{A_0} \frac{r_j}{r_{0j}} \right) \mathrm{d}\Omega_j \quad . \; . \; . \; (5.1.6) \end{aligned}$$

In the second term of the first integral one recognizes, for the situation (5.1.4), the kernel $P_{0i;j}$ from (4.5.2). Strictly speaking the device from (4.1.4) – (4.1.6) will have to be applied to (5.1.5), with a derivation like the one for (5.1.6). The result is then again (4.1.8).

5.2 "Inverse" Green integrals

Differentiate (5.1.5) with respect to r_i for P_i outside S* and put:

$$-r_i \frac{\partial V_i}{\partial r_i} = \Phi_i \quad , \quad -r_i \frac{\partial A_i}{\partial r_i} = B_i \quad , \quad \text{with } \frac{A_i}{A_0} \simeq \frac{B_i}{A_0} \simeq \frac{r_0}{r_i} \quad . \; . \; . \; . \; (5.2.1)$$

This gives, with (1.5.1):

$$\Phi_i - \frac{B_i}{A_0} V_0 = \frac{1}{4\pi} \iint \left[\Phi_i \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} - V_{ij} \left\{ \left(\frac{r_j}{r_{ij}}\right)^2 + \frac{1+\delta_{ij}}{2} \frac{1-3\delta_{ij}}{2} \right\} \frac{r_j}{r_{ij}} + \right.$$
$$+ \frac{B_i}{A_0} V_{0j} \frac{1+\delta_{0j}}{2} \frac{r_j}{r_{0j}} + r_i \frac{\partial \Phi_i}{\partial r_i} \left(\frac{r_j}{r_{ij}} - \frac{A_i}{A_0} \frac{r_j}{r_{0j}} \right) +$$
$$\left. + \Phi_{ij} \left(\frac{1-\delta_{ij}}{2} \frac{r_j}{r_{ij}} - \frac{B_i}{A_0} \frac{r_j}{r_{0j}} \right) \right] d\Omega_j \quad . \; . \; (5.2.2)$$

The term with $\frac{\partial \Phi_i}{\partial r_i}$ vanishes, in view of (5.1.3). Consequently, there are no difficulties when P_i passes on to S*.

Now substract one half times (5.1.5) from (5.2.2):

$$\Phi_i - \tfrac{1}{2} V_i - \frac{2B_i - A_i}{2A_0} V_0 =$$
$$= \frac{1}{4\pi} \iint \left[\Phi_i \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} - V_{ij} \left\{ \left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4} \delta_{ij} (1+\delta_{ij}) \right\} \frac{r_j}{r_{ij}} + \right.$$
$$\left. + \frac{2B_i - A_i}{2A_0} V_{0j} \frac{1+\delta_{0j}}{2} \frac{r_j}{r_{ij}} + \Phi_{ij} \left\{ \tfrac{1}{2}(-\delta_{ij}) \frac{r_j}{r_{ij}} - \frac{2B_i - A_i}{2A_0} \frac{r_j}{r_{0j}} \right\} \right] d\Omega_j$$

(5.2.3)

The discontinuities in the situation $i \leftrightarrow j$ vanish because of the coefficients V_{ij} and Φ_{ij}. If this is also assumed for B_i, Φ_{ij} can be replaced by Φ_j. With V_0 from (4.1.1) we then obtain for (5.2.3):

$$\boxed{\Phi_i - \tfrac{1}{2} V_i = \frac{1}{4\pi} \iint \left[\Phi_i \frac{1+\delta_{ij}}{2} \frac{r_j}{r_{ij}} + \tfrac{1}{2} \Phi_j (-\delta_{ij}) \frac{r_j}{r_{ij}} + - V_{ij} \left\{ \left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4} \delta_{ij} (1+\delta_{ij}) \right\} \frac{r_j}{r_{ij}} \right] d\Omega_j} \quad (5.2.4)$$

For points P_i and P_j on a spherical surface with centre P_M and radius R $\{r_i = r_j = R; \delta_{ij} = 0\}$, and with (1.7.1), one obtains the relation of MOLODENSKII {see also [HM, (1-97)] and [KOCH, 1967[b], p. 19]}:

$\frac{\partial V_i}{\partial r_i} + \frac{V_i}{R} = \frac{1}{2\pi} \iint_{\text{sphere}} \frac{V_{ij}}{r_{ij}^3} R^2 \, d\Omega_j$	P_i on sphere

. . . . . . (5.2.5)

Now apply to (5.2.4) the device from (4.1.4) – (4.1.6):

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)-\tfrac{1}{2}\Delta\left(\frac{V_i}{\Phi_0}-\frac{r_0}{r_i}\right)=$$

$$=\frac{1}{4\pi}\iint\left[\frac{1+\delta_{ij}}{2}\,\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)+\tfrac{1}{2}(-\delta_{ij})\,\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{ij}}\,d\Omega_j+$$

$$+\frac{1}{4\pi}\iint\left[\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)\right]\left\{\left(\frac{r_j}{r_{ij}}\right)^2-\frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}\,d\Omega_j$$

The first integral in the right hand member becomes, with (1.7.1) and (3.2.5″):

$$\tfrac{1}{2}\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)$$

Now subtract $\left(-\dfrac{r_0}{r_i}\right)$ times the equation applied to $i\to 0$. Then one obtains for P_i and P_0 outside or on S*:

$$\boxed{\begin{aligned}&\tfrac{1}{2}\left[\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)-\left\{\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)\right\}\right]=\\&=\frac{1}{4\pi}\iint\left[\left\{\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}\left\{\left(\frac{r_j}{r_{ij}}\right)^2-\frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}+\right.\\&\qquad\left.+\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)\left\{\left(\frac{r_j}{r_{0j}}\right)^2-\frac{3}{4}\,\delta_{0j}\,(1+\delta_{0j})\right\}\frac{r_0}{r_i}\,\frac{r_j}{r_{0j}}\right]d\Omega_j\end{aligned}}\tag{5.2.6}$$

From the form of the left hand member it follows that (5.2.6) can neither be considered as the inverse relation of (4.1.8), nor as the inverse relation of (4.2.9) or (4.2.10).
In left- and right hand member there occur the difference quantities $\Delta\left(\dfrac{r_0}{r_i}\right)$ and $\Delta\left(\dfrac{r_0}{r_j}\right)$ respectively, and $\Delta\left(\dfrac{V_{0i}}{\Phi_0}\right)$ and $\Delta\left(\dfrac{V_{0j}}{\Phi_0}\right)$ respectively, so that the application gives no advantage, neither on land nor at sea {compare section 4.3}. Also in view of the coefficients the application of (5.2.6) is not attractive.
From the derivation of (5.2.6) follows:

$$\left\{\left(\frac{r_j}{r_{ij}}\right)^2-\frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}=r_i r_j^2\left\{\frac{\partial}{\partial r_j}\left(\frac{1}{r_{ij}}\right)\right\}+\tfrac{1}{2}r_j^2\,\frac{\partial}{\partial r_j}\left(\frac{1}{r_{ij}}\right)=$$

$$=-r_i\,\frac{\partial}{\partial r_i}\left\{\frac{1+\delta_{ij}}{2}\,\frac{r_j}{r_{ij}}\right\}-\tfrac{1}{2}\,\frac{1+\delta_{ij}}{2}\,\frac{r_j}{r_{ij}}$$

Because (5.2.6) is valid for P_i outside S*, the series (1.6.1) may be used for this situation:

$$\left\{\left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}} = -\sum_{n=1}^{\infty}(2n+1)\,\frac{n}{2}\left(\frac{r_j}{r_i}\right)^{n+1} Y_i'^{(n)}\,Y_j'^{(n)} \qquad (5.2.7')$$

$$-\left\{\left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}} + \left\{\left(\frac{r_j}{r_{0j}}\right)^2 - \frac{3}{4}\,\delta_{0j}\,(1+\delta_{0j})\right\}\frac{r_0}{r_i}\,\frac{r_j}{r_{ij}} =$$

$$= \tfrac{1}{2}\,Q_{0i;j} = \frac{r_j}{r_i}\sum_{n=1}^{\infty}(2n+1)\,\frac{n}{2}\left\{\left(\frac{r_j}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{r_j}{r_0}\right)^n Y_0'^{(n)}\right\} Y_j'^{(n)} \qquad (5.2.7'')$$

(5.2.7) into (5.2.6) gives $\{P_i$ outside S*; $r_j \simeq R\}$:

$$\left.\begin{aligned}
&\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) - \left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)\right\} = \\
&\qquad = \frac{1}{4\pi}\iint Q_{0i;j}\,\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) d\Omega_j \\
&\text{In which it is possible to replace:} \\
&\Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) \text{ by: } \Delta\left(\frac{V_{0j}}{\Phi_0} - \frac{r_0}{r_j}\right) - \frac{r_0 - r_j}{r_j}\,\Delta\left(\frac{V_0}{\Phi_0}\right)
\end{aligned}\right\} \qquad (5.2.8)$$

Write (4.2.9) as:

$$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right) = \Delta X_{0i}$$

$$\Delta X_{0i} = \frac{1}{4\pi}\iint S_{0i;j}^{(n+1)}\,\Delta X_{0j}\,d\Omega_j +$$

$$+ \frac{1}{4\pi}\iint S_{0i;j}^{(n+1)}\left\{\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) - \Delta X_{0j}\right\} d\Omega_j$$

and substitute (5.2.8). The result is:

$$\Delta X_{0i} = \frac{1}{4\pi}\iint P_{0i;j}\,\Delta X_{0j}\,d\Omega_j$$

or the first Poisson integral in (4.5.2).

Although this derivation is only valid for P_i outside S*, the conclusion seems acceptable that

(5.2.6) is linearly dependent on (4.2.9) {and hence on (4.2.10)} so that also for this reason one can renounce the use of (5.2.6).

In geodetic literature {e.g. [KOCH, 1967[b]]} the problem treated here is coupled with the derivation of formulae for the deflection of the vertical, the small angle between the direction of the vertical {plumb line} and the ellipsoidal normal of a reference ellipsoid. In the first approximation one obtains Vening Meinesz's formulae, e.g. by differentiating Stokes' formula [HM, section 2-22]. Because of the form of the left hand member of our corresponding formula (4.2.10), this differentiation does not make sense in our case.
Besides, we have started from a model of approximate values in which a reference ellipsoid is not essential any more. Because of this, the significance of the concept "deflection of the vertical" vanishes. In spatial geometric quaternion theory this concept does not occur either; astronomical observations are included in this theory and have the task to strengthen the relative orientations of local coordinate systems.

5.3 First and second derivatives of the gravitational potential

Apply (1.74) – (1.7.6) to the harmonic function $\Phi_i = r_i\left(-\dfrac{\partial V_i}{\partial r_i}\right)$. Now it is important to note what was said in section 1.7 on observations of $\dfrac{\partial \Phi_j}{\partial r_j}$ on S* and not on {but near} S. Similar problems as in section 5.1 are met, so that in choosing the kernel one has to take account of this. The result is {see section 3.1}:

$$\Phi_i = \frac{1}{4\pi}\iint\left[\Phi_{ij}\frac{\partial}{\partial n_j}\left(\frac{1}{r_{ij}}\right) - \frac{1}{r_{ij}}\frac{\partial \Phi_j}{\partial n_j}\right] dS_j^* \quad \ldots\ldots\ldots\ldots \quad (5.3.1)$$

$$\Phi_i = \frac{1}{4\pi}\iint\left[-\Phi_{ij}\frac{1+\delta_{ij}}{2} + r_j\left(-\frac{\partial \Phi_j}{\partial r_j}\right)\right]\frac{r_j}{r_{ij}}\, d\Omega_j$$

$$r_j\left(-\frac{\partial \Phi_j}{\partial r_j}\right) = r_j\frac{\partial}{\partial r_j}\left(r_j\frac{\partial V_j}{\partial r_j}\right) = r_j\frac{\partial V_j}{\partial r_j} + r_j^2\frac{\partial^2 V_j}{\partial r_j^2} = -\Phi_j + \Psi_j$$

$$\Phi_i = \frac{1}{4\pi}\iint\left[\Phi_i\frac{1+\delta_{ij}}{2} - \Phi_j\frac{3+\delta_{ij}}{2} + \Psi_j\right]\frac{r_j}{r_{ij}}\, d\Omega_j$$

Application of the device (4.1.4) – (4.1.6) gives:

$$\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) = \frac{1}{4\pi}\iint\left[\frac{1+\delta_{ij}}{2}\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) + \tfrac{1}{2}(-\delta_{ij})\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{ij}}\, d\Omega_j +$$

$$+ \frac{1}{4\pi}\iint\left[-\frac{3}{2}\Delta\left(\frac{\Phi_j}{\Phi_0} - \frac{r_0}{r_j}\right) + \Delta\left(\frac{\Psi_j}{\Phi_0} - 2\frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{ij}}\, d\Omega_j$$

or, with (1.7.1) and (3.2.5):

$$\tfrac{1}{2}\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint\left[-\frac{3}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_j}{\Phi_0}-2\frac{r_0}{r_j}\right)\right]\frac{r_j}{r_{ij}}\,\mathrm{d}\Omega_j \tag{5.3.2}$$

A better kernel is obtained by subtracting $\frac{r_0}{r_i}$ times (5.3.2) with $i\to 0$ {see (4.1.9)}:

$$\tfrac{1}{2}\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint\left[-\frac{3}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\right.$$

$$\left.+\Delta\left(\frac{\Psi_j}{\Phi_0}-2\frac{r_0}{r_j}\right)\right]\left(\frac{r_j}{r_{ij}}-\frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right)\mathrm{d}\Omega_j \tag{5.3.3}$$

The last equations in (3.1.7) and (3.4.7) indicate a possible improvement by replacing Ψ_j by Ψ_{0j}, which is possible because of the kernel (4.1.9) in (5.3.3). For brevity we shall omit the term $\frac{r_0-r_j}{r_j}\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$.

Then from (5.3.3) the following integral equation is obtained:

$$\boxed{\tfrac{1}{2}\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint\left[-\frac{3}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right]\left(\frac{r_j}{r_{ij}}-\frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right)\mathrm{d}\Omega_j}$$

P_i outside or on S*; P_0, P_j on S*; $r_0\simeq r_j\simeq R$

(5.3.4)

Like in section (4.1), one can arrive at different solutions. For example, one can eliminate $Y^{(1)}$-terms under the integral sign {and, curiously, one then also eliminates $\Delta\left(\frac{r_0}{r_j}\right)$}, or with a view to (5.4.3) one can choose another possibility. The following derivation serves the purpose:

$$\left[-\frac{3}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right]=$$

$$=3\left[\tfrac{1}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\frac{1}{3}\left\{-3\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}\right]$$

$$=\left[\tfrac{1}{2}\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\left\{-2\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}\right]$$

With the series expansion (3.1.7″) one then obtains the solutions {compare (4.2.9) and (4.2.10)}:

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)-2\frac{r_0}{r_i}\left\{\frac{R}{r_i}\,\Delta Y_i^{(1)}-\frac{R}{r_0}\,\Delta Y_0^{(1)}\right\} =$$

$$\text{or: }\frac{r_0}{r_i}\sum_{n=2}^{\infty}(n+1)\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} =$$

$$=\frac{1}{4\pi}\iint S_{0i;j}^{(n-1)}\left\{-3\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}d\Omega_j \qquad (5.3.5)$$

$$S_{0i;j}^{(n-1)}=\frac{R}{r_i}\sum_{n=2}^{\infty}\frac{2n+1}{n-1}\left\{\left(\frac{R}{r_i}\right)^n Y_i^{\prime(n)}-\left(\frac{R}{r_0}\right)^n Y_0^{\prime(n)}\right\}Y_j^{\prime(n)}$$

P_i outside, on {or inside?} S*; P_0, P_j on S*; r_0, $r_j \simeq R$

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right) =$$

$$\text{or: }\frac{r_0}{r_i}\,2\left\{\frac{R}{r_i}\,\Delta Y_i^{(1)}-\frac{R}{r_0}\,\Delta Y_0^{(1)}\right\}+\frac{r_0}{r_i}\sum_{n=2}^{\infty}(n+1)\left\{\left(\frac{R}{r_i}\right)^n\Delta Y_i^{(n)}-\left(\frac{R}{r_0}\right)^n\Delta Y_0^{(n)}\right\} = \qquad (5.3.6)$$

$$=\frac{1}{4\pi}\iint S_{0i;j}^{(n)}\left\{-2\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}d\Omega_j$$

$$S_{0i;j}^{(n)}=\frac{R}{r_i}\sum_{n=1}^{\infty}\frac{2n+1}{n}\left\{\left(\frac{R}{r_i}\right)^n Y_i^{\prime(n)}-\left(\frac{R}{r_0}\right)^n Y_0^{\prime(n)}\right\}Y_j^{\prime(n)}$$

These are only examples of possible solutions, therefore we shall not discuss further interpretation and application {compare section 4.3}.

(4.1.8) and (5.3.4) must be considered as being linearly independent, also because different quantities occur. Therefore one may form combinations of these integral equations, e.g. to eliminate $\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)$.

Like in section 4.5, it is also possible to form combinations of (5.3.5) or (5.3.6) with the third Poisson integral in (4.5.2). Thus a number of possibilities is created in practice.

5.4 The "inverse" Green integral. The constant $\Delta\left(\dfrac{\Psi_0}{\Phi_0}\right)$

The gist of the considerations in sections 5.1 and 5.2 is that (5.3.1) can be differentiated with respect to r_i without further preface. On the analogy of (5.2.4), one obtains:

$$\left(-r_i\,\frac{\partial\Phi_i}{\partial r_i}\right) - \tfrac{1}{2}\Phi_i = \frac{1}{4\pi}\iint \left[\left(-r_i\,\frac{\partial\Phi_i}{\partial r_i}\right)\frac{1+\delta_{ij}}{2}\,\frac{r_j}{r_{ij}} + \tfrac{1}{2}\left(-r_j\,\frac{\partial\Phi_j}{\partial r_j}\right)\left(-\delta_{ij}\right)\frac{r_j}{r_{ij}} + \right.$$
$$\left. -\Phi_{ij}\left\{\left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}\right]\mathrm{d}\Omega_j$$

Or again with:

$$\left(-r_i\,\frac{\partial\Phi_i}{\partial r_i}\right) = -\Phi_i + \Psi_i$$

$$\boxed{\begin{aligned}(-\Phi_i+\Psi_i) - \tfrac{1}{2}\Phi_i = \frac{1}{4\pi}\iint &\left[(-\Phi_i+\Psi_i)\,\frac{1+\delta_{ij}}{2}\,\frac{r_j}{r_{ij}} + \tfrac{1}{2}\,(-\Phi_j+\Psi_j)\,(-\delta_{ij})\,\frac{r_j}{r_{ij}} + \right.\\ &\left. -\,\Phi_{ij}\left\{\left(\frac{r_j}{r_{ij}}\right)^2 - \frac{3}{4}\,\delta_{ij}\,(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}\right]\mathrm{d}\Omega_j\end{aligned}} \qquad (5.4.1)$$

For points P_i and P_j on a spherical surface with centre P_M and radius $R\{r_i = r_j = R\,;\,\delta_{ij} = 0\}$, one obtains with (17.1) the second relation of MOLODENSKII {[MOLODENSKI, 1958, (1.6)], [MOLODENSKII et al, 1962, (III.1.10)], [HM, (2-217)]}:

$$-2\Phi_i + \Psi_i = \frac{1}{2\pi}\iint (-\Phi_{ij})\left(\frac{R}{r_{ij}}\right)^3\mathrm{d}\Omega_j$$

$$-\Phi_i = R\,\frac{\partial V_i}{\partial r_i} \quad ; \quad \Psi_i = R^2\,\frac{\partial^2 V_i}{\partial r_i^2} \quad ; \quad -\Phi_{ij} = R\left(\frac{\partial V_j}{\partial r_j} - \frac{\partial V_i}{\partial r_i}\right)$$

$\dfrac{2}{R}\,\dfrac{\partial V_i}{\partial r_i} + \dfrac{\partial^2 V_i}{\partial r_i^2} = \dfrac{1}{2\pi}\displaystyle\iint_{\text{sphere}}\dfrac{1}{r_{ij}^3}\left(\dfrac{\partial V_j}{\partial r_j} - \dfrac{\partial V_i}{\partial r_i}\right)R^2\,\mathrm{d}\Omega_j$	P_i on sphere

(5.4.2)

The application of the device (4.1.4) – (4.1.6) to (5.4.1), with a further elaboration as for (5.2.6) then gives, for P_i outside or on S*:

$$\boxed{\begin{aligned}&\tfrac{1}{2}\left\{\Delta\left(\frac{\Psi_{0i}}{\Phi_0}-2\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\Delta\left(\frac{\Psi_0}{\Phi_0}\right)-2\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)\right\}=\\&=\frac{1}{4\pi}\iint\left[\left\{\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)-\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}\left\{\left(\frac{r_j}{r_{ij}}\right)^2-\frac{3}{4}\delta_{ij}(1+\delta_{ij})\right\}\frac{r_j}{r_{ij}}+\right.\\&\qquad\left.+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\left\{\left(\frac{r_j}{r_{0j}}\right)^2-\frac{3}{4}\delta_{0j}(1+\delta_{0j})\right\}\frac{r_0}{r_i}\frac{r_j}{r_{0j}}\right]d\Omega_j\end{aligned}}$$

(5.4.3)

One might consider (5.4.3) as the inverse of (5.3.6).

(5.4.3) substituted into (5.3.6) gives with (5.2.7) for P_i outside S*:

$$\Delta\left(\frac{\Phi_i}{\Phi_0}-\frac{r_0}{r_i}\right)=\frac{1}{4\pi}\iint S^{(n)}_{0i;j}\left[\frac{1}{4\pi}\iint Q_{0j;k}\,\Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)d\Omega_k\right]d\Omega_j=$$

$$=\frac{1}{4\pi}\iint P_{0i;k}\,\Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)d\Omega_k$$

or the second Poisson integral in (4.5.2). This indicates a linear dependence between (5.4.3) and (5.3.6), so that the use of (5.4.3) can be renounced*) {compare what was said to (5.2.6)}.

The constant $\Delta\left(\dfrac{\Psi_0}{\Phi_0}\right)$

Supplementary to section 4.4, it follows from (5.4.1) for $i\to 0$ with $\Delta\left(\dfrac{\Phi_0}{\Phi_0}\right)=\Delta\left(\dfrac{r_0}{r_0}\right)=0$ and $\delta_{0j}\simeq 0$ when the region around P_0 is horizontal and plane:

P_0 on S* ; $r_j\simeq r_0\simeq R$; $\delta_{0j}=\dfrac{r_j^2-r_0^2}{r_{0j}^2}$, $\delta_{0j}\simeq 0$?	(5.4.4)
$\tfrac{1}{2}\Delta\left(\dfrac{\Psi_0}{\Phi_0}\right)=\dfrac{1}{4\pi}\iint\left\{-\Delta\left(\dfrac{\Phi_j}{\Phi_0}-\dfrac{r_0}{r_j}\right)\right\}\left\{\left(\dfrac{r_j}{r_{0j}}\right)^2-\dfrac{3}{4}\delta_{0j}(1+\delta_{0j})\right\}\dfrac{r_j}{r_{0j}}\,d\Omega_j$	

Also in this integral the singular case of $\dfrac{1}{r_{0j}}$ for $j\to 0$ vanishes because $\Delta\left(\dfrac{\Phi_j}{\Phi_0}-\dfrac{r_0}{r_j}\right)$ vanishes for this situation.

*) Unless perhaps for gradiometer measurements from spacecraft.

Remark

An attempt to establish a connection between the formulae in this section and e.g. formula (1046) in [PICK et al, 1973] has remained unsuccessful.

5.5 Concluding remarks on Poisson- and Green integrals

In chapters 3-5 it has been tried to illuminate the background and the origin of Poisson- and Green integrals which form the core of gravimetric geodesy. The approach followed should be improved mathematically, and it will also be necessary to make extensions with a view to new instrumentation. In the computing model, refinements are possible [MORITZ, 1974].
But is has been shown that systems of difference equations are obtained that can be transformed according to the requirements following from the measurements available, and treated in the same way as difference equations pertaining to terrestrial or spatial networks.
An important aspect is also the establishment of a kernel that permits regional application of Poisson- and Green integrals. The Stokes-function $S_{0i;j}^{(n-1)}$ is given a more modest task restricted to land areas. The corresponding task for sea areas is taken over by the sum-function $S_{0i;j}^{(n+1)}$. But whereas the computation of the mass of the earth is to be considered as no more than an appendix to the adjustment computations, the coordinates (X_C, Y_C, Z_C) of the centre of mass of the earth will as a rule be an essential part of the computing problem. Although the indicated coupling of satellite and terrestrial data is interesting, it must once more be pointed out that a further analysis of dynamic satellite methods is necessary.
The considerations in chapter 5 are more dubious. It has been tried to find a connected generalization of the two so-called inverse relations of Molodenskii. The preference for other relations has been argued. Nevertheless, the formulae with $\frac{\partial^2 V}{\partial r^2}$ should be considered as the result of "toying" with the theory. But it is essential that Vening Meinesz's formulae do not find a place in the theory designed.

6. ADDITIONAL CONSIDERATIONS

6.1 Free air reduction and geoid

From spirit levelling follows:

$$\frac{W_{0i}}{r_0 g_0} = - \frac{1}{r_0 g_0} \sum_{j=0}^{i-1} g_{j.j+1}\, h_{j.j+1}$$

W_{0i} and $h_{j,j+1}$ are small in comparison with r and g, hence {see (2.6.7)}:

$$\frac{\Delta W_{0i}}{r_0 g_0} = - \frac{1}{r_0 g_0} \sum_{j=0}^{i-1} g_{j.j+1}\, \Delta h_{j,j+1} \quad \ldots\ldots \quad (6.1.1)$$

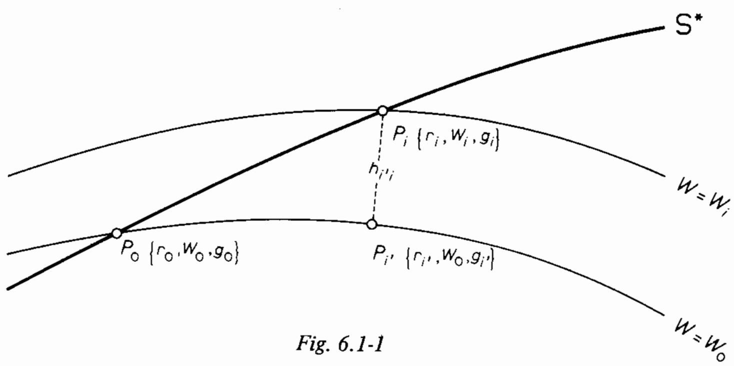

Fig. 6.1-1

Choose the fictitious auxiliary point $P_{i'}$, on the plumb line of P_i and in the equipotential surface through P_0; see figure 6.1-1. Then we have:

$$\sum_{j=0}^{i-1} g_{j.j+1}\, h_{j,j+1} = \bar{g}_i h_{i'i} \quad , \quad g_i < \bar{g}_i < g_{i'}$$

and: $$\sum_{j=0}^{i-1} g_{j.j+1}\, \Delta h_{j,j+1} = \bar{g}_i \Delta h_{i'i} \quad \ldots\ldots \quad (6.1.2')$$

with reasonable approximation: $\bar{g}_i \simeq \sqrt{g_i g_{i'}}$ . . . . (6.1.2″)

Further: $$\frac{r_i g_i}{r_0 g_0} \simeq \frac{\Phi_i}{\Phi_0} \simeq \frac{r_0}{r_i} \quad , \quad \text{hence:} \quad \frac{g_{i'}}{g_0} \simeq \left(\frac{r_0}{r_i}\right)^2 \quad \ldots\ldots \quad (6.1.2''')$$

(6.1.1) with (6.1.2) gives:

$$\frac{\Delta W_{0i}}{r_0 g_0} \simeq - \frac{r_0}{r_i} \frac{\Delta h_{i'i}}{r_{i'}} \simeq \frac{r_0}{r_i} \Delta \ln \left(1 - \frac{h_{i'i}}{r_{i'}}\right) \quad \ldots\ldots\ldots\ldots (6.1.3)$$

$$r_i \simeq r_{i'} + h_{i'i} = r_{i'} \left(1 + \frac{h_{i'i}}{r_{i'}}\right) \quad , \quad \text{hence:}$$

$$r_{i'} \simeq r_i \left(1 - \frac{h_{i'i}}{r_{i'}}\right) \quad \ldots\ldots\ldots\ldots (6.1.4)$$

With (6.1.3) and (6.1.4) one obtains:

$$\frac{\Delta W_{0i}}{r_0 g_0} + \frac{r_0}{r_i} \Delta \left(\ln \frac{r_i}{r_0}\right) \simeq \frac{r_0}{r_i} \Delta \left\{\ln \frac{r_i}{r_0} \left(1 - \frac{h_{i'i}}{r_{i'}}\right)\right\}$$

or: $$\boxed{\frac{\Delta W_{0i}}{r_0 g_0} + \frac{r_0}{r_i} \Delta \left(\ln \frac{r_i}{r_0}\right) \simeq \frac{r_0}{r_i} \Delta \left(\ln \frac{r_{i'}}{r_0}\right)} \quad \ldots\ldots\ldots (6.1.5)$$

With the well-known approximation formula [HM, (2-150)]:

$$g_i \simeq g_{i'} \left(1 - \frac{2h_{i'i}}{r_{i'}}\right) \quad , \quad \text{or:}$$

$$g_{i'} \simeq g_i \left(1 + \frac{2h_{i'i}}{r_{i'}}\right) \quad \ldots\ldots\ldots\ldots (6.1.6)$$

With (6.1.3) and (6.1.6):

$$-2 \frac{\Delta W_{0i}}{r_0 g_0} + \frac{r_0}{r_i} \Delta \left(\ln \frac{g_i}{g_0}\right) \simeq \frac{r_0}{r_i} \Delta \left\{\ln \frac{g_i}{g_0} \left(1 + \frac{2h_{i'i}}{r_{i'}}\right)\right\}$$

or: $$\boxed{-2 \frac{\Delta W_{0i}}{r_0 g_0} + \frac{r_0}{r_i} \Delta \left(\ln \frac{g_i}{g_0}\right) \simeq \frac{r_0}{r_i} \Delta \left(\ln \frac{g_{i'}}{g_0}\right)} \quad \ldots\ldots\ldots (6.1.7)$$

The formulae (6.1.5) and (6.1.7) might be interpreted as a "free air reduction" of r_i and g_i to $r_{i'}$ and $g_{i'}$.

If the difference between P_M and P_C {$\overline{Y}^{(1)}$-terms} is neglected, then from (3.4.6) – (3.4.8), together with (6.1.5) and (6.1.7) follows {see also the remarkable resemblance with the geometrical interpretation of Molodenskii's approximate solution in [HM, section 8-8]}:

P_i, P_0 on S* ; $r_i \simeq r_0 \simeq R$
$\Delta\left(\dfrac{V_{0i}}{\Phi_0}-\dfrac{r_0}{r_i}\right) \simeq \dfrac{\Delta W_{0i}}{r_0 g_0}+\dfrac{r_0}{r_i}\,\Delta\left(\ln\dfrac{r_i}{r_0}\right) \simeq \dfrac{r_0}{r_i}\left(\Delta\ln\dfrac{r_{i'}}{r_0}\right)$
$-2\Delta\left(\dfrac{V_{0i}}{\Phi_0}-\dfrac{r_0}{r_i}\right)+\Delta\left(\dfrac{\Phi_i}{\Phi_0}-\dfrac{r_0}{r_i}\right)= \simeq -2\,\dfrac{\Delta W_{0i}}{r_0 g_0}+\dfrac{r_0}{r_i}\,\Delta\left(\ln\dfrac{g_i}{g_0}\right) \simeq \dfrac{r_0}{r_i}\left(\Delta\ln\dfrac{g_{i'}}{g_0}\right)$

. . . (6.1.8)

In (6.1.8), one recognizes the difference quantities occurring in (4.2.10). The influence of ΔW_{0i} *may* therefore be interpreted as a "free air reduction", but this is only an interpretation by which the formulae can be compared with the conventional theory. In reality one does *not* reduce; ΔW_{0i} is, as a stochastic observational quantity, part of the computing process. In this way the place and the meaning of levelling, gravity measurement and geometric elements in the theory are clear and operational.

In the interpretation mentioned, it is remarkable that the "free air reduction" is done with respect to an equipotential surface through the datum point P_0 {see figure 6.1-1}. This equipotential surface thus assumes the task of the "geoid" in conventional theory. But if a transition to another datum point is made, then the "free air reduction" is automatically done with respect to the equipotential surface through the new datum point. The "geoid" therefore moves along with the datum point!

The concept of "geoid" is therefore confusing, and since it can be avoided it is better to eliminate it from the theory. In applications at sea {see (4.2.9)}, the above-mentioned interpretation is not possible anyway, and one would need far-fetched arguments to give the "geoid" a place in the theory.

6.2 The fundamental equation of physical geodesy

We follow the treatment given in [HM, section 2-13]. But with a slight difference, in order to be able to leave remaining influences of the centrifugal potential on $(W_i - W_i^{\text{appr}})$ out of consideration {sections 1.7 and 3.4}. Instead, we start from:

$$\Delta V_i = V_i - V_i^{\text{appr}} \quad ; \quad -\frac{\partial V_i}{\partial r_i} \underset{\text{say}}{=} \gamma_i \qquad \text{(6.2.1)}$$

In the coordinate system of section 1.3, the point P_i^{appr} can be represented by $(X_i^{\text{appr}}, Y_i^{\text{appr}}, Z_i^{\text{appr}})$.

The following quantities pertain to this point:

$$r_i^{\text{appr}} \quad , \quad V_i^{\text{appr}} \quad , \quad \gamma_i^{\text{appr}} \qquad \{r_i^{\text{appr}}\,\gamma_i^{\text{appr}} = \Phi_i^{\text{appr}}\} \qquad \text{(6.2.2)}$$

According to chapter 4 one obtains by measurements and computations:

$$r_i = r_i^{\text{appr}} + \Delta r_i \quad , \quad V_i = V_i^{\text{appr}} + \Delta V_i \quad , \quad \gamma_i = \gamma_i^{\text{appr}} + \Delta\gamma_i \quad . \; . \; . \; . \; . \; (6.2.3)$$

which may be assigned to a point P_i, obtained by moving P_i^{appr} over a distance Δr_i along the line $P_M P_i^{\text{appr}}$. See figure 6.2-1.

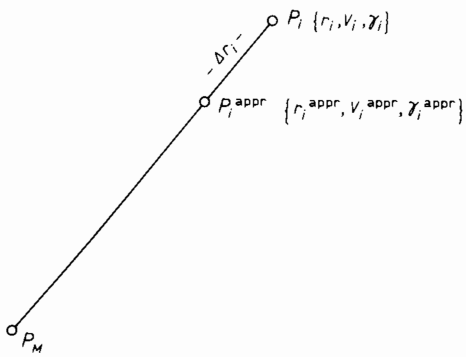

Fig. 6.2-1

But within the model of approximate values, one can also from (6.2.2) by Taylor's series, compute for P_i:

$$\left.\begin{aligned} P_i : \quad & V_i^{\text{appr}} + \left(\frac{\partial V_i}{\partial r_i}\right)^{\text{appr}} \Delta r_i = V_i^{\text{appr}} - \gamma_i^{\text{appr}} \Delta r_i \\ & \gamma_i^{\text{appr}} + \left(\frac{\partial \gamma_i}{\partial r_i}\right)^{\text{appr}} \Delta r_i \simeq \gamma_i^{\text{appr}} - 2 \left(\frac{\gamma_i}{r_i}\right)^{\text{appr}} \Delta r_i \end{aligned}\right\} \quad (6.2.4)$$

With (6.2.3) and (6.2.4) one thereby obtains in P_i the difference quantities:

$$\left.\begin{aligned} T_i = \delta V_i = V_i - \left\{ V_i^{\text{appr}} - \gamma_i^{\text{appr}} \Delta r_i \right\} = \Delta V_i + \gamma_i^{\text{appr}} \Delta r_i \\ \delta\gamma_i = \gamma_i - \left\{ \gamma_i^{\text{appr}} - 2 \left(\frac{\gamma_i}{r_i}\right)^{\text{appr}} \Delta r_i \right\} = \Delta\gamma_i + 2 \left(\frac{\gamma_i}{r_i}\right)^{\text{appr}} \Delta r_i \end{aligned}\right\} \quad (6.2.5')$$

From (6.2.5′) with (6.2.4):

$$\frac{\partial T_i}{\partial r_i} = -\frac{\partial V_i}{\partial r_i} + \left(\frac{\partial V_i}{\partial r_i}\right)^{\text{appr}} - \left(\frac{\partial \gamma_i}{\partial r_i}\right)^{\text{appr}} \Delta r_i = \delta\gamma_i \quad . \; . \; . \; . \; . \; . \; (6.2.5'')$$

Further omitting the suffix "appr" in the coefficients of difference quantities, it follows from (6.2.5) with $\Phi_i = \gamma_i r_i$:

$$\left.\begin{aligned}
&\frac{T_i}{\Phi_i} = \frac{\Delta V_i}{\Phi_i} + \Delta(\ln r_i) \\
&\frac{\left(-r_i \dfrac{\partial T_i}{\partial r_i}\right)}{\Phi_i} = \Delta(\ln \gamma_i) + 2\Delta(\ln r_i) = \Delta(\ln \Phi_i) + \Delta(\ln r_i)
\end{aligned}\right\} \quad (6.2.6)$$

Hence also:

$$\frac{T_i}{\Phi_i} - \frac{\Phi_i}{\Phi_0}\frac{T_0}{\Phi_i} = \frac{\Delta V_i}{\Phi_i} - \frac{\Phi_i}{\Phi_0}\frac{\Delta V_0}{\Phi_i} + \Delta\left(\ln \frac{r_i}{r_0}\right)$$

$$\frac{\left(-r_i \dfrac{\partial T_i}{\partial r_i}\right)}{\Phi_i} - \frac{\Phi_i}{\Phi_0}\frac{\left(-r_0 \dfrac{\partial T_0}{\partial r_0}\right)}{\Phi_i} = \Delta\left(\ln \frac{\Phi_i}{\Phi_0}\right) + \Delta\left(\ln \frac{r_i}{r_0}\right)$$

Multiplication by $\dfrac{\Phi_i}{\Phi_0} \simeq \dfrac{r_0}{r_i}$ gives, with (3.4.8):

$$\boxed{\begin{aligned}
&\frac{T_{0i}}{\Phi_0} - \frac{r_0 - r_i}{r_i}\frac{T_0}{\Phi_0} = \Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right) \\
&\frac{\left(-r_i \dfrac{\partial T_i}{\partial r_i}\right) - \left(-r_0 \dfrac{\partial T_0}{\partial r_0}\right)}{\Phi_0} - \frac{r_0 - r_i}{r_i}\frac{\left(-r_0 \dfrac{\partial T_0}{\partial r_0}\right)}{\Phi_0} = \Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right)
\end{aligned}} \quad (6.2.7)$$

The introduction of quantities like in (6.2.5) is unusual in geodetic adjustment problems, and it is not necessary either, in view of (6.2.7), where the more usual difference quantities from chapter 3 and 4 occur.

For a further comparison with existing literature, Δr_i from the second equation of (6.2.6) can be eliminated by means of the first equation. After some derivations one obtains:

$$\boxed{\Delta r_i = \frac{T_i - \Delta V_i}{\gamma_i}} \quad \text{. . . . . (6.2.8)}$$

$$\boxed{\Delta \gamma_i = -\frac{\partial T_i}{\partial r_i} - 2\,\frac{T_i - \Delta V_i}{r_i}} \quad \text{. . . . . (6.2.9)}$$

In spherical approximation, (6.2.8) can be compared with "Bruns's formula" $N = \dfrac{T}{\gamma}$ [HM, (2-144)], and (6.2.9) with the "fundamental equation of physical geodesy" $\Delta g = -\dfrac{\partial T}{\partial r} - \dfrac{2}{r} T$ [HM, (2-154)].

The difference in the formulae is due to the quantity ΔV_i. In the literature the following reasoning is made: Starting from observations or estimates r_i, T_i, $\dfrac{\partial T_i}{\partial r_i}$, γ_i, V_i one *chooses* r_i^{appr} and hence P_i^{appr} in such a way that $V_i^{\text{appr}} = V_i$. Then the estimate $\Delta V_i = 0$. Essentially, this is a very difficult problem of computing technique, and it does not fit well in our line of thought.

The set of points P_i^{appr} thus found is called "telluroid". But the estimates V_i and hence the points P_i^{appr} and the telluroid belong to one sample. If this were done for all possible samples, the approximate values and also the telluroid would become stochastic, which is impractical and therefore should be rejected.

If one sticks to the procedure described, i.e. to one sample, then for any other sample $\Delta V_i \neq 0$, hence for variates $\underline{\Delta V_i} \neq 0$ and for mean values $E\,\{\underline{\Delta V_i}\} \neq 0$. This means that neither "Bruns's formula" nor the "fundamental equation of physical geodesy" may be considered as model laws. One should replace them by e.g. (6.2.8) and (6.2.9), but these equations, according to (6.2.7), only mean a translation of quantities T into quantities V and r, as indicated in (6.2.7)*).

In the previous chapters it has been shown that the equations concerned are not necessary either. When they are eliminated from the theory this implies also the elimination of "the determination of T as a third boundary-value problem of potential theory" [HM, p. 86].

6.3 Green integrals and surface layer

Now that in section 6.2 the "fundamental equation of physical geodesy" has been eliminated from the theory, and is no more acceptable as a boundary condition, the question arises whether the so-called anomalous potential T may be expressed as the potential of a surface layer or coating on the earth's surface [HM, sections 6-5, 8-6]. In section 6.2 it has been shown that T and $\dfrac{\partial T}{\partial r}$ are functions of Δr, ΔV and $\Delta\Phi$. It is therefore logical to investigate whether the difference equations found in chapter 4 can be seen as an equivalent of the above-mentioned problem.

For this investigation, write the integral equation (4.1.11) in the following form:

$$
\left.
\begin{aligned}
&\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right) \underset{\text{say}}{=} \Delta X_{0i} \\
&-2\Delta X_{0i} + \Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) \underset{\text{say}}{=} \Delta\overline{X}_{0i} \\
&-(\tfrac{3}{2}\Delta X_{0i} + \Delta\overline{X}_{0i}) + \frac{3}{4\pi}\iint(\tfrac{3}{2}\Delta X_{0j} + \Delta\overline{X}_{0j})\left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\,\frac{r_j}{r_{0j}}\right)\mathrm{d}\Omega_j + \\
&\qquad\qquad + \Delta\overline{X}_{0i} = 0
\end{aligned}
\right\} \qquad (6.3.1)
$$

*) I think that in principle this also applies to the sharpened approach in [MORITZ, 1977, section 2], which only came to my notice after this section had been written.

or in a notation analogous to [HM, (8-32), (8-35)]:

$$\boxed{\begin{aligned} &0 = -\ 2\pi\phi_{0i} + \tfrac{3}{2}\iint \phi_{0j} \left(\frac{r_j}{r_{ij}} - \frac{r_0}{r_i}\ \frac{r_j}{r_{0j}}\right) \mathrm{d}\Omega_j + \Delta\overline{X}_{0i} \\ &\phi_{0i} = \frac{1}{2\pi}\ (\tfrac{3}{2}\Delta X_{0i} + \Delta\overline{X}_{0i}) \end{aligned}} \quad . \ . \ . \ . \ . \ . \ (6.3.2)$$

so that the equivalent presentation mentioned is indeed possible.
The solution of this integral is completely parallel to the solution of (4.1.11). One finds:

$$(\tfrac{3}{2}\Delta X_{0i} + \Delta\overline{X}_{0i}) = \Delta\overline{X}_{0i} +$$

$$+\tfrac{3}{2}\left[\frac{r_0}{r_i}\left\{\frac{R}{r_i}\ \Delta Y_i^{(1)} - \frac{R}{r_0}\ \Delta Y_0^{(1)}\right\} + \frac{1}{4\pi}\iint \mathrm{S}_{0i;j}^{(n-1)}\ \Delta\overline{X}_{0j}\ \mathrm{d}\Omega_j\right] \quad . \quad (6.3.3)$$

in analogy with [HM, (8-37)]. But (6.3.3) is in fact the same result as (4.2.10), so that the use of the surface layer method does not offer any advantage. In this respect the paper [KOCH, POPE, 1972] is curious, or would the course of thought of the authors support the theory developed here?

6.4 Height and height variation

In conventional geodesy, many concepts of height are in use. An excellent survey is found in [HM, chapter 4]. In the author's approach "the concept" triangulated {or trigonometric} height" has been incorporated in the spatial geometric quaternion theory, and the question arises whether a special concept of "height" may be introduced beside the concepts introduced earlier.
Consider, as an example, a situation on land, with points P_i, P_0 and P_j on S*, so that:

$$r_i \simeq r_0 \simeq r_j \simeq R \simeq 0.64 \cdot 10^7 \text{ m} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (6.4.1)$$

Then from (4.2.10) or (4.3.6) with (3.4.7) and (2.6.7) follows:

$$\left\{\frac{\Delta W_{0i}}{Rg_0} + \frac{\omega^2 R}{g_0}\left(\overline{Y}_i^{(1)} - \overline{Y}_0^{(1)}\right)\right\} + \left\{\Delta\left(\ln\frac{r_i}{r_0}\right) - \left(\Delta Y_i^{(1)} - \Delta Y_0^{(1)}\right)\right\} =$$

$$= \frac{1}{4\pi}\iint \mathrm{S}_{0i;j}^{(n-1)}\left\{-2\ \frac{\Delta W_{0j}}{Rg_0} + \Delta\left(\ln\frac{g_j}{g_0}\right) - 3\ \frac{\omega^2 R}{g_0}\left(\overline{Y}_j^{(1)} - \overline{Y}_0^{(1)}\right)\right\}\ \mathrm{d}\Omega_j \tag{6.4.2}$$

In order to emphasize spirit levelling we introduce as an abbreviated notation a kind of "dynamic height" {compare (2.6.4)}:

$$\frac{\Delta W_{0i}}{g_0} = -\sum_{k=0}^{i-1} \frac{g_{k.k+1}}{g_0} \Delta h_{k,k+1} \underset{\text{say}}{=} -\Delta H_{0i} \quad \ldots \ldots \ldots \ldots \quad (6.4.3)$$

Now assume additionally that in the situation (6.4.1) the following holds for use in the coefficients of difference quantities:

$$\left.\begin{array}{ll} & g_i \simeq g_0 \simeq g_j \simeq G \simeq 9.8 \text{ ms}^{-2} \\ \text{hence:} & \dfrac{R}{G} \simeq 0.65 \cdot 10^6 \text{ s}^2 \end{array}\right\} \quad (6.4.4)$$

then (6.4.2) with (6.4.3) and (6.4.4) becomes:

$$\boxed{\begin{array}{l} \left\{-\dfrac{\Delta H_{0i}}{R} + \dfrac{\omega^2 R}{G}\left(\overline{Y}_i^{(1)} - \overline{Y}_0^{(1)}\right)\right\} + \left\{\dfrac{\Delta r_i - \Delta r_0}{R} - \left(\Delta Y_i^{(1)} - \Delta Y_0^{(1)}\right)\right\} = \\ = \dfrac{1}{4\pi} \displaystyle\iint S_{0i;j}^{(n-1)} \left\{+2\,\frac{\Delta H_{0j}}{R} + \frac{\Delta g_j - \Delta g_0}{G} - 3\,\frac{\omega^2 R}{G}\left(\overline{Y}_j^{(1)} - \overline{Y}_0^{(1)}\right)\right\} d\Omega_j \,. \\ \hline (3.3.4), (3.4.5)\text{: } \dfrac{\Delta r_i - \Delta r_0}{R} - \left(\Delta Y_i^{(1)} - \Delta Y_0^{(1)}\right) = \dfrac{\Delta r_i' - \Delta r_0'}{R} \quad \{\text{with respect to } P_C\} \\ \hline P_0 \text{ datum point means: } \Delta r_0 \text{ and } \Delta g_0 \text{ are taken to be zero.} \end{array}} \quad (6.4.5)$$

(6.4.5) means in principle, that from levelling and gravity measurements the metric height for P_i follows as the distance r_i' to the centre of mass of the earth $\{P_C\}$, provided that the small ω^2-terms can be computed or neglected. With P_0 as datum point, this concept of height becomes in fact r_i'/r_0', in other words this height concept is nothing but a radial component of the X, Y, Z-system in a spatial S-system. In this view, the introduction of some extra concept of "height" is *not* necessary.

Some examples for illustration:

I. In section 4.3 under item I, mention was made of inexplicable differences between geodetic and oceanic levelling. In order to find out whether there are sources of systematic errors in geodetic levelling, it is investigated in the U.S.A. to what extent V.L.B.I. or other very precise geometric network elements may be helpful. (6.4.5) shows that this concerns data for Δr_i, which in combination with ΔH_{0j} and Δg_j provide condition equations giving the possibility to test for an alternative hypothesis {connected with the suspected source of error}.

In this case one must have data on the relative position of P_M and P_C, because the distances between points P_i are of the order of 1000 km or more. Theoretically, the problem can be clearly analyzed, but the practical solution will be difficult.

II. In a spatial photogrammetric network, computed by a bundle method from photographs with 60% forward and lateral overlap, one meets a curious {theoretical} difficulty. From the ellipsoidal data for "given points", cartesian spatial coordinates are computed, the "height" in the above-mentioned sense usually being rather dubious [HM, (5-5)]. The area of the network is now rather small, with a diameter of e.g. 100 km. In (6.4.5) the ω^2-terms can then be neglected if P_0 is chosen in or near the network. If there are enough levelling and gravity measurements, (6.4.5) provides the possibility to compute corrections of "heights" $\Delta r_i'$ for the "given points".

III. Consider the problem of a regional subsidence. Choose S_a {and hence S_a^*} as a part of S in such a way that outside S_a there is no change of H, r or g, and assume that the centre of gravity of the earth remains unchanged. Choose P_0 outside, but close to S_a; see figure 6.4-1.

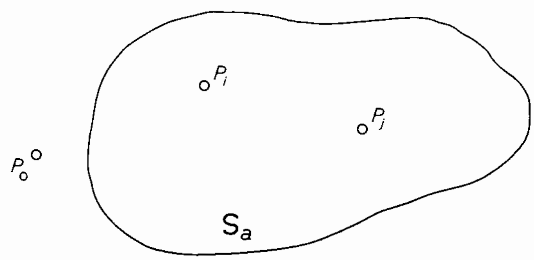

Fig. 6.4-1

If the same approximate values are used for measurement in the first period t_{I} and in a later period t_{II}, (6.4.5) gives {in the system with P_0 as datum point}:

$$\boxed{\begin{aligned} & -\frac{H_{i,\mathrm{II}}^{(0)} - H_{i,\mathrm{I}}^{(0)}}{R} + \frac{r_{i,\mathrm{II}}^{(0)} - r_{i,\mathrm{I}}^{(0)}}{R} = \\ & = \frac{1}{4\pi} \iint\limits_{\Omega_a} S_{0i,j}^{(n-1)} \left\{ +2\, \frac{H_{j,\mathrm{II}}^{(0)} - H_{j,\mathrm{I}}^{(0)}}{R} + \frac{g_{j,\mathrm{II}}^{(0)} - g_{j,\mathrm{I}}^{(0)}}{G} \right\} d\Omega_j \end{aligned}} \quad . \; . \; . \quad (6.4.6)^*)$$

From (6.4.6), metric height differences $r_{i,\mathrm{II}}^{(0)} - r_{i,\mathrm{I}}^{(0)}$ can be computed. If water levels are to be compared, $H_{i,\mathrm{II}}^{(0)} - H_{i,\mathrm{I}}^{(0)}$ is more useful {levelling only}.

Now suppose that mass is removed from, or added to the subsoil {extraction of oil, flooding of old mines}, and assume that the structure of the soil is such that $r_{i,\mathrm{II}}^{(0)} - r_{i,\mathrm{I}}^{(0)} = 0$. Then (6.4.6) indicates that from $g_{j,\mathrm{II}}^{(0)} - g_{j,\mathrm{I}}^{(0)} \neq 0$ follows: $H_{i,\mathrm{II}}^{(0)} - H_{i,\mathrm{I}}^{(0)} \neq 0$. This means that one should be cautious in inferring subsidence or uplift from levelling results only.

*) In [STRANG VAN HEES, 1977], an interesting check on (6.4.6) is described. Starting from corrections for free air reduction, these are subsequently interpreted as observation variates, i.e. the approach of section 6.1 in the opposite way. An objection to this method of derivation is that the variates are not defined with sufficient exactness.

7. REMARKS ON COLLOCATION

7.1 Isotropic covariance functions

Many interesting studies have been published on the theoretical complex in which isotropic covariance functions find a place, the theory of stationary stochastic processes {[HM, chapter 7], [KRARUP, 1969], [MEISSL, 1971], [MORITZ, 1973], the studies by MORITZ published in the series of the Department of Geodetic Science of Ohio State University, etc.}. We shall not here go into these problems; it is only mentioned that the following covariance function belongs to an isotropic stochastic process on the unit sphere {[MEISSL, 1971], [GRAFAREND, 1976]}:

$$C\{\psi_{ik}\} = \sum_{n=0}^{\infty} \bar{c}^{(n)} P^{(n)} (\cos \psi_{ik}) \quad , \quad \bar{c}^{(n)} > 0 \tag{7.1.1'}$$

or with [HM, (1-82')] in the notation (1.4.3), compare [KRARUP, 1969, p. 23]:

$$C\{\psi_{ik}\} = \sum_{n=0}^{\infty} \bar{c}^{(n)} Y_i'^{(n)} Y_k'^{(n)} \quad , \quad \bar{c}^{(n)} > 0 \tag{7.1.1''}$$

The coefficients $\bar{c}^{(n)}$ can be interpreted as 'degree variances" on the analogy of [HM, p. 259].

Assume now the three types of difference quantities in the left hand members of (3.1.6) to be as many stochastic processes outside the sphere $\bar{S}$ with centre P_M and radius R, e.g. with an interpretation as in [KRARUP, 1969, p. 21 ff]. Then the covariance function (7.1.1) {with a further specified completion of the coefficients} is valid for the situation: $r_0 \simeq r_i \simeq r_k \simeq R$

It is essential to couple this assumption with the difference quantities, because for these quantities the surface S or S* may be replaced by the sphere $\bar{S}$. This gives the possibility to define a stochastic process 'invariant with respect to rotations around the centre P_M".

From (7.1.1) then follows for the difference quantities in (3.1.6), compare [KRARUP, 1969, p. 23]:

$r_0, r_i, r_k \gtrsim R$	$C\{\psi_{ik}\}$	
$\Delta\left(\frac{V_i}{\Phi_0} - \frac{r_0}{r_i}\right)$	$\frac{r_0}{r_i}\frac{r_0}{r_k}\sum_{n=0}^{\infty} \bar{c}^{(n)} \left(\frac{R}{r_i}\right)^n \left(\frac{R}{r_k}\right)^n Y_i'^{(n)} Y_k'^{(n)}$	
$\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right)$	$\frac{r_0}{r_i}\frac{r_0}{r_k}\sum_{n=0}^{\infty} (n+1)^2 \bar{c}^{(n)} \left(\frac{R}{r_i}\right)^n \left(\frac{R}{r_k}\right)^n Y_i'^{(n)} Y_k'^{(n)}$	(7.1.2)
$\Delta\left(\frac{\Psi_i}{\Phi_0} - 2\frac{r_0}{r_i}\right)$	$\frac{r_0}{r_i}\frac{r_0}{r_k}\sum_{n=0}^{\infty} (n+2)^2 (n+1)^2 \bar{c}^{(n)} \left(\frac{R}{r_i}\right)^n \left(\frac{R}{r_k}\right)^n Y_i'^{(n)} Y_k'^{(n)}$	

But likewise one can compute the covariance between other types of difference quantities, the computing formalism being the same as in the law of propagation of covariances in adjustment theory. For example:

$$\text{Covariance}\left\{\Delta\left(\frac{V_i}{\Phi_0}-\frac{r_0}{r_i}\right) \quad , \quad \Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)\right\} =$$

$$= \frac{r_0}{r_i}\,\frac{r_0}{r_k}\sum_{n=0}^{\infty}(n+1)\,\bar{c}^{(n)}\left(\frac{R}{r_i}\right)^n\left(\frac{R}{r_k}\right)^n Y_i'^{(n)}\,Y_k'^{(n)}\,. \quad (7.1.3)$$

The quantities mentioned in (7.1.2) can only be seen as abstractions; they are not estimable in the sense of chapter 2. This means that the quantities from (3.1.7) must take the place of (3.1.6). For a number of examples this has been elaborated in (7.1.4) {see next page}.

The partial covariance matrices of every type of quantity ΔX_{0i} in (7.1.4) have no rank deficiency because of the introduction of the datum point P_0 {the case $i = 0$ is excluded}; the positive definiteness can be proved as in [KRARUP, 1969, p. 23]. Between the different types of quantities ΔX_{0i} there are, however, functional relationships {Poisson- and Green integrals such as (4.5.2), (4.2.9), (4.2.10), (4.5.3), (5.3.5)} which can serve to investigate questions of rank deficiency. One can, consequently, compute covariances by (3.1.7), but also by these functional relationships [MORITZ, 1973]. This has importance for section 7.4.

Remark 1

Perhaps it is advisable in (7.1.4) always to join the first degree terms with ΔX_{0i}. Thereby one obtains e.g., see (4.2.10):

$$\text{Covariance}\left[\left\{\Delta\left(\frac{V_{0i}}{\Phi_0}-\frac{r_0}{r_i}\right)-\frac{r_0-r_i}{r_i}\,\Delta\left(\frac{V_0}{\Phi_0}\right)-\frac{r_0}{r_i}\left(\frac{R}{r_i}\,\Delta Y_i^{(1)}-\frac{R}{r_0}\,\Delta Y_0{}^{(1)}\right)\right\}\right. ,$$

$$\left.\left\{-2\left(\Delta\left(\frac{V_{0k}}{\Phi_0}-\frac{r_0}{r_k}\right)-\frac{r_0-r_k}{r_k}\,\Delta\left(\frac{V_0}{\Phi_0}\right)\right)+\Delta\left(\frac{\Phi_k}{\Phi_0}-\frac{r_0}{r_k}\right)\right\}\right] =$$

$$= \frac{r_0}{r_i}\,\frac{r_0}{r_k}\sum_{n=2}^{\infty}(n-1)\,\bar{c}^{(n)}\left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}\left\{\left(\frac{R}{r_k}\right)^n Y_k'^{(n)}-\left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}$$

All summations in (7.1.4) then run from n = 2.

Remark 2

The considerations in this section only give an indication of theoretical possibilities. For methods aimed at applications see e.g. [MORITZ, 1976, 1977], [TSCHERNING, 1978].

$\sum_{n=a}^{\infty} A_n$ from:	$C\{\psi_{ik}\} = \frac{r_0}{r_i}\frac{r_0}{r_k}\sum_{n=a}^{\infty} A_n \bar{c}^{(n)} \left\{\left(\frac{R}{r_i}\right)^n Y_i'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\} \left\{\left(\frac{R}{r_k}\right)^n Y_k'^{(n)} - \left(\frac{R}{r_0}\right)^n Y_0'^{(n)}\right\}$				
ΔX_{0i} \ ΔX_{0k}					
$\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)$	$\sum_{n=1}^{\infty} 1$	$\sum_{n=1}^{\infty}(n+1)$	$\sum_{n=2}^{\infty}(n-1)$	$\sum_{n=1}^{\infty}(n+2)(n+1)$	$\sum_{n=2}^{\infty}(n-1)(n+1)$
$\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right)$		$\sum_{n=1}^{\infty}(n+1)^2$	$\sum_{n=2}^{\infty}(n-1)(n+1)$	$\sum_{n=1}^{\infty}(n+2)(n+1)^2$	$\sum_{n=2}^{\infty}(n-1)(n+1)^2$
$-2\left\{\Delta\left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\Delta\left(\frac{V_0}{\Phi_0}\right)\right\} + \Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right)$			$\sum_{n=2}^{\infty}(n-1)^2$	$\sum_{n=2}^{\infty}(n-1)(n+2)(n+1)$	$\sum_{n=2}^{\infty}(n-1)^2(n+1)$
$\Delta\left(\frac{\Psi_{0i}}{\Phi_0} - 2\frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$				$\sum_{n=1}^{\infty}(n+2)^2(n+1)^2$	$\sum_{n=2}^{\infty}(n-1)(n+2)(n+1)^2$
$-3\Delta\left(\frac{\Phi_i}{\Phi_0} - \frac{r_0}{r_i}\right) + \left\{\Delta\left(\frac{\Psi_{0i}}{\Phi_0} - 2\frac{r_0}{r_i}\right) - \frac{r_0 - r_i}{r_i}\Delta\left(\frac{\Psi_0}{\Phi_0}\right)\right\}$					$\sum_{n=2}^{\infty}(n-1)^2(n+1)^2$

(7.1.4)

7.2 Isotropy and criterion covariance functions

Is it possible to describe the global, regional or local behaviour of functions ΔX_{0i} from (7.1.4) by means of *isotropic* stochastic processes? Sampling speaks against this, see e.g. the analysis of data of ocean and continental gravity in [GAPOSCHKIN, 1973] and [WILLIAMSON and GAPOSCHKIN, 1975] and other analyses mentioned in [KEARSLEY, 1977].

By authors like KRARUP and MORITZ it is repeatedley stated that in least-squares collocation the use of homogeneous and isotropic covariance functions {but now interpreted as a kernel function in Hilbert space} is not objectionable; see also [SANSÒ, 1978], [MORITZ, 1978]. Perhaps, an argument for this is also that the kernel functions in our Poisson- and Green integrals have this same character, and that these integral formulae can be considered as limiting cases of collocation [MORITZ, 1975].

In order to get a better agreement between sampling and isotropic covariance functions, several authors advocate the separation of a deterministic "trend" in the data in such a way that the experimental covariance functions thus derived also have a sufficiently isotropic character; see e.g. [DREWES, 1976]. This separation of the trend is, however, applied in a rather opportunistic way and does not really seem to fit well into the total theoretic approach. In our present approach, however, this "separation of the trend" can get a clear place and purpose.

First let us remark that gravity potential, gravity, etc. only occur in one realization on earth. Observations therefore cannot be changed, contrary to geometric networks where by the choice of network points one can pursue isotropy in the covariance matrix of coordinates [BAARDA, 1977].

But our theory of gravimetric geodesy is constructed from difference quantities, and consequently observations relate to difference variates. This means that the possibility to give corrections to these difference variates {and hence the application of a "trend separation"} must be found in the model of approximate values. The word model must be emphasized here: simple corrections for topography and isotasy are not sufficient, it will be necessary to include geophysical hypotheses in the approximate potential model in an attempt to describe irregularities in the mass distribution of the earth, as stated in section 1.2. The concept of "separation of trend" is thus replaced by model building, by way of the introduction of geophysical hypotheses as alternative hypotheses. This means the possibility of choice between different geophysical conjectures, founded in part on non-geodetic observations. Now this choice can be partly determined by the requirement that experimental covariance functions {or -matrices} of the observed variates ΔX_{0i} from (7.1.4) are sufficiently in agreement with *criterion covariance functions* {or *-matrices*}, constructed on the basis of homogeneity and isotropy; also the order of magnitude for ΔX_{0i}, required in section 1.2, can then be taken care of stochastically. In fact this means that the model of approximate values is equally well adapted to the observational data in all directions over the earth.

It is a matter for further theoretical research to work this out for practical application. Several of the publications referred to already give an initiation to this research. In essence, the testing of experimental covariance functions or -matrices with respect to criterion covariance functions or -matrices can be done along the lines already worked out for geometric networks.

In following this approach one could kill three birds with one stone:

1. the requirements of section 1.2 can be met;
2. a sound base for cooperation with geophysicists is found;

3. in least-squares collocation, many theoretical and practical difficulties are avoided by the use of the {theoretical} criterion covariance functions or -matrices.

It will now be interesting to make a comparison with the approach in the publications [MORITZ, 1976, 1977], [TSCHERNING and FORSBERG, 1978].

Now that a good base has been found for the establishment and the use of covariance functions of the deterministic part of variates ΔX_{0i}, it has become possible, on the one hand, to apply test methods {including the noise part of ΔX_{0i}} also in gravimetric geodesy for the indication and detection of measurement blunders or model deformations, and, on the other hand, to make a comparison between geometric {or more generally: network-} geodesy and gravimetric geodesy:

	geometric {or network} geodesy	gravimetric geodesy
choice of approximate values	not essential {in adjustment by condition equations}	essential; dependent on alternative hypotheses on mass distribution in the earth
experimental covariance functions or -matrices	dependent on choice of network points	dependent on choice of model of approximate values
homogeneity and isotropy	network precision equivalent in all directions; test by criterion covariance matrices	approximate values in all directions equally well adapted; test by criterion covariance functions or -matrices
increase of coefficients in variance formulae {e.g. degree variances}	for network coordinates: from higher order to lower order	for ΔX_{0i}: from global to local fitting (?)

7.3 Averages of difference quantities over S*

A possible objection to the approach of section 7.2 is that the averages of difference quantities ΔX_{0i} over S* {and therefore in our approximation: over $\overline{S}$} are not zero. See e.g. [KOCH, 1977]. According to [HM, (7-1)] we have with (3.1.7) and (3.1.6):

$$\overline{E}\left\{\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}=\frac{1}{4\pi}\iint\sum_{n=1}^{\infty}\left\{\Delta Y_j^{(n)}-\Delta Y_0^{(n)}\right\}d\Omega_j=$$

$$=-\sum_{n=1}^{\infty}\Delta Y_0^{(n)}=\Delta\left(\frac{M}{\Phi_0 r_0}\right)-\Delta\left(\frac{V_0}{\Phi_0}\right)\qquad\ldots\ldots(7.3.1')$$

Thus one obtains:

P_0, P_j on S*	$\bar{E}\{\Delta X_{0j}\}$
$\bar{E}\left\{\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}$	$\Delta\left(\frac{M}{\Phi_0 r_0}\right)-\Delta\left(\frac{V_0}{\Phi_0}\right)$
$\bar{E}\left\{\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}$	$\Delta\left(\frac{M}{\Phi_0 r_0}\right)$
$\bar{E}\left\{-2\Delta\left(\frac{V_{0j}}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)\right\}$	$-\Delta\left(\frac{M}{\Phi_0 r_0}\right)+2\Delta\left(\frac{V_0}{\Phi_0}\right)$
$\bar{E}\left\{\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}$	$2\Delta\left(\frac{M}{\Phi_0 r_0}\right)-\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$
$\bar{E}\left\{-3\Delta\left(\frac{\Phi_j}{\Phi_0}-\frac{r_0}{r_j}\right)+\Delta\left(\frac{\Psi_{0j}}{\Phi_0}-2\frac{r_0}{r_j}\right)\right\}$	$-\Delta\left(\frac{M}{\Phi_0 r_0}\right)-\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$
Compute constants	**from**
$\Delta\left(\frac{M}{\Phi_0 r_0}\right)$	(4.4.3), (4.4.4)
$\Delta\left(\frac{V_0}{\Phi_0}\right)$	(4.4.6), (4.4.7)
$\Delta\left(\frac{\Psi_0}{\Phi_0}\right)$	(5.4.4)

(7.3.1″)

From the line of thought of chapter 2 it is understandable that $\bar{E}\,\{\Delta X_{0j}\}$ is dependent on the data of the datum point P_0.

According to (7.3.1) the three "constants" needed can be computed, via integral formulae, from observations. These observations are heterogeneous in accuracy {precision *and* reliability} and with respect to their position on earth. It is just this heterogeneity that was to be avoided or made less harmful by applying the collocation method.

It is not quite clear to what extent the testing of alternative geophysical hypotheses according to section 7.2 is hampered by this. Perhaps it is sufficient to increase criterion covariance matrices by a constant amount for each type ΔX_{0i}, and to join together the deterministic or signal part and the noise part.

7.4 Collocation or integral formulae*)

Least-squares collocation computes the signal value of ΔX_{0i}, making use of a smoothing process by means of signal + noise of same or other types ΔX_{0j}.
Consider now, as an example, the situations in (4.3.4) and (4.3.6). In the first place one computes for both situations:

$$\Delta \left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) \quad \ldots \quad (7.4.1)$$

but after this the function value of (7.4.1) has to be split up:

$$(4.3.4)\colon\ \Delta \left(\frac{V_{0i}}{\Phi_0}\right) = \Delta \left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) + \Delta \left(\frac{r_0}{r_i}\right) \quad \ldots \quad (7.4.2')$$

$$(4.3.6)\colon\ \Delta \left(\frac{r_0}{r_i}\right) = -\Delta \left(\frac{V_{0i}}{\Phi_0} - \frac{r_0}{r_i}\right) + \Delta \left(\frac{V_{0i}}{\Phi_0}\right) \quad \ldots \quad (7.4.2'')$$

(7.4.2) can only be computed by a least-squares method, making use of the noise-properties {probability distributions} of the variates in the right hand members. Therefore it is important to develop a procedure which is well-connected with the least-squares method of adjustment.
The present author has always wondered why, during the last decade, the computation by collocation methods is preferred to the computation by integral equations. Some remarks may serve to illustrate his reluctance to accept this.
Consider again, as an example, the integral equations (4.3.4) or (4.3.6), and write them in the form:

$$\Delta X_{0i} = L_{ij}\, \Delta X_{0j} \quad \ldots \quad (7.4.3)$$

On account of these *functional relationships* the relation between the signal-covariances of section 7.1 can be written symbolically as:

$$\left.\begin{aligned} C_{ii'} &= L_{ij}\, C_{jj'}\, L_{j'i'} \\ C_{ij'} &= L_{ij}\, C_{jj'} \end{aligned}\right\} \quad \ldots \quad (7.4.4)$$

For practical computations the integration in (7.4.3) is replaced by summation. If vectors or matrices are denoted by (. . .), and L_{ij} are taken to be coefficients instead of operators, (7.4.3) can be written as:

$$(\Delta X_{0i}) = (L_{ij})\,(\Delta X_{0j}) \quad \ldots \quad (7.4.5)$$

For a similar conversion of (7.4.4) it has to be presumed that a sufficiently dense and regular field of stations over the earth is available; only in that case the following is valid with a sufficient approximation:

---

*) Section 7.4 has been re-written after the insertion of section 1.8.

$$\left.\begin{aligned}&(C_{ii}) = (L_{ij})\,(C_{jj})\,(L_{ji})\\ &(C_{ij}) = (L_{ij})\,(C_{jj})\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7.4.6)$$

If the ΔX_{0j} in (7.4.5) are algebraically independent, it is always possible to write:

$$\left.\begin{aligned}&(L_{ij}) = (H_{ij})\,(H_{jj})^{-1} \qquad \text{, with}\\ &(H_{jj}) \text{ an arbitrary positive definite matrix}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7.4.7)$$

From (7.4.7) follows for (7.4.6) and (7.4.5):

$$(L_{ij}) = (C_{ij})\,(C_{jj})^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4.8')$$

$$(\Delta X_{0i}) = (C_{ij})\,(C_{jj})^{-1}\,(\Delta X_{0j}) \quad \ldots\ldots\ldots\ldots\ldots \quad (7.4.8'')$$

Hence (7.4.8') and (7.4.8'') are collocation formulae like the ones treated in [MORITZ, 1975, section 3], with the purpose to consider integral formulae as limiting cases of collocation.
In the train of thought of the present publication it is, moreover, possible that observations for (ΔX_{0i}) *and* (ΔX_{0j}) are available, i.e. observations of the {noise-} variates $(\underline{\Delta X}_{0i})$ and $(\underline{\Delta X}_{0j})$. In that case the functional relationships (7.4.3) or (7.4.5) are *only* valid for the expectations $(E\{\underline{\Delta X}_{0i}\})$ and $(E\{\underline{\Delta X}_{0j}\})$:

$$(E\{\underline{\Delta X}_{0i}\}) - (L_{ij})\,(E\{\underline{\Delta X}_{0j}\}) = (0) \quad \ldots\ldots\ldots\ldots\ldots \quad (7.4.9)$$

On account of the mixed composition of the variates $\underline{\Delta X}_{0i}$ and $\underline{\Delta X}_{0j}$, the noise covariance matrix:

$$\begin{pmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4.10)$$

is, in general, a regular, non-diagonal matrix. In contrast, the signal covariance matrix:

$$\begin{pmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4.11)$$

is singular under the assumptions made with (7.4.8).

Now the least-squares adjustment based on the condition model (7.4.9) can be executed with (7.4.10) or with (7.4.10) plus (7.4.11). In both cases the same solution is obtained, because the influence of (7.4.11) is cancelled by the functional relationship (7.4.5). In this way one therefore obtains a consistent system, without contradiction between collocation- and adjustment methods and without hindrance from the problems treated in section 7.3. But it should be noted that (7.4.8′) will only be a coarse approximation for a field of points that has a low density and is irregularly distributed over the earth, if the elements of (7.4.11) are computed by (7.4.4). In such a case the choice of method seems to be a matter of taste. The results are not likely to be very reliable.

The approach just sketched has a second advantage. From section 1.8 it is evident that, among others, the integral equations (4.3.4) or (4.3.6) are approximations, acceptable in some cases, inacceptable in others. But it follows that the same applies to (7.4.5) as well as to (7.4.8). In other words, also in this respect there is no difference between solutions based on the integral formulae mentioned and solutions based on collocation. Under the assumptions made, they are equally good or equally bad. However, the possibility remains to correct the integral equations and, consequently, their solutions.

In the collocation methods now in use, in (7.4.8) (C_{jj}) is replaced by $(C_{jj} + D_{jj})$. This is founded on the assumption that:

$$E\{\underline{\Delta X}_{0j}\} = 0 \qquad , \qquad \begin{array}{l} \bar{E}\{\Delta X_{0j}\} = 0 \\ \bar{E}\{\Delta X_{0i}\} = 0 \end{array}$$

But according to section 7.3 these assumptions are *not* valid as far as they pertain to the averages $\bar{E}$. In other words, the following is *not* necessarily valid:

$$(\Delta X_{0i}) = (C_{ij})\,(C_{jj} + D_{jj})^{-1}\,(\Delta X_{0j}) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7.4.12)$$

This could be one of the reasons to question the deduction in [MORITZ, 1975, section 4] to the effect that coefficients in Green integrals can be dependant on noise, although the deduction as such is logical. A possible contradiction with the present author's theory of linking-up mathematical models is then avoided; this theory states that functional relationships must always pertain to expected values of the noise-variates. See [BAARDA, 1967, chapter 4].

One could possibly correct (7.4.12) for the $\bar{E}$-values, but then one meets the difficulties mentioned in section 7.3. Besides, these $\bar{E}$-values are functions of observations, so that the computation of the noise-covariance matrix (7.4.10) will be very complicated. Equally complicated problems are then met in connection with a condition model like (7.4.9). A conjecture might even be that a smoothing process is already brought about by an adjustment procedure, as was stated in connection with (7.4.9) {see also section 4.3}. If so, collocation methods with signal + noise would be superfluous in this respect. It is a hesitating conjecture, which also touches upon the fascinating theory of Integrated Geodesy by T. KRARUP.

Finally we return once more to the situation where areas with a high point density alternate with areas where density is low and distribution of points is irregular. In [MORITZ, 1975, section 6] it is proposed to apply integral formulae and collocation alternately.
In the existing theory the application of integral formulae is always concerned with local irregularities in position and density of stations over the whole earth. In our theory, however, it is possible to apply integral formulae {the Green integrals} to regional areas, by means of the

good choice of a datum point. These regions can be chosen in accordance with the regularity and density of stations. Of course, the difficulty remains that regions of this type alternate with regions without many stations, so that the connection of local or regional systems to form a global system cannot be done without loss of overall precision and reliability. But this inevitable objection is valid both for integral formulae and for collocation.

Thus we reach the *conclusion* that the theory of difference quantities developed in the present publication makes it possible to use the computation by integral formulae in many more situations than the existing theory admits.

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