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Errata (Vol. 2, No. 4)

Page	Instead of	Read
23, line 19	(2.3.16)	(2.2.16)
25, formula (3.2.9)	$ \nabla_0 = \frac{\lambda_0}{N}$	$ \nabla_0 = \sqrt{\left(\frac{\lambda_0}{N}\right)}$
27, formula (3.2.16)	„	„
30, formula (3.2.24)	„	„

PREFACE

In this paper the starting point is formed by some elementary applications of statistical methods to geodetic adjustment problems. In its line of thought the paper is connected to earlier publications, such as [BAARDA 1960, 62]. An indication is given of some difficulties of principle which are inherent in these applications, and which often make the value of tests illusory. This leads to what the author in recent years has come to consider as the central problem of geodesy, viz. the introduction (linking-up) of mathematical models for the description of measured quantities and their relations, including the possibility of choice between different but related models. These thoughts have been directive for the design of several new geodetic theories, one of which, the so-called “polygon theory in the complex plane” is now nearly complete both from a theoretical and practical point of view. An impression of this theory has been given in the paper „A Generalization of the Concept Strength of Figure”, which was written in 1962 for the I.A.G. Special Study Group No. 1: 14, now included as an appendix. This paper also gives an impression of the possibilities (and difficulties) of a quantification of the objectives of geodesy. Other aspects of this theory are treated by KRIJGER [1966].

Mention should be made of a different approach to these problems, which is based on decision theory and of which ALBERDA [1966] gives an outline. An interesting problem for the years to come will be to find the line of thought connecting these approaches.

The author is much indebted to Ir. J. E. ALBERDA for his translation of the manuscript and for his stimulating remarks in discussing the underlying theory.

Delft, June 1966

W. BAARDA

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1 INTRODUCTION

In a discussion on the application of methods of mathematical statistics to geodesy, it is necessary to indicate very precisely what actually is the subject of the discussion.

In general, mathematical statistics, in connection with mathematical physics, aims at the description of a chain (in time) of physical events, which has been observed by man, by a mathematical model, so that an analysis becomes possible, and it can be hoped to obtain a possibility for predicting future events. After first experiences one can try to lend nature a helping hand to accentuate connections and relations; the stage of experimentation is entered. A geodesist taking part in this game is not different from scientists in other professions, and he, too, will be forced to make himself acquainted with the discipline of mathematical statistics, for which he has many excellent handbooks at his disposal. Many articles in the present-day geodetic periodicals bear witness of such studies and treat a part of mathematical statistics which the author judges to be applicable in his field of work. An example is [BAARDA 1960].

Until recently, the description was usually considered to have been evolved from series of repeated measurements, so that from the frequency theory, by further abstraction, one arrived at the theory of probability. Characteristic of this line of thought was the work of the English school (FISHER, NEYMAN, PEARSON) and the theory presented by VON MISES. A beautiful mathematical rounding-off linked to the abstract-axiomatic theory of the Russian school was given by CRAMÉR [1946]. See also the approach in [BAARDA 1960, 62].

In opposition to this approach there has of old been the approach using a more personal, subjective concept of probability, which is being increasingly appealed to in these last decades after the development of decision theory by WALD. Whereas many mathematical formulations in this theory are in agreement with the first mentioned theory, the interpretation can be completely different. And it is exactly this circumstance which makes it so difficult to generalize on the application of mathematical statistics in geodesy. One will have to choose, and indicate explicitly what part of mathematical statistics is under discussion and which theoretical approach one wishes to follow with regard to the interpretation.

From a purely mathematical point of view, a difference in the foundations (basic axioms) does not make any difference in the question whether a theory is true or false. However, it is an entirely different matter if one wishes to provide mathematical quantities with labels establishing a connection with measurable characteristics of physical quantities. Naturally it is possible to do this labelling carelessly, it is only a matter of names. But if a geodesist wishes to arrive at a description or an analysis of physical events, thereby aiming at the possibility to predict, he is essentially leaving pure mathematics, and enters geodesy. It is generally agreed that only the execution of experiments makes it possible to choose a mathematical descriptive model. The choice of the design of experiments is a personal one; this explains the

existence of conflicting views among scientists working in the same field. For everyone will aim at a design which looks most appropriate to *him*, and on the basis of his experiment he will arrive at the choice of a mathematical theory which is most acceptable to himself: every theory is *true* in itself. Thus the situation is possible, that a mathematical theory is true, but the theory plus its application to measurements *false*. But *false* is an elastic concept; depending on the situation it may mean: good enough, almost good enough, definitely unsatisfactory, entirely useless. This means that one of the tasks of scientists working in the same field, like geodesists, is to find criteria for judging the degree of falsehood. But how must we formulate these criteria, and from what should they be derived?

When working on a practical problem, the geodesist has in principle only a *single* measurement at his disposal for each of the quantities observed. He is then faced with the difficult problem how to connect the numbers resulting from the measurements, with mathematical relations, describing on the one hand functional relationships, on the other hand randomness. The observations are no part of an experiment, because in geodetic practice, measurements are in principle not repeated. The connection with a mathematical model describing functional relationships and randomness must therefore be established with the aid of information which has been obtained in addition to the collection of observational numbers. This information is given by the description of the measuring procedure and the circumstances, such as the weather, the character of the terrain etc. In [BAARDA 1960, 62] this has been called the “registration” R . Let the observational numbers form the vector (x^i) ; then the total observation is given by:

$$\{(x^i); R\}$$

One can now compare R with previous experiments, possibly executed with special reference to the measurement at hand. Usually, such experiments are very restricted in size, but by the comparison one arrives at the statement:

- a. (x^i) is a sample (of size one) of the vector of stochastic variables (x^i) . The probability distribution of the stochastic variables is determined up to at most a number of unknown parameters.
- b. Between the unknown expectations $E\{x^i\} = \tilde{x}^i$ there exist a certain number of functional relationships, which can be derived from a consistent functional model.

To prevent misunderstanding, it should be noted that this formulation cannot be seen as a generally accepted one. The problem of the “linking-up of a mathematical model or formalism” is hardly mentioned in the literature on mathematical statistics. Perhaps the formulation must be seen as a personal one, but it has enabled the author to arrive at a system of interconnected conclusions pertaining to different areas of geodesy.

The requirement of a very careful comparison of registrations R implies a very sharp analysis of measuring procedures and circumstances. The statements under a. en b. imply that a well-directed analysis is made of experiments, affording the possibility of making such statements. Special attention should be given to the attachment of labels to mathematical concepts. This construction of a mathematical

model, proceeding step by step and accompanied by careful analysis, can be considered as an application of *operational defining*. The latter name applies in this case more to the method followed than to the choice of a philosophical system.

Once the combination of statements a. and b. has been made, the processing of the observations by the geodesist has been reduced to an application of the theory of estimation from mathematical statistics, in particular the part that is often denoted by the misleading name of regression-analysis.

One is naturally free to choose for the probability distribution under a. a distribution which is not a Laplace-Gauss distribution, but then one cannot without further preface apply the theory of regression analysis as it has been elaborated in mathematical statistics. In many cases, such an adoption of non-normal distributions is harmless provided the correct covariance matrix is used, i.e. the matrix of second reduced moments about the means ($E\{(x^i - \bar{x}^i)(x^j - \bar{x}^j)\}$). Regression analysis is then essentially the same as the method of adjustment by least squares, and the possibilities of testing by F -tests as well as the theory of power functions have been completely worked out. If one wishes to estimate elements of the covariance matrix, repeated measurements are needed, and one has to use the mathematically more involved theory of multivariate analysis, where instead of the F -distribution the Wishart-distribution is used. The latter technique is not yet used in geodesy for a simple economic reason: a large geodetic network can only be measured once. The apparent repetition of direction- and distance measurements is mainly done to indicate the possible presence of certain disturbing functional effects ("systematic errors") or to eliminate them by taking the average.

Whereas the choice under a. can be done within rather wide limits without materially affecting the estimation, the choice under b. has more drastic consequences, especially when the size of the geodetic networks considered is increased. The ideas of MARUSSI and HOTINE published in the last decades have no doubt caused the geodesist to consider his choice much more carefully than he used to do. As the size of the network increases he can choose Euclidean plane geometry, then geometry on one curved surface, such as sphere or ellipsoid (with all "reduction" difficulties) and finally spatial Euclidean geometry with as variants different possibilities of curvilinear coordinate systems. When networks reach a certain size, potential theory *must* be applied in some form or other; since the last publications by HOTINE even the applicability of Newtonian mechanics is doubted.

Each of the mathematical theories mentioned forms a consistent whole, each has been developed by abstraction from scarce experience; from a mathematical point of view each theory is true, though there are marked differences in complexity. Only the link to observational numbers from a certain measuring procedure can explain a preference for one of the theories. Recourse to an experiment will seldom lead to an unambiguous decision, because economically feasible experiments are only concerned with networks which are very limited in size. Consequently a choice will have to be made, and one must try to predict where a check on the acceptability of the choice can be made by means of the computed value of some quantity which has been derived using the theory. How difficult this complex problem is, is apparent from the examples constructed by HOTINE, from which hardly

resulted a distinct difference in coordinates between his geometric method and the classical ellipsoidal computation. The problem becomes even more difficult because of the fact that apparent differences can be generated by different types of rounding errors, and, in particular, because the functional relations derived from a mathematical theory are almost always non-linear. Regression analysis, and consequently the method of least squares, is only consistent if these relations are linear ones. With non-linear relations, there appears a freedom of choosing the method of computation, i.e. a non-uniqueness which is already hidden in the classical methods of adjustment, and which can certainly give rise to differences in the results (although the differences are small and will only make themselves felt appreciably in large networks). Given one mathematical theory, it even makes a difference which system of functional relationships one chooses, so that isomorphy in theory does not guarantee equality of results of computations.

In an earlier era of geodesy it was the custom to give, before measurement, only a provisional indication of the statements a. and b.; after the measurement, changes in the set-up of the computation were made if indicated by the observational results. In the present era of computers, a careful preliminary planning will be made on the basis of concrete statements a. and b., taking account of all consequences, aimed at reaching the purpose in an economical way. Measurement and computation are then executed strictly according to design. The tests, also executed according to plan, can give an indication how far the observational numbers or the mathematical model theories are acceptable, and possibly they may result in the necessity of new observations, or of a revision of the model theory and with it the whole plan.

The statements under a. and b., which must be incorporated in the planning are indicated by the (historically grown) name of null hypothesis, H_0 . Tests and power functions can only be applied and worked out if preliminary guesses can be made concerning possible alternative statements a. and b., which are indicated as alternative hypotheses H_{a_1}, H_{a_2}, \dots

This formulation means that in this paper the discussion will be based on the statistical theories of NEYMAN and PEARSON, dating from about 1935, in which have been incorporated many elements from the theory of R. A. FISHER dating from about 1925 and containing a number of ideas going back to GAUSS and HELMERT.

This is a restriction because the theories mentioned can be considered as special cases of decision theory, founded by WALD during the last war and since developed by many others. Some aspects of it are treated by ALBERDA [1966]. The restriction may be useful, however, because many applications have led to reflection on the many difficulties, often of principle, which are connected with the linking-up of mathematical model theories by the geodesist. It was partly by these difficulties that the author was led to design modifications to existing geodetic methods of work.

2 POINT ESTIMATION

2.1 Notation

The notation is a mixed ricci- and matrix notation.

- () indicates vector or matrix, but ordinary brackets when used for indices
- ()* transpose of vector
- { } brackets, especially in function notation
- (x^i) vector of measured numbers (readings) from a practical geodetic problem interpreted as a random sample, sometimes to avoid misunderstanding indicated as (x_s^i)
- R registration of the measurement process applied in this geodetic problem
- {(x^i); R } or {(x_s^i); R } total outcome of practical measurement
- (\underline{x}^i) vector of variates or stochastic variables; samples indicated as (x_s^i), $S = \dots$
- (\tilde{x}^i) vector of means (mathematical expectations) of the \underline{x}^i
- (σ_{x^i, x^j}) covariance matrix; ($\sigma_{x^i, x^j} = \sigma^2 \cdot \overline{(x^i), (x^j)^*}$)
- σ^2 variancefactor („variance of unit weight“)
- $\overline{(x^i), (x^j)^*}$ matrix of weight-coefficients, sometimes indicated as (g^{ij})
- (\underline{X}^i) vector of estimators of (\tilde{x}^i)

2.2 Null hypothesis and adjustment

If in a practical geodetic problem the following observational results have been obtained:

$$\{(x^i); R\} \quad (i, j = 1, \dots, m) \quad \dots \dots \dots (2.2.1)$$

compare then R with the registration of previous experiments, and use this comparison to link the results to a mathematical model which consists of a probability distribution of the same number of variates (\underline{x}^i), between whose means a number of functional relationships (“laws of nature”) are assumed to exist. The relationships are derived from a consistent mathematical model. For simplicity we shall only consider a probability distribution of Gauss-Laplace (“normal”) probability distribution, whereas to begin with the “laws of nature” are taken to be linear relations. (x^i) is then considered as one of the possible samples (x_s^i) of (\underline{x}^i). This model, linked to (2.2.1), is called “null hypothesis” H_0 ; the form we choose here is:

H_0	(\tilde{x}^i) to be estimated	
Covariance matrix (rank m), variance factor σ^2 :		
$\sigma^2 \cdot \overline{(x^i), (x^j)^*} = \sigma^2 \cdot (g^{ij}) \quad (i, j = 1, \dots, m)$	 (2.2.2)
“Laws of nature”, (u_i^0) rank b , (u_0^0) vector:		
$(u_i^0) (\tilde{x}^i) - (u_0^0) = (0) \quad (\rho, \tau = 1, \dots, b)$		

A second form of the “laws of nature” is possible.

Determination of the vector (a^i) from:

$$(u_i^a) (a^i) = (0)$$

gives $(m-b)$ linearly independent vectors, forming the matrix with the rank $m-b$:

$$\text{Hence: } \left. \begin{array}{l} (a_\alpha^i) \quad (\alpha, \beta = b+1, \dots, m) \\ (u_i^a) (a_\alpha^i) = (0) \end{array} \right\} \dots \dots \dots (2.2.3)$$

The vector (a_0^i) is then determined (up to $m-b$ degrees of freedom) from:

$$(u_i^a) (a_0^i) = (u_0^a) \dots \dots \dots (2.2.4)$$

Then the “laws of nature” can also be written in a parametric form, with the parameters (\hat{y}^α) :

$$(\hat{x}^i - a_0^i) = (a_\alpha^i) (\hat{y}^\alpha) \dots \dots \dots (2.2.5)$$

from which, with (2.2.3), the first form follows in a modified representation:

$$(u_i^a) (\hat{x}^i - a_0^i) = (0) \dots \dots \dots (2.2.6)$$

whence, with (2.2.4), again follows (2.2.2).

Mixed forms are possible, but we will not go into these details.

The classical problem of “least squares adjustment” can be interpreted as a point estimation problem, giving estimates for the unknown values (\hat{x}^i) and, where appropriate, (\hat{y}^α) . Estimates are then again considered as a random sample of stochastic variables, these variables being called estimators.

The form in which the “laws of nature” has been introduced implies the choice of the vector (x^i) from the (sub-)space covered by the (\hat{x}^i) , which must be introduced into the functional model in order to obtain consistent relations. The choice (which can actually be motivated) is (\hat{x}^i) . This implies that in an application of an adjustment procedure, the variables in the chosen functional mathematical model must be provided with a tilde \sim .

However, these quantities are unknown because they are just the ones that must be estimated. If one wants to make computations with the model relations as if one were working with the means, then the relations valid for the means must also be valid for the estimators.

For both forms of the “laws of nature” this leads to the establishment of the “condition equations”, in the terminology of TIENSTRA:

1st standard problem	$(u_i^a) (\hat{X}^i - a_0^i) = (0)$ (2.2.7)
2nd standard problem	$(\hat{X}^i - a_0^i) = (a_\alpha^i) (\hat{Y}^\alpha)$	

(Y^α) sometimes indicated as “unknowns”

(\hat{Y}^α) then to be indicated as “unknown-variates”

It is curious that although (2.2.7) is not valid for the (\hat{x}^i) and in general (\hat{y}^α) is not even defined, it is always possible to define so-called derived variates:

or:
$$\left. \begin{aligned} (x^r) &= (A_i^r) (x^i) + (A_0^r) \quad (r, s = \dots) \\ (x^r - a_0^r) &= (A_i^r) (x^i - a_0^i) \end{aligned} \right\} \dots \dots \dots (2.2.8)$$

They may be entirely arbitrary functions, but in general they are relations derived within the chosen functional model, such as coordinates computed from angles and distances in two-dimensional Euclidean geometry.

In the case of linear functions (2.2.8), the law of propagation of the means gives:

$$(\tilde{x}^r - a_0^r) = (A_i^r) (\tilde{x}^i - a_0^i) \dots \dots \dots (2.2.9)$$

which is in accordance with the choice of the vector in the “laws of nature”, so that for these relations the relations between the estimators can be introduced:

$$(\underline{X}^r - a_0^r) = (A_i^r) (\underline{X}^i - a_0^i) \dots \dots \dots (2.2.10)$$

This in its turn has as a consequence that in estimation problems the relation (2.2.10) between estimators is always joined to (2.2.8).

In the theory of adjustment, several types of these derived variates are introduced, such as:

- (y^o) : vector of misclosure variates, with $(\tilde{y}^o) = (0)$
- (y_{β}) : vector of reciprocal unknown-variates, $(\tilde{y}_{\beta}) \neq (0)$

whereas in fact all estimators can be considered as derived variates.

After (2.2.6) it was mentioned that mixed forms exist. In correspondence with this, there are problems forming a mixture of 1st and 2nd standard problems. An analysis shows that all these forms can be reduced to one of the two standard problems, for which special techniques have been developed with a view to the present possibilities of computer technique. We will therefore only indicate the algorithms of the two standard problems, in a form which brings out the parallelism in the solutions.

(see (2.2.11), (2.2.12) and (2.2.13) on page 12 and 13)

1st standard problem	2nd standard problem
condition equations	
$(0) = (u_i^e) (\underline{X}^i - a_0^i)$	$(\underline{X}^i - a_0^i) = (a_a^i) (\underline{Y}^a)$
$(0) = (u_i^e) (\underline{X}^i) - (u_0^e)$	$(\underline{X}^i) = (a_a^i) (\underline{Y}^a) + (a_0^i)$
1st part of the adjustment: estimators	
$(\underline{y}^e) = (u_i^e) (\underline{x}^i - a_0^i)$	$(\underline{g}_{ji}) = (g^{ij})^{-1}$
$(\underline{x}^i), (\underline{y}^r)^* \equiv (g^{ir})$	$(\underline{y}_\beta) = (a_\beta^i)^* (\underline{g}_{ji}) (\underline{x}^i - a_0^i)$
$(g^{ir}) = (g^{ij}) (u_j^r)^*$	$(\underline{x}^i), (\underline{y}_a)^* = (a_a^i)$
$(\underline{y}^e), (\underline{y}^r)^* = (g^{er})$	$(\underline{y}_\beta), (\underline{y}_a)^* = (g_{\beta a})$
$(g^{er}) = (u_i^e) (g^{ij}) (u_j^r)^*$	$(g_{\beta a}) = (a_\beta^i)^* (\underline{g}_{ji}) (a_a^i)$
$(\underline{g}_{\tau e}) = (g^{er})^{-1}$	$(\underline{g}^{\alpha\beta}) = (g_{\beta a})^{-1}$
$(\underline{\varepsilon}^i) = (g^{ir}) (\underline{g}_{\tau e}) (-\underline{y}^e)$	$(\underline{X}^i - a_0^i) = (a_a^i) (\underline{g}^{\alpha\beta}) (\underline{y}_\beta)$
$(\underline{X}^i) = (\underline{x}^i) + (\underline{\varepsilon}^i)$	$(\underline{\varepsilon}^i) = (\underline{X}^i - a_0^i) - (\underline{x}^i - a_0^i)$
correlate-variates (not needed)	unknown-variates (sometimes needed)
$(\underline{K}_\tau) = (\underline{g}_{\tau e}) (-\underline{y}^e)$	$(\underline{Y}^a) = (\underline{g}^{\alpha\beta}) (\underline{y}_\beta)$
$(\underline{\varepsilon}^i) = (g^{ir}) (\underline{K}_\tau)$	$(\underline{X}^i - a_0^i) = (a_a^i) (\underline{Y}^a)$
controls	
$(0) = (\underline{y}^e) + (u_i^e) (\underline{\varepsilon}^i)$	$(0) = (a_\beta^i)^* (\underline{g}_{ji}) (\underline{\varepsilon}^i)$
derived variates	
$(\underline{X}^r - a_0^r) = (A_i^r) (\underline{X}^i - a_0^i)$	$(\underline{X}^r - a_0^r) = (A_i^r) (a_a^i) (\underline{Y}^a)$
	$= (A_a^r) (\underline{Y}^a)$

(2.2.11)

1st standard problem	2nd standard problem
2nd part of the adjustment: weight-coefficients	
$(\underline{y}^e), (\underline{y}_a)^* = (0)$	
$(\underline{\varepsilon}^i), (\underline{X}^j)^* = (0)$	$(\underline{x}^i), (\underline{X}^j)^* = (\underline{X}^i), (\underline{X}^j)^*$
$(\underline{X}^i), (\underline{X}^j)^* \equiv (G^{ij})$	$(\underline{X}^i), (\underline{Y}^\beta)^* \equiv (G^{i\beta})$
$(\underline{X}^r), (\underline{X}^j)^* \equiv (G^{rj})$	$(\underline{X}^r), (\underline{Y}^\beta)^* \equiv (G^{r\beta})$
	$(\underline{Y}^a), (\underline{Y}^\beta)^* \equiv (G^{a\beta})$
$(G^{ij}) = (g^{ij}) - (g^{ir}) (\underline{g}_{\tau e}) (g^{ej})$	$(G^{ij}) = (a_a^i) (\underline{g}^{\alpha\beta}) (a_\beta^j)^*$
$(G^{rj}) = (A_i^r) (G^{ij})$	$(G^{i\beta}) = (a_a^i) (\underline{g}^{\alpha\beta})$
	$(G^{r\beta}) = (A_a^r) (\underline{g}^{\alpha\beta})$
	$(G^{a\beta}) = (\underline{g}^{\alpha\beta})$

(2.2.12)

1st standard problem	2nd standard problem	(2.2.13)
3rd part of the adjustment: shifting variate		
$\underline{E} = (\underline{\varepsilon}^j) * (\bar{g}_{ji}) (\underline{\varepsilon}^i)$		
$\underline{E} = (-\underline{y}^r) * (\bar{g}_{r\alpha}) (-\underline{y}^\alpha)$ $= (\underline{y}^r) * (\bar{g}_{r\alpha}) (\underline{y}^\alpha)$	$\underline{E} = (\underline{x}^j - a_0^j) * (\bar{g}_{ji}) (\underline{x}^i - a_0^i) +$ $-(\underline{y}_\alpha) * (\bar{g}^{\alpha\beta}) (\underline{y}_\beta)$	

If (X^i) has been computed from the sample (x^i) according to (2.2.11) and (x^i) and (X^i) are represented as “measured point” P and “adjusted point” P' in the standardized x^i -sample space, then the distance $\overline{PP'}$ is given by:

$$\overline{PP'} = \left\{ \frac{1}{\sigma^2} \underline{E} \right\}^{1/2} \dots \dots \dots (2.2.14)$$

or it can be said that the adjustment shifts point P to point P' .

For theoretical considerations it is better to take the formulas of the two standard problems together. For example, it follows from (2.2.11) and (2.2.12):

$$\left. \begin{aligned} (\underline{X}^i - a_0^i) &= (G^{ij}) (\bar{g}_{ji}) (\underline{x}^i - a_0^i) \\ (\underline{Y}^\alpha - 0) &= (G^{\alpha j}) (\bar{g}_{ji}) (\underline{x}^i - a_0^i) \\ (\underline{X}^r - a_0^r) &= (G^{rj}) (\bar{g}_{ji}) (\underline{x}^i - a_0^i) \end{aligned} \right\} \dots \dots \dots (2.2.15')$$

These can be taken together in:

$$(\underline{X}^R - a_0^R) = (G^{Rj}) (\bar{g}_{ji}) (\underline{x}^i - a_0^i) \quad (R = i, \alpha, r) \dots \dots \dots (2.2.15'')$$

$$(S = j, \beta, s)$$

Instead of (2.2.11)–(2.2.13) we then get:

Adjustment		
1st part	$(\underline{X}^R - a_0^R) = (G^{Rj}) (\bar{g}_{ji}) (\underline{x}^i - a_0^i)$	(2.2.16)
2nd part	$\overline{(\underline{X}^R)}, \overline{(\underline{X}^S)} * \equiv (G^{RS})$ $(G^{RS}) = (G^{Rj}) (\bar{g}_{ji}) (G^{iS})$	
3rd part	$\underline{E} = (\underline{x}^j - a_0^j) * (\bar{g}_{ji}) (g^{ij'} - G^{ij'}) (\bar{g}_{j'i}) (\underline{x}^i - a_0^i)$	

It is a fine quality of the computing system of the method of least squares that – provided all equations are linear – the system is a consistent whole, so that one can arbitrarily transfer from the 1st standard problem to the 2nd standard problem and vice versa.

In the linear case, (\underline{y}^α) will always approximately have a Laplace-Gauss probability distribution, even if this is not strictly so for (\underline{x}^i) ; this follows from the central limit theorem. Under the null hypothesis H_0 , we have then $(\hat{y}^\alpha) = (0)$.

This implies for the positive definite quadratic form of rank b , see (2.2.13):

$$\underline{E} = (\underline{y}^r)^* (\underline{\tilde{g}}_{rv}) (\underline{y}^r)$$

that $\left\{ \frac{1}{\sigma^2} \frac{\underline{E}}{b} \right\}$ has the $F_{b, \infty}$ -distribution.

Because:

$$E\{F_{b, \infty}\} = 1$$

the following estimator can be introduced for σ^2 :

$$\left. \begin{array}{l} \left[\hat{\sigma}^2 = \frac{\underline{E}}{b} \right] \\ \text{so that under the condition } H_0: \\ \left\{ \frac{\hat{\sigma}^2}{\sigma^2} \mid H_0 \right\} = F_{b, \infty} \\ E\{\hat{\sigma}^2 \mid H_0\} = \sigma^2 \end{array} \right\} \dots \dots \dots (2.2.17)$$

2.3 Adjustment in steps (phases)

Decompose (\underline{y}^r) in a number of partial vectors:

$$\left. \begin{array}{l} \left(\begin{array}{c} y^{rI} \\ y^{rII} \\ y^{rIII} \\ y^{rIV} \end{array} \right) \quad \left. \begin{array}{l} (rI = 1, \dots, b^I) \\ (rII = b^I + 1, \dots, b^I + b^{II}) \\ (rIII = b^I + b^{II} + 1, \dots, b^I + b^{II} + b^{III}) \\ (rIV = b^I + b^{II} + b^{III} + 1, \dots, b) \\ (b = b^I + b^{II} + b^{III} + b^{IV}) \end{array} \right\} \quad (2.3.1)$$

Adjustment in steps according to TIENSTRA means essentially that the partial vectors $(\underline{y}^{rI}), \dots, (\underline{y}^{rIV})$ are orthogonalized with respect to the probability distribution of (\underline{y}^r) . In fact this can be performed by the generalized reduction procedure of Gauss.

TIENSTRA's interpretation is that the adjustment is split up into (in our case four) partial adjustments which are consecutively executed; the first is the adjustment on the first b^I condition equations, the second adjusts the estimators of the first together with new observations to the next b^{II} condition equations, etc. Or, the procedure of adjustment in steps can be seen as a procedure which can practically follow step by step the progress of measurement of a geodetic network. An extreme situation is that all y^r -variates are orthogonalized, i.e. that every time when new observations make possible the establishment of one more condition equation, the corresponding phase in the adjustment is computed.

Although the method of adjustment in steps, too, has other theoretical aspects, the preceding gives a certain directive for the design of an adjustment in steps. We need not go into the different possibilities of technique: every problem will have to be considered separately.

If we leave this directive suggested by adjustment in steps out of consideration, it is difficult to arrive at a single unambiguous rule for the design of an adjustment. This difficulty is caused by the many possibilities of ordering the orthogonalizations. Modern computer technique brings this difficulty more to the fore, because the size of networks that can be adjusted in one step is continually increasing. There is room for doubt whether this is going in the right direction and whether the computing facilities do not tempt us to forget techniques that work on better principles. Especially in testing problems, this question turns out to be very important, although sound directives for a method to be followed have not yet been found.

For the problems connected with testing, a decomposition of \underline{E} corresponding with the adjustment in steps is important. It is not easy to find a fitting notation; the following choice is made:

After step	$\overline{(X^i)}, \overline{(X^j)}^*$	In step	\underline{E}	Rank	(2.3.2)
I	$(G^{ij \cdot I})$	I	\underline{E}^I	b^I	
II	$(G^{ij \cdot II})$	II	\underline{E}^{II}	b^{II}	
III	$(G^{ij \cdot III})$	III	\underline{E}^{III}	b^{III}	
IV	$(G^{ij \cdot IV})$	IV	\underline{E}^{IV}	b^{IV}	
IV	$(G^{ij}) = (G^{ij \cdot IV})$	$\underline{E} = \underline{E}^I + \dots + \underline{E}^{IV}$	$b = b^I + \dots + b^{IV}$		

Then the following can be shown to be valid, compare (2.2.16):

$\underline{E} = (\underline{x}_j - a_0^j) * (\underline{g}_{j' i'}) (g^{i' j'} - G^{i' j'}) (\underline{g}_{j' i}) (\underline{x}^i - a_0^i)$	(2.3.3)
$\underline{E} = \underline{E}^I + \underline{E}^{II} + \underline{E}^{III} + \underline{E}^{IV}$	
$\underline{E}^I = (\underline{x}^j - a_0^j) * (\underline{g}_{j' i'}) (g^{i' j'} - G^{i' j' \cdot I}) (\underline{g}_{j' i}) (\underline{x}^i - a_0^i)$	
$\underline{E}^{II} = (\underline{x}^j - a_0^j) * (\underline{g}_{j' i'}) (G^{i' j' \cdot I} - G^{i' j' \cdot II}) (\underline{g}_{j' i}) (\underline{x}^i - a_0^i)$	
$\underline{E}^{III} = (\underline{x}^j - a_0^j) * (\underline{g}_{j' i'}) (G^{i' j' \cdot II} - G^{i' j' \cdot III}) (\underline{g}_{j' i}) (\underline{x}^i - a_0^i)$	
$\underline{E}^{IV} = (\underline{x}^j - a_0^j) * (\underline{g}_{j' i'}) (G^{i' j' \cdot III} - G^{i' j'}) (\underline{g}_{j' i}) (\underline{x}^i - a_0^i)$	

Instead of (2.2.7) one then has to introduce, with stochastically independent $F_{b^I, \infty}, \dots, F_{b^{IV}, \infty}$:

(see (2.3.4) on page 16)

$\hat{\sigma}^2 = \frac{1}{b} \cdot \underline{E}$	$\frac{\hat{\sigma}^2}{\sigma^2} = \underline{F}_{b, \infty}$	$E\{\underline{F}_{b, \infty}\} = 1$	(2.3.4)
$b \cdot \hat{\sigma}^2 = b^I \cdot \{\hat{\sigma}^I\}^2 + \dots + b^{IV} \cdot \{\hat{\sigma}^{IV}\}^2$		H_0 valid	
$\{\hat{\sigma}^I\}^2 = \frac{1}{b^I} \cdot \underline{E}^I$	$\frac{\{\hat{\sigma}^I\}^2}{\sigma^2} = \underline{F}_{b^I, \infty}$	$E\{\underline{F}_{b^I, \infty}\} = 1$	
$\{\hat{\sigma}^{II}\}^2 = \frac{1}{b^{II}} \cdot \underline{E}^{II}$	$\frac{\{\hat{\sigma}^{II}\}^2}{\sigma^2} = \underline{F}_{b^{II}, \infty}$	$E\{\underline{F}_{b^{II}, \infty}\} = 1$	
$\{\hat{\sigma}^{III}\}^2 = \frac{1}{b^{III}} \cdot \underline{E}^{III}$	$\frac{\{\hat{\sigma}^{III}\}^2}{\sigma^2} = \underline{F}_{b^{III}, \infty}$	$E\{\underline{F}_{b^{III}, \infty}\} = 1$	
$\{\hat{\sigma}^{IV}\}^2 = \frac{1}{b^{IV}} \cdot \underline{E}^{IV}$	$\frac{\{\hat{\sigma}^{IV}\}^2}{\sigma^2} = \underline{F}_{b^{IV}, \infty}$	$E\{\underline{F}_{b^{IV}, \infty}\} = 1$	

But also we have, for example ($\underline{F}_{b^I, b^I}, \dots, \underline{F}_{b^{IV}, b^I}$ not being stochastically independent):

$\frac{\{\hat{\sigma}^{II}\}^2}{\{\hat{\sigma}^I\}^2} = \underline{F}_{b^I, b^I}$	$E \left\{ \frac{b^I - 2}{b^I} \cdot \underline{F}_{b^I, b^I} \right\} = 1$	\dots \dots \dots (2.3.5)
$\frac{\{\hat{\sigma}^{III}\}^2}{\{\hat{\sigma}^I\}^2} = \underline{F}_{b^I, b^I}$	$E \left\{ \frac{b^I - 2}{b^I} \cdot \underline{F}_{b^I, b^I} \right\} = 1$	
$\frac{\{\hat{\sigma}^{IV}\}^2}{\{\hat{\sigma}^I\}^2} = \underline{F}_{b^I, b^I}$	$E \left\{ \frac{b^I - 2}{b^I} \cdot \underline{F}_{b^I, b^I} \right\} = 1$	
$b^I > 2$		

From (2.3.5) it appears that adjustment in steps makes it possible to use other members of the family of F -distributions. At the same time the effect of greater or smaller uncertainty about the value of σ^2 , which in H_0 is assumed to be given, is eliminated. But this elimination will in later formulas turn out to be only apparent. Besides, if the tests based on (2.3.5) are to be effective, it is required that $b^I \gg 2$, which means in practice that the first step in the adjustment can only be executed after the completion of an appreciable part of the measurements. This is contrary to the directive mentioned previously.

It is important that if an appropriate partition of the adjustment into steps has been made, it suffices to use F -distributions only, provided that the covariance matrix of (\underline{x}^i) can be considered as known.

All this sounds beautiful. Unfortunately, the practical execution is subject to many disturbing influences and cannot be done by just following clear-cut directives; the author and his collaborators are still searching for better directives for the the application of the theory.

2.4 Critical regions. Choice of α

No experiment gives infinite ranges for (x^i) , only the mathematical model of the probability distribution of Laplace-Gauss does.

This means that a part of the sample space of (x^i) , outside a boundary surface situated around (\bar{x}^i) on a certain distance, contains vectors (x_s^i) which are non-realistic if they are taken to represent samples on (x^i) .

This region we indicate by the name:

critical region, $K^{(i)}$.

This implies that the sub-sample space of the (y^e) has a similar critical region:

critical region, $K^{(e)}$.

For several reasons one takes:

$K^{(e)}$ = the region outside a b -dimensional ellipsoid, a locus of points with the same probability density, whose centre is $(\bar{y}^e) = (0)$

and, writing P for probability:

$$P\{(y^e) \in K^{(e)}|H_0\} = \alpha \dots \dots \dots (2.4.1)$$

in which α is sufficiently small, but can curiously not be brought in connection with the range of stochastic variables in an experiment. Here, much is left to the user. In practice one uses the values:

$$\alpha = 0.05 \quad \text{or} \quad 0.01 \quad \text{or} \quad 0.001$$

On the basis of this line of thought it is intuitively felt that a certain observed sample (x_s^i) is not in agreement with H_0 if:

$$(x_s^i) \rightarrow (y_s^e); \quad (y_s^e) \in K^{(e)} \dots \dots \dots (2.4.2)$$

One is then, with different nuances of conviction, led to rejection of the assumption that (x_s^i) is in agreement with H_0 .

This is called the execution of a test; it will have to be carefully investigated *what* actually is being tested.

The formulation given here leads to a different and also supplementary terminology:

$$\left. \begin{array}{l} K^{(e)}: \quad \text{rejection region} \\ B^{(e)} = \text{non-}K^{(e)}: \text{ acceptance region} \\ P\{(y^e) \in K^{(e)}|H_0\} = \alpha \\ P\{(y^e) \in B^{(e)}|H_0\} = 1 - \alpha \end{array} \right\} \dots \dots \dots (2.4.3)$$

By mathematical deduction it is shown that (2.4.3) corresponds with the right hand tail region of the $F_{b, \infty}$ -distribution; the so-called critical value $F_{1-\alpha; b, \infty}$ can be found in tables of percentiles of the F -distribution:

$$\left. \begin{array}{l} K^{(e)}: \quad \left\{ \frac{\hat{\sigma}^2}{\sigma^2} = F_{b, \infty} | H_0 \right\} > F_{1-\alpha; b, \infty} \\ B^{(e)}: \quad \left\{ \frac{\hat{\sigma}^2}{\sigma^2} = F_{b, \infty} | H_0 \right\} < F_{1-\alpha; b, \infty} \end{array} \right\} \dots \dots \dots (2.4.4)$$

However it is a nuisance that $K^{(e)}$ and consequently $B^{(e)}$ is not invariant with respect to a partition of the adjustment into steps.

If for $K^{(e_1)}, \dots, K^{(e_{IV})}$ regions are taken as defined in (2.4.1), with successive dimensions b^I, \dots, b^{IV} and with, instead of α , successively $\alpha^I, \dots, \alpha^{IV}$, then one can in correspondence with (2.3.4) establish the following table, analogous to (2.4.4):

$B^{(e)}$	$\frac{\hat{\sigma}^2}{\sigma^2} < F_{1-\alpha; b, \infty}$	
$B^{(e_I)}$	$\frac{\{\hat{\sigma}^I\}^2}{\sigma^2} < F_{1-\alpha^I; b^I, \infty}$	
$B^{(e_{II})}$	$\frac{\{\hat{\sigma}^{II}\}^2}{\sigma^2} < F_{1-\alpha^{II}; b^{II}, \infty}$ (2.4.5)
$B^{(e_{III})}$	$\frac{\{\hat{\sigma}^{III}\}^2}{\sigma^2} < F_{1-\alpha^{III}; b^{III}, \infty}$	
$B^{(e_{IV})}$	$\frac{\{\hat{\sigma}^{IV}\}^2}{\sigma^2} < F_{1-\alpha^{IV}; b^{IV}, \infty}$	

Because $F_{b^I, \infty}, \dots, F_{b^{IV}, \infty}$ are stochastically independent, the multiplication rule for probabilities can be applied. As indicated in par. 3 note IV of [BAARDA 1960], the regions $B^{(e_1)}, \dots, B^{(e_{IV})}$ generate in the sample space of (y^e) a product space $\bar{B}^{(e)}$, formed by the intersection of four ellipsoidal cylinders. Then we have, cf. (2.4.3):

adjustment in one step	$P\{(y^e) \in B^{(e)} H_0\} = 1 - \alpha$	
adjustment in four steps	$P\{(y^e) \in \bar{B}^{(e)} H_0\} =$ $= \{1 - \alpha^I\}\{1 - \alpha^{II}\}\{1 - \alpha^{III}\}\{1 - \alpha^{IV}\}$	(2.4.6)

$\bar{B}^{(e)}$ will never coincide completely with $B^{(e)}$, so that the outcome of a test connected with an adjustment in one step may be different from the outcome of tests connected with an adjustment in more steps.

Theoretically a certain degree of coincidence can be attained, by making the α 's subject to the condition:

$$1 - \alpha = \{1 - \alpha^I\}\{1 - \alpha^{II}\}\{1 - \alpha^{III}\}\{1 - \alpha^{IV}\} \dots \dots \dots (2.4.7)$$

In order to be more or less independent of the chosen grouping of the condition equations, one might choose:

$1 - \alpha = \{1 - \bar{\alpha}\}^b$	$B^{(e)} \approx \bar{B}^{(e)}$	
$1 - \alpha^I = \{1 - \bar{\alpha}\}^{b^I}$	$\bar{\alpha}$ as parameter or the α -value for a step with only one condition equation	
$1 - \alpha^{II} = \{1 - \bar{\alpha}\}^{b^{II}}$		(2.4.8)
$1 - \alpha^{III} = \{1 - \bar{\alpha}\}^{b^{III}}$		
$1 - \alpha^{IV} = \{1 - \bar{\alpha}\}^{b^{IV}}$		

However, it follows with certainty from our consideration that the different critical regions based on the F -distribution do not form a consistent system, although the adjustment itself is consistent with respect to a decomposition into steps.

Our discussion so far was based on (2.3.4), but we can also establish critical regions based on (2.3.5). The treatment will be somewhat more complicated, because $\underline{F}_{b^I, \infty}, \underline{F}_{b^{II}, b^I}, \dots, \underline{F}_{b^{IV}, b^I}$ are not stochastically independent.

It is not entirely clear what will be the best line to follow. I am convinced that it will remain necessary to split up the adjustment into steps. Although one is not always aware of it, this method is often practised, e.g. when the adjustment of geodetic networks is split up into firstly the station adjustments and secondly the adjustment on triangle- and side conditions. But it is an open question how far one should go. The subjects treated in the following paragraphs will play an important rôle in the answer.

Perhaps one should search for an entirely different approach to the problem of testing. The present-day theory of mathematical statistics undoubtedly offers possibilities for this.

3 ERROR THEORY

3.1 Alternative hypotheses

The null hypothesis H_0 in (2.2.2) is chosen on the basis of factual knowledge concerning foreseen situations.

Apart from this, one usually knows very well that errors in measurements cannot be avoided entirely, whereas factors like refraction may also be present for which it may be impossible to give a realistic mathematical description. Besides, the "laws of nature" are just as relative as all laws made by man.

Because of all such influences it can very easily happen that, see (2.2.11)–(2.2.13):

$$(E\{y^e\}) \neq \{(y^e) = (0) \text{ from } H_0\} \dots \dots \dots (3.1.1)$$

$$\overline{(y^e), (y^e)^*} \neq \{(g^{e\tau}) \text{ from } H_0\} \dots \dots \dots (3.1.2)$$

$$\sigma^2\text{-value in } H_0 \text{ doubtful} \dots \dots \dots (3.1.3)$$

To examine the influence of this, one will try to formulate his doubt in a mathematical form, i.e. to formulate one or more alternative hypotheses:

$$H_{a_1}, H_{a_2}, H_{a_3}, \dots \dots \dots (3.1.4)$$

and to compare these with, or oppose them to, H_0 .

For alternative hypotheses concerning (3.1.1), a theory has been developed by NEYMAN and PEARSON.

Instead of:

$$(E\{y^e|H_0\}) = (0)$$

an alternative hypothesis H_a is introduced:

$$\left. \begin{aligned} (E\{y^e|H_a\}) &= (\tilde{V}y^e) \neq (0) \\ (y^e|H_a) &= (y^e|H_0) + (\tilde{V}y^e) \end{aligned} \right\} \dots \dots \dots (3.1.5)$$

We denote by an accent the non-central F -distribution which replaces in this case the F -distribution, see [BAARDA 1960]. Instead of (2.2.17) we then have:

$$\left. \begin{aligned} \left\{ \frac{\hat{\sigma}^2}{\sigma^2} | H_a \right\} &= \underline{F}'_{b, \infty, \lambda} \\ E\{\hat{\sigma}^2 | H_a\} &= \sigma^2 \left(1 + \frac{\lambda}{b} \right) \\ \lambda &= \frac{1}{\sigma^2} (\tilde{V}y^e)^* (\tilde{g}_{\tau e}) (\tilde{V}y^e) \end{aligned} \right\} \dots \dots \dots (3.1.6)$$

(3.1.5) means a translation of the probability distribution of (y^e) in the standardized sample space over the distance $\sqrt{\lambda}$.

Instead of (2.4.3) and (2.4.4) we have, under H_a :

$\beta = P\{(\underline{y}^e) \in K^{(e)} H_a\} = P\left\{\frac{\hat{\sigma}^2}{\sigma^2} > F_{1-\alpha; b, \infty} H_a\right\}$ (3.1.7)
$1-\beta = P\{(\underline{y}^e) \in B^{(e)} H_a\} = P\left\{\frac{\hat{\sigma}^2}{\sigma^2} < F_{1-\alpha; b, \infty} H_a\right\}$	

β is a monotonic increasing function of λ (α and β being given) and is called the *power function* of the test with critical region $K^{(e)}$. See also [BAARDA 1960, 62]. This makes it possible to compute the size λ must at least have to let it be indicated with a probability β_0 (e.g. 80%) by the test. Tables are available for:

$$\lambda_0 = \lambda\{\alpha, \beta_0, b, \infty\} \dots \dots \dots (3.1.8)$$

Generally we are less interested in this than in the computation of the vector $(\nabla_0 \tilde{y}^e)$ which generates this value λ_0 .

From (3.1.6) follows the relation:

$$\lambda_0 = \frac{1}{\sigma^2} (\nabla_0 \tilde{y}^e)^* (\tilde{g}_{\tau e}) (\nabla_0 \tilde{y}^e) \dots \dots \dots (3.1.9)$$

but we can only obtain an unique solution from this if the ratios of the $\tilde{\nabla} \tilde{y}^e$ are given, as a further specification of H_a . This can e.g. be done in the following form:

H_a	$\left(\frac{\tilde{\nabla} \tilde{y}^e}{\sigma}\right) = (c^e) \cdot \nabla$	$\sigma = + \sigma^2$ (3.1.10)
	(c^e) a vector	∇ a parameter	

From (3.1.9) with (3.1.10) it follows that:

$$\lambda_0 = (c^\tau)^* (\tilde{g}_{\tau e}) (c^e) \cdot \{\nabla_0\}^2$$

$ \nabla_0 = \sqrt{\frac{\lambda_0}{(c^\tau)^* (\tilde{g}_{\tau e}) (c^e)}} \left(\frac{\tilde{\nabla} \tilde{y}^e}{\sigma}\right) = (c^e) \cdot \nabla_0 $ (3.1.11)
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Remark

Starting from a given vector $(\tilde{\nabla} \tilde{y}^e)$, one can just as with (2.3.1) decompose the \underline{y}^e into groups, with subsequent orthogonalization. This means a transformation of the problem, similar to the one used in the adjustment in steps.

Analogously as with \underline{E} in (2.3.3) we then obtain:

$$\lambda = \lambda^I + \lambda^{II} + \lambda^{III} + \lambda^{IV} \dots \dots \dots (3.1.12)$$

from which it follows that λ , being a positive definite quadratic form, in general increases with b .

(3.1.12) can be elucidated, by writing in (3.1.6):

$$E\{\hat{\sigma}^2|H_a\} = E\left\{\frac{\underline{E}}{b}H_a\right\} = \sigma^2 + \sigma^2 \frac{\lambda}{b}$$

or, with (2.2.17):

$$\frac{1}{\sigma^2} \cdot E\{\underline{E}|H_a\} = \frac{1}{\sigma^2} \cdot E\{\underline{E}|H_0\} + \lambda \dots \dots \dots (3.1.13)$$

3.2 Errors (mistakes) in measurements

A very striking example of (3.1.1) are (gross) errors in measurements, errors which in spite of all care are always being made. They may be mistakes in pointing when measuring angles, errors in distance measurement, wrong position of signals or instruments, but also influences of lateral refraction and errors of given coordinates. This does not necessarily invalidate the covariances of (x^i) , so that a mathematical description can be established by putting, as in (3.1.5):

$$\left. \begin{aligned} (E\{x^i|H_a\}) &= (\tilde{x}^i) + (\tilde{\nabla}x^i) \\ \text{or: } (x^i|H_a) &= (x^i|H_0) + (\tilde{\nabla}x^i) \end{aligned} \right\} \dots \dots \dots (3.2.1)$$

For the same reason as in (3.1.10), H_a is made more specific, viz.:

H_a	$\left(\frac{\tilde{\nabla}x^i}{\sigma}\right) = (c^i) \cdot \nabla$	$\sigma = +\sqrt{\sigma^2}$	$\dots \dots \dots (3.2.2)$
	(c^i) a vector	∇ a parameter	

Application of the law of propagation of means on derived variates in the adjustment process:

(2.2.11): $(y^g) = (u_i^g) (x^i - a_0^i)$

(2.3.16): $(X^R - a_0^R) = (G^{Rj}) (\tilde{g}_{ji}) (x^i - a_0^i)$

With (3.2.1) and (3.2.2) these give:

$$\left. \begin{aligned} (E\{y^g|H_a\}) &= (y^g = 0) + (\tilde{\nabla}y^g) \\ \left(\frac{\tilde{\nabla}y^g}{\sigma}\right) &= (u_i^g) (c^i) \cdot \nabla \end{aligned} \right\} \dots \dots \dots (3.2.3)$$

$$\left. \begin{aligned} (E\{X^R|H_a\}) &= (\tilde{x}^R) + \tilde{\nabla}X^R \\ \left(\frac{\tilde{\nabla}X^R}{\sigma}\right) &= (G^{Rj}) (\tilde{g}_{ji}) (c^i) \cdot \nabla \end{aligned} \right\} \dots \dots \dots (3.2.4)$$

Substitution of (3.2.3) in (3.1.6), or, in view of (3.1.13), direct substitution of (3.2.1) into \underline{E} in (2.2.16) gives the formula, which is appropriate for theoretical considerations:

$$\lambda = (c^j)^* (\tilde{g}_{ji'}) (g^{i'j'} - G^{i'j'}) (\tilde{g}_{j'i}) (c^i) \cdot \nabla^2 \dots \dots \dots (3.2.5)$$

Subsequently the line of thought of (3.1.7)–(3.1.11) can be applied again, so that:

$$\lambda_0 \rightarrow |\nabla_0|$$

and, from (3.2.3) and (3.2.4):

$$\left. \begin{aligned} \left(\frac{\widetilde{\nabla}_0 y^e}{\sigma}\right) &= (a_i^e) (c^i) \cdot |\nabla_0| \\ \left(\frac{\widetilde{\nabla}_0 X^R}{\sigma}\right) &= (G^{Rj}) (\tilde{g}_{ji}) (c^j) \cdot |\nabla_0| \end{aligned} \right\} \dots \dots \dots (3.2.6)$$

With (3.2.6) the effect of a minimum value of ∇ in (3.2.2) on derived variates can be computed, which according to (3.1.7) can be found with probability β_0 in a test with the critical region K^e .

The results (3.2.6) are of the utmost importance, for they give an indication of the *reliability* of the results of measurement and adjustment.

For example, a computed coordinate may have a standard deviation of 10 cm, but a value $\widetilde{\nabla}_0 X^R$ of 100 cm. This means that influences $\widetilde{\nabla}_0 X^R$ which generate in the final result an amount of $\widetilde{\nabla}_0 X^R < 100$ cm, in 80% of the cases will *not* be noticed in the computation (inclusive of test). Often, these 100 cm are much more detrimental to the purposes of the measurement than the standard deviation of 10 cm, so that the “strength” of a network is determined by the values $(\widetilde{\nabla}_0 X^R)$ rather than by the covariance matrix $\sigma^2 \cdot \overline{(X^R)}, \overline{(X^S)^*} \dots \sigma^2 \cdot (G^{RS})$.

From (3.2.5) it follows that in a geodetic network without supernumerous (redundant) observations (and consequently without condition equations so that $(G^{ij}) = (g^{ij})$), for every value ∇ , one obtains $\lambda = 0$. Conversely, it follows that $|\nabla_0|$ is indefinite, or, in (3.2.6) one must introduce $|\nabla_0| = \infty$. This again means that there is no check on errors in measurement or computation, or, in other words, in this case there is no *error control*.

This is an extreme example; current geodetic practice shows constructions with a great number of condition equations (often $b \simeq \frac{1}{2}$ m), so that it may be stated that the geodesist usually aims at a reasonable error control. Computations like (3.2.6), however, are seldom or never found in geodetic literature, so that quantitative data about values $(\widetilde{\nabla}_0 X^R)$ are lacking. But some published research deals with these problems though in a different form; a typical example is [REICHENEDER 1941].

Research computations have shown that there are great differences in error control between different types of geodetic networks, and one of the aims the author wanted to set for his work in I.A.G. Special Study Group No. 1 : 14, was a sharper and more general formulation of the concept “Strength of Figure” of a geodetic network, see also the annexed publication [BAARDA 1962]. A number of theoretical and practical difficulties have prevented him from actually executing this idea. Some of these questions will here be treated in more detail.

Most adjustment problems in geodesy are executed in one way or another by splitting the adjustment into steps. Therefore we will first follow the line of par. 2.3 and 2.4 to find the consequences of this. λ from (3.2.5) is then split up as indicated in (3.1.12).

Introducing for brevity:

$N = (c^j)^* (\tilde{g}_{j'j'}) (g^{ij'} - G^{ij'}) (\tilde{g}_{ji}) (c^i)$	$N, N^I, \dots, N^{IV} \geq 0$ (3.2.7)
$N = N^I + N^{II} + N^{III} + N^{IV}$		
$N^I = (c^j)^* (\tilde{g}_{j'j'}) (g^{ij'} - G^{ij'.I}) (\tilde{g}_{ji}) (c^i)$		
$N^{II} = (c^j)^* (\tilde{g}_{j'j'}) (G^{ij'.I} - G^{ij'.II}) (\tilde{g}_{ji}) (c^i)$		
$N^{III} = (c^j)^* (\tilde{g}_{j'j'}) (G^{ij'.II} - G^{ij'.III}) (\tilde{g}_{ji}) (c^i)$		
$N^{IV} = (c^j)^* (\tilde{g}_{j'j'}) (G^{ij'.III} - G^{ij'}) (\tilde{g}_{ji}) (c^i)$		

we get, see also (3.1.7):

$\lambda = N \cdot \nabla^2$	$\beta = P \left\{ \frac{\hat{\sigma}^2}{\sigma^2} > F_{1-\alpha, b, \infty} H_a \right\}$	(3.2.8)
$\lambda = \lambda^I + \dots + \lambda^{IV}$		
$\lambda^I = N^I \cdot \nabla^2$	$\beta^I = P \left\{ \frac{\{\hat{\sigma}^I\}^2}{\sigma^2} > F_{1-\alpha^I, b^I, \infty} H_a \right\}$	
$\lambda^{II} = N^{II} \cdot \nabla^2$	$\beta^{II} = P \left\{ \frac{\{\hat{\sigma}^{II}\}^2}{\sigma^2} > F_{1-\alpha^{II}, b^{II}, \infty} H_a \right\}$	
$\lambda^{III} = N^{III} \cdot \nabla^2$	$\beta^{III} = P \left\{ \frac{\{\hat{\sigma}^{III}\}^2}{\sigma^2} > F_{1-\alpha^{III}, b^{III}, \infty} H_a \right\}$	
$\lambda^{IV} = N^{IV} \cdot \nabla^2$	$\beta^{IV} = P \left\{ \frac{\{\hat{\sigma}^{IV}\}^2}{\sigma^2} > F_{1-\alpha^{IV}, b^{IV}, \infty} H_a \right\}$	

This implies, that (3.1.6) is split into four parts; the four non-central F -variates:

$$\underline{F}_{b^I, \infty, \lambda^I}, \dots, \underline{F}_{b^{IV}, \infty, \lambda^{IV}}$$

are independently distributed.

The line of thought (3.1.7)–(3.1.11) or (3.2.3)–(3.2.6) can therefore be applied four times, once for each step:

$\lambda_0 = \lambda\{\alpha, \beta_0, b, \infty\}$	$ \nabla_0 = \frac{\lambda_0}{N}$ (3.2.9)
$\lambda_0^I = \lambda\{\alpha^I, \beta_0^I, b^I, \infty\}$	$ \nabla_0^I = \frac{\lambda_0^I}{N^I}$	
$\lambda_0^{II} = \lambda\{\alpha^{II}, \beta_0^{II}, b^{II}, \infty\}$	$ \nabla_0^{II} = \frac{\lambda_0^{II}}{N^{II}}$	
$\lambda_0^{III} = \lambda\{\alpha^{III}, \beta_0^{III}, b^{III}, \infty\}$	$ \nabla_0^{III} = \frac{\lambda_0^{III}}{N^{III}}$	
$\lambda_0^{IV} = \lambda\{\alpha^{IV}, \beta_0^{IV}, b^{IV}, \infty\}$	$ \nabla_0^{IV} = \frac{\lambda_0^{IV}}{N^{IV}}$	

The question arises what is the (practical) significance of the $|\nabla_0|$ -values in the right-hand column of (3.2.9).

To see this consider (2.4.6), but now under the alternative hypothesis H_a from (3.2.2):

adjustment in one step	$P\{(y^a) \in B^{(a)} H_a\} = 1 - \beta$	(3.2.10)
adjustment in four steps	$P\{(y^a) \in \bar{B}^{(a)} H_a\} =$ $= \{1 - \beta^I\}\{1 - \beta^{II}\}\{1 - \beta^{III}\}\{1 - \beta^{IV}\}$	
β 's and λ 's from (3.2.8), as functions of:	$\beta = \beta\{a, b, \infty, \lambda\}$	
	$\beta^I = \beta\{a^I, b^I, \infty, \lambda^I\}$	
	$\beta^{II} = \beta\{a^{II}, b^{II}, \infty, \lambda^{II}\}$	
	$\beta^{III} = \beta\{a^{III}, b^{III}, \infty, \lambda^{III}\}$	
	$\beta^{IV} = \beta\{a^{IV}, b^{IV}, \infty, \lambda^{IV}\}$	

Now the situation is possible that:

$$1 - \beta = \{1 - \beta^I\}\{1 - \beta^{II}\}\{1 - \beta^{III}\}\{1 - \beta^{IV}\} \dots \dots \dots (3.2.11)$$

which, in view of the ratios between the λ 's given by (3.2.8), requires a certain relation between the a 's.

Consequently, starting from (2.4.7) with (2.4.8), (3.2.11) will be fulfilled with some approximation.

(3.2.11) suggests an approach for the converse problem in view of (3.2.9); $\beta_0^I, \dots, \beta_0^{IV}$ must be determined with respect to an adopted value β_0 , making use of:

$$1 - \beta_0 = \{1 - \beta_0^I\}\{1 - \beta_0^{II}\}\{1 - \beta_0^{III}\}\{1 - \beta_0^{IV}\} \dots \dots \dots (3.2.12)$$

If, for example, the a 's are chosen according to (2.4.8) and the ratios of the λ_0 's according to (3.2.8), one can compute values $\beta_0^I, \dots, \beta_0^{IV}$. However, then this is not necessary any more, because from the ratios of $\lambda_0, \lambda_0^I, \dots, \lambda_0^{IV}$ from (3.2.8) it follows with (3.2.9) that:

$$|\nabla_0| = |\nabla_0^I| = |\nabla_0^{II}| = |\nabla_0^{III}| = |\nabla_0^{IV}| \dots \dots \dots (3.2.13)$$

Possibly one can also attain (3.2.13) by computing $\beta_0^I, \dots, \beta_0^{IV}$ in the same way from β_0 as a^I, \dots, a^{IV} were computed from a according to (2.4.8). If then the ratios of the values λ_0 from (3.2.8) are used again, values a^I, \dots, a^{IV} will have to be labouriously computed, so that the critical regions are in fact determined by the corresponding λ 's and consequently by the special alternative hypothesis H_a from (3.2.2).

The two theoretical possibilities leading to (3.2.13) have the practical (and actually even theoretical) difficulty that it is tacitly assumed that it is possible to have a complete view of the total adjustment in one step.

But if one considers a geodetic network as a work which grows over a period of many years, during which it is impossible to get a total picture in all details, then this assumption turns out to be a fiction.

Therefore it is in *practice* admissible that besides (2.4.10) one makes the simplifying assumption:

$$\left. \begin{aligned} \beta_0 = \beta_0^I = \beta_0^{II} = \beta_0^{III} = \beta_0^{IV} \\ \text{e.g.} = 0.80 \end{aligned} \right\} \dots \dots \dots (3.2.14)$$

(3.2.9) with (3.2.14) will then in general result in:

$$|\nabla_0| \neq |\nabla_0^I| \neq |\nabla_0^{II}| \neq |\nabla_0^{III}| \neq |\nabla_0^{IV}| \dots \dots \dots (3.2.15)$$

which gives rise to the question *which* of these values (3.2.15) should be substituted into (3.2.6).

An analogous problem was treated already in [ALBERDA 1963], where it was tried to reach a minimum value in (3.2.5) by choosing a certain group of condition equations for each alternative hypothesis specified there. In this case, the measurement of the whole network had been completed, so that the formulation of the problem was not entirely equivalent to the one here treated.

An extreme illustration of (3.2.15) is given in the following case:

<i>H_a</i> , special case		
(3.2.7)	N ^I = N ^{II} = N ^{III} = 0, therefore: N ^{IV} = N	
(2.4.10)	α ^I = α ^{II} = α ^{III} = α ^{IV} = α	
(3.2.4)	β ₀ ^I = β ₀ ^{II} = β ₀ ^{III} = β ₀ ^{IV} = β ₀	
(3.2.9)	λ ₀ ^I , λ ₀ ^{II} , λ ₀ ^{III} , λ ₀ ^{IV} ≠ λ ₀	
	∇ ₀ ^I = ∇ ₀ ^{II} = ∇ ₀ ^{III} = ∞	
	{ ∇ ₀ ^{IV} = $\frac{\lambda_0^{IV}}{N}$ } ≠ { ∇ ₀ = $\frac{\lambda_0}{N}$ }	(3.2.16)

In this example, no check on the occurrence of *H_a* is possible in the steps I-III, this possibility is only provided by step IV.

In fact one is faced with a decision problem:

$$\left. \begin{aligned} \text{either adjustment in one step} &\rightarrow |\nabla_0| \\ \text{or adjustment in four steps} &\rightarrow |\nabla_0^{IV}| \\ \text{including the choice of } \alpha\text{'s and } \beta_0\text{'s} & \end{aligned} \right\} \dots \dots \dots (3.2.17)$$

At this stage it is not clear what should be the basis of the solution of this decision problem, if it is not the directive to adapt the adjustment in steps to the order of execution of the measurements in a certain geodetic network.

For illustrative purposes this will be worked out for a more general case of *H_a*. For the tests, the following decision scheme can be established:

H_a , general case: $N^I, N^{II}, N^{III}, N^{IV} \neq 0$					
Sequence	$\frac{\{\hat{\sigma}^I\}^2}{\sigma^2}$	$\frac{\{\hat{\sigma}^{II}\}^2}{\sigma^2}$	$\frac{\{\hat{\sigma}^{III}\}^2}{\sigma^2}$	$\frac{\{\hat{\sigma}^{IV}\}^2}{\sigma^2}$	Action
1°	$\in K^{(e_I)}$ $\in B^{(e_I)}$				reject (x'_S) in step I go to 2°
2°	$\in B^{(e_I)}$ $\in B^{(e_I)}$	$\in K^{(e_{II})}$ $\in B^{(e_{II})}$			reject (x'_S) in steps I, II go to 3°
3°	$\in B^{(e_I)}$ $\in B^{(e_I)}$	$\in B^{(e_{II})}$ $\in B^{(e_{II})}$	$\in K^{(e_{III})}$ $\in B^{(e_{III})}$		reject (x'_S) in steps I-III go to 4°
4°	$\in B^{(e_I)}$ $\in B^{(e_I)}$	$\in B^{(e_{II})}$ $\in B^{(e_{II})}$	$\in B^{(e_{III})}$ $\in B^{(e_{III})}$	$\in K^{(e_{IV})}$ $\in B^{(e_{IV})}$	reject (x'_S) in steps I-IV accept (x'_S)

(3.2.18)

This scheme is remarkable in several respects:

a. In H_a of (3.2.2) there is only one parameter ∇ and yet one is working in steps. But because in every step new observations are being added, one has the possibility to check the occurrence of H_a at an earlier stage, so that in the case of rejection the number of x^i to be remeasured is a minimum.

b. It is possible that the null hypothesis is rejected in the different steps in spite of the fact that H_a does not occur. The probability to take such a wrong action (NEYMAN and PEARSON: errors of the first kind or Type I errors) are, successively:

$$\left. \begin{aligned} &\alpha^I \\ &1 - \{1 - \alpha^I\}\{1 - \alpha^{II}\} \\ &1 - \{1 - \alpha^I\}\{1 - \alpha^{II}\}\{1 - \alpha^{III}\} \\ &1 - \{1 - \alpha^I\}\{1 - \alpha^{II}\}\{1 - \alpha^{III}\}\{1 - \alpha^{IV}\} \end{aligned} \right\} \dots \dots \dots (3.2.19)$$

c. It is possible that in the different steps one does not reject the null hypothesis, in spite of the fact that H_a occurs. The probabilities to take such a wrong action (NEYMAN and PEARSON: errors of the second kind or Type II errors) are, successively:

$$\left. \begin{aligned} &1 - \beta^I \\ &\{1 - \beta^I\}\{1 - \beta^{II}\} \\ &\{1 - \beta^I\}\{1 - \beta^{II}\}\{1 - \beta^{III}\} \\ &\{1 - \beta^I\}\{1 - \beta^{II}\}\{1 - \beta^{III}\}\{1 - \beta^{IV}\} \end{aligned} \right\} \dots \dots \dots (3.2.20)$$

d. There is some similarity between (3.2.18) and so-called „sequential tests”. There are, however, characteristic differences: for the reasons mentioned before, it is particularly difficult to indicate in advance values for α and β_0 to which (3.2.18) can be adapted, whereas the measuring process is finished when the vector (x'_S) is complete (except in the case of remeasurements).

(3.2.19) may give rise to quite high values for α , as has already been remarked with reference to table (2.4.9). If the cost of remeasuring is also considered, the hesitation of practical geodesists to follow consequently the scheme (3.2.18), is very understandable. The choice of $\alpha^I, \dots, \alpha^{IV}$ becomes then a question of conflict between conscience and cost!

In addition one is faced with the also increasing values (3.2.20), in which the β 's increase as the α 's are chosen smaller. This is the next conflict of conscience for the geodesist who wants his errors $\{H_a\}$ to be indicated.

Using (3.2.9) he can compute the magnitude of the errors in (x_s^i) which, with a certain probability, will be just detected, but in practice he is then faced with the situation indicated in (3.2.15). It seems difficult to find a satisfactory solution for this. The whole scheme (3.2.18) must be worked through, so that one finally has an acceptance region $\bar{B}^{(a)}$ from (2.4.6), which, choosing e.g. $\alpha^I = \dots = \alpha^{IV}$, may be quite different from $B^{(a)}$ from (2.4.6), even if α is computed from (2.4.7). For this reason, the comparison between tests based on adjustment in four steps and tests based on adjustment in one step will be very difficult if not totally impossible. And because of this, it becomes practically impossible to compute a unique value for $|\nabla_0|$, and consequently for $(\nabla_0 \tilde{X}^R)$ from (3.2.6).

There is no need to be too pessimistic about this conclusion. Theory is just theory; sometimes it may sharpen our intuitive insight, sometimes it does not. Many years of experience in the application of this theory have taught us that a sharpening of intuitive insight has resulted, even if the final decision does not always conform to the theory.

Every geodesist who has a practical mind will of course ask: where should I start looking for errors in case (3.2.18)?

This question leads to a refinement of the formulation of H_a in (3.2.2). For, by definition, (3.2.2) does not give an answer to this question because on account of the introduction of only one vector (c^i) there is a dependence between all $(\tilde{\nabla} x^i)$.

From a practical point of view, (3.2.2) is therefore nonsense: no functional relationship can be indicated for most types of measuring errors.

There, (3.2.2) is replaced by the following composite alternative hypothesis:

H_{a_p}	$\left(\frac{\tilde{\nabla}_p x^i}{\sigma}\right) = (c_p^i) \cdot \nabla_p$	$\sigma = +\sqrt{\sigma^2}$	(3.2.21)
$p = 1,2,3,4$	(c_p^i) vectors	∇_p parameters	

The line of thought is now for each H_{a_p} the same as for H_a . The notation becomes:

(3.2.3):	$\left(\frac{\tilde{\nabla}_p y^a}{\sigma}\right);$	(3.2.4):	$\left(\frac{\tilde{\nabla}_p X^R}{\sigma}\right)$	} (3.2.22)
(3.2.5):	$\lambda_p, \lambda_{p,0} \rightarrow \nabla_{p,0} $				
(3.2.6):	$\left(\frac{\tilde{\nabla}_{p,0} y^a}{\sigma}\right),$	$\left(\frac{\tilde{\nabla}_{p,0} X^R}{\sigma}\right)$			

$$\left. \begin{aligned}
 (3.2.7): & N_p, N_p^I, \dots, N_p^{IV} \\
 (3.2.8): & \lambda_p^I, \text{ etc.}; \beta_p, \beta_p^I, \text{ etc.} \\
 (3.2.9): & \lambda_{p,0}^I, \text{ etc.}; |\nabla_{p,0}^I|, \text{ etc.}
 \end{aligned} \right\} \dots \dots \dots (3.2.22 \text{ cont.})$$

Now put the case that a decomposition into four steps can be made, so that, compare (3.2.16):

N_p from (3.2.7), with (3.2.21) and (3.2.22)					
	$p = 1$	$p = 2$	$p = 3$	$p = 4$	
N_p^I	N_1^I	0	0	0 (3.2.23)
N_p^{II}	N_1^{II}	N_2^{II}	0	0	
N_p^{III}	N_1^{III}	N_2^{III}	N_3^{III}	0	
N_p^{IV}	N_1^{IV}	N_2^{IV}	N_3^{IV}	N_4^{IV}	
Sum = N_p	N_1	N_2	N_3	N_4	

Now the indicated difficulty of interpreting (3.2.15) can be circumvented by making a choice of different $|\nabla_0|$ to be computed, such as:

Influence of only		in step	choice of α, β_0	λ_0 (3.2.9)	selection of $ \nabla_0 $	$(\nabla_0 \tilde{X}^R)$ (3.2.6)
if not	then					
	H_{a_1}	I	$\alpha^I, \beta_{1,0}^I$	$\lambda_{1,0}^I$	$ \nabla_{1,0}^I = \frac{\lambda_{1,0}^I}{N_1^I}$	$(\nabla_{1,0} \tilde{X}^R)$
H_{a_1}	H_{a_2}	II	$\alpha^{II}, \beta_{2,0}^{II}$	$\lambda_{2,0}^{II}$	$ \nabla_{2,0}^{II} = \frac{\lambda_{2,0}^{II}}{N_2^{II}}$	$(\nabla_{2,0} \tilde{X}^R)$
H_{a_1}, H_{a_2}	H_{a_3}	III	$\alpha^{III}, \beta_{3,0}^{III}$	$\lambda_{3,0}^{III}$	$ \nabla_{3,0}^{III} = \frac{\lambda_{3,0}^{III}}{N_3^{III}}$	$(\nabla_{3,0} \tilde{X}^R)$
$H_{a_1}, H_{a_2}, H_{a_3}$	H_{a_4}	IV	$\alpha^{IV}, \beta_{4,0}^{IV}$	$\lambda_{4,0}^{IV}$	$ \nabla_{4,0}^{IV} = \frac{\lambda_{4,0}^{IV}}{N_4^{IV}}$	$(\nabla_{4,0} \tilde{X}^R)$

(3.2.24)

(3.2.24) is now used in connection with (3.2.18), whereby a certain indication for the detection of errors is now present. But this will be no more than an *indication*, because, essentially, the remarks given after (3.2.18) will be valid for the situation in (3.2.23) too.

(3.2.21) with (3.2.23) only give an utterly simplified picture of the real situation in practice. It might be said that the four steps are in fact four groups each consisting of many partial steps. In addition, (3.2.21) cannot be established in reality either, so that, on account of the fact that the four vectors (c_p^i) cannot be established, in practice one often makes as many alternative hypotheses as there are x_s^i . Every

separate hypothesis $H_{a(i)}$ consists then of the hypothesis that only x_s^i has a measuring error, whose size is $\tilde{\nabla}_{(i)} x^i$. This makes the schemes (3.2.18), (3.2.23) and (3.2.24) very large and complicated, so that one usually restricts oneself to a limited choice out of all possible combinations. Experience has shown that in this way it is possible to get an overall picture of the decision problem, but that in practice it is very difficult to actually make a decision.

A typical example of the situation (3.2.23) is the classical triangulation network, where the measurement of directions is spread over two or more nights in order to check on possible lateral refraction. One gets the following arrangement:

Step I Every station adjustment per station per night can be seen as a partial step of step I. H_{a_1} is concerned with reading mistakes. The problem is then to decide whether a certain series of direction measurements can be accepted or must be rejected. If the measurement has been done with great care, H_{a_1} can often be left out of consideration, and the computed $\hat{\sigma}^2$ of all partial steps serve to test if the adopted σ^2 of (2.2.2) has an acceptable order of magnitude. This may e.g. be done by means of confidence intervals.

The partial steps mentioned here occur during the gradual progress of the measurement of the network, alternating with partial steps of step II and step III, which is simplified by the practical absence of stochastic dependence between the partial steps within step I .

Step II The joining of the different station adjustments made for each night for the same station can be considered as a partial step of step II. H_{a_2} is then concerned with influences of lateral refraction. The establishment of vectors (c_2^i) from (3.2.21) meets with great difficulties, or is impossible even per partial step. This makes it very difficult to estimate $|\nabla_{2,0}^{\text{II}}|$ from (3.2.24), so that in this respect the theory developed offers relatively few possibilities for practical application.

We were very disagreeably reminded of this when the base extension network which was measured in 1965 in the North of our country clearly showed signs of lateral refraction influences which could not be localized, although experiments during the whole 1964 season seemed to indicate that such influences were not occurring there.

Step III Triangle- and side-conditions and sometimes base- and Laplace-conditions, grouped in partial steps or not, can be considered as step III. H_{a_3} is then concerned with pointing errors in different form, such as e.g., are caused by the centering of instrument and signals. The decomposition into partial steps, which is necessary for the analysis of the progressing measuring process, is here hampered by the (often strong) stochastic dependence between the misclosure variates. Because variates corresponding to directions may occur in many condition equations and consequently in a relatively large number of partial steps, one is in step III forcibly confronted with the theoretical difficulties of the decision problem mentioned in connection with (3.2.17). In addition, we have here also the problem of establishing

the vector or vectors (c_3^i) per partial step, so that usually one just decomposes H_{a_3} into as many alternative hypotheses as there are direction variates.

The difficulty of actually deciding to reject or accept is well illustrated by the base extension network mentioned before, where we actually do not know how to make a decision which is justifiable economically *and* scientifically.

Step IV Fitting the net to given coordinates by establishing coordinate conditions is the last step; these conditions may be grouped in partial steps. H_{a_4} is now concerned with errors in these coordinates or with faulty signals in the corresponding points. Again we meet the difficult and often unsatisfactorily solvable problem of the establishment of the vector or vectors (c_4^i) per partial step. The test problem is of an extraordinary difficulty because as a rule so little is known about the covariance matrix of given coordinates. This problem, which we shall touch upon in par. 3.3, is almost daily met in smaller local networks. But also in large geodetic networks this problem turns up, as e.g. was evident from the filling up of the first order chain net in the U.S.A.

Does the preceding lead to a negative conclusion about the applicability of methods of testing? Certainly not, we could not do without them any more. But it appears that our line of thought must be ordered better, perhaps the true simplicity is lacking.

3.3 Effects of the assumption of a covariance matrix

In par. 2.2 it has been sketched how one can arrive at the choice of the covariance matrix of (x^i) :

$$(\sigma_{x^i x^j}) = \sigma^2 \cdot (g^{ij}) \quad \quad (3.3.1)$$

in H_0 , see (2.2.2).

The question may be asked if it is important that the choice of (3.3.1) is done as carefully as possible. Research and practical experience show that this choice is of fundamental importance for the interpretation of the results of measurement and computation, with the restriction that it suffices to ascertain the elements of (3.3.1) in only a few (two) digits. This is a consequence of the limitations of estimating covariances from an (always restricted) experiment consisting of repeated measurements. Furthermore, the comparison between the registration R of the measurement of a geodetic network and the supposedly equivalent registration of such an experiment contains so many doubtful elements that it would be self-deception to put the elements of (3.3.1) down in more than two digits.

This set-up, i.e. the assignment of a covariance-matrix to a vector (x_s^i) (obtained from the measurement of a geodetic network) by means of a comparison of the registration R , could in principle be replaced by a set-up where experiment and measurement are executed together. In that case the adjustment by the method of least

squares with tests based on the F -distribution could be replaced by the so-called multivariate analysis with tests based on the Wishart-distribution. The theory concerned has been completely elaborated and is available, although tables have not been elaborated to a sufficient extent. Principle is one thing, application another: it will rarely happen that a geodesist is able to measure a geodetic network, covering one or more countries, more than once! Therefore we can restrict our discussion to the set-up of par. 2.2 ff.

In (3.3.1) the usual decomposition of $(\sigma_{x_i x_i})$ into the variance factor σ^2 and the matrix of weight coefficients (g^{ij}) has been indicated.

The adoption of a value for σ^2 turns out to have no influence on the 1st part of the adjustment (2.2.11), the computation of the least squares estimators. Nor has this choice any influence on the 2nd part of the adjustment (2.2.12), the computation of the matrix of weight coefficients of these estimators. Only when we transfer to the covariance matrix, σ^2 has the function of a kind of scale factor. Also in the 3rd part of the adjustment (2.2.13), the computation of the shifting variate, the adoption of σ^2 has a similar influence. However, there is a serious influence if one considers tests with power functions; at first sight it seems that σ^2 has again an effect like a scale factor, but in reality a wrong assumption for σ^2 may have a ruinous effect on the interpretation of results of the computation. This is also true for the situation where no alternative hypothesis needs to be assigned to Step I of the adjustment, so that in testing-formulas σ^2 can be replaced by the estimate $\{\hat{\sigma}^2\}$. See the remarks referring to this at the end of par. 2.4. It would lead us too far if we discussed this further.

It is particularly troublesome that just this latter influence of σ^2 turns out to be a disturbing factor for the testing theory. For, in the way described in par. 2.2, experience shows that the ratios of elements of $\sigma_{x_i x_j}$ can be established with a greater reliability than their actual numerical value. Or in other words, it might be said that the matrix of weight coefficients (g^{ij}) can be established more reliably than $(\sigma_{x_i x_j})$, and consequently that the assumption of a value for σ^2 is always a more or less precarious affair, see (3.1.3). This question will always have to be taken into account in the decision problem of the testing theory.

More drastic are the consequences of an incorrect assumption of the matrix (g^{ij}) , from which (g^{er}) follows, see (3.1.2).

The elements of (g^{ij}) , just as those of $(\sigma_{x_i x_j})$, will never have a realistic number of digits higher than 2. As is easily shown from the geometrical illustration of the adjustment and testing process in the sample space of (x^i) , the consequences of this restriction are hardly appreciable. LINKWITZ has quantitatively studied these consequences [LINKWITZ 1961] and found an admissible deviation of 10% for the elements of (g^{ij}) , which checks reasonably with the aforementioned restriction to two digits. I often have the feeling that we should be happy if the possible deviation is within 30%. We therefore definitely do not have to worry about "true" weight coefficients, as long as we have acceptable ones.

In ordinary practice one takes much less care, even to-day. It still happens that for (g^{ij}) a unit matrix is assumed without more ado, whereas often the probability distribution of given coordinates is ignored or forgotten. Such a matrix, which (often unjustifiably) is put in the place of (g^{ij}) will be called a matrix of pseudo

weight coefficients, to be denoted by:

$$(h^{ij}) \dots \dots \dots (3.3.2)$$

so that as the pseudo covariance matrix is introduced:

$$\sigma^2 \cdot (h^{ij}) \text{ instead of } \sigma^2 \cdot (g^{ij}) \dots \dots \dots (3.3.3)$$

If this is done thoughtlessly, one is easily led to an automatic application of the algorithm of the method of least squares, i.e. (2.2.11)-(2.2.13). But one obtains pseudo least squares estimators, which will be classified as unbiased linear estimators, denoted by:

$$(\underline{X}_{(h)}^R) \dots \dots \dots (3.3.4)$$

Accordingly one gets pseudo weight coefficients of the estimators and a pseudo shifting variate.

Summarizing, and using a notation which is in accordance with (3.3.2)-(3.3.4) we get in analogy with (2.2.16):

Adjustment with (h^{ij})		
1st part	$(\underline{X}_{(h)}^R - a_0^R) = (H^{Rj}) (\bar{h}_{ji}) (\underline{x}^j - a_0^j)$	(3.3.5')
2nd part	$(H^{RS}) = (H^{Rj}) (\bar{h}_{ji}) (H^{iS})$	(3.3.5'')
3rd part	$\underline{E}_{(h)} = (\underline{x}^j - a_0^j)^* (\bar{h}_{jv}) (h^{ij} - H^{ij}) (\bar{h}_{ji}) (\underline{x}^i - a_0^i)$	(3.3.5''')

As far as the computation of estimators is concerned, the introduction of (3.3.2) need not be harmful. Many satisfactory approximate adjustment methods have been developed, and it can be proved [BAARDA 1961] that always a matrix (h^{ij}) can be found with which the estimators from the approximate method can be interpreted as pseudo least squares estimators, *provided* that the same "laws of nature" (2.2.2) or (2.2.5) are used. The algebraic technique developed for this has only rarely been used, because the results (3.3.5'') and (3.3.5''') usually cannot be interpreted.

To illustrate this we oppose to each other the:

$$\left. \begin{array}{l} \text{least squares estimators } (\underline{X}^R) \\ \text{pseudo least squares estimators } (\underline{X}_{(h)}^R) \end{array} \right\} \dots \dots \dots (3.3.6)$$

Consider now a linear function like (2.2.9) with an *arbitrary* row vector (A_R) :

$$\tilde{f} = (A_R)(\tilde{x}^R) \dots \dots \dots (3.3.7)$$

Then from (3.3.7) with (3.3.6) follow the estimators:

$$\underline{F} = (A_R)(\underline{X}^R) \dots \dots \dots (3.3.8')$$

$$\underline{F}_{(h)} = (A_R)(\underline{X}_{(h)}^R) \dots \dots \dots (3.3.8'')$$

An application of the law of propagation of weight coefficients to (3.3.5'') and (3.3.8) with $\overline{(x^i)}, \overline{(x^j)^*} = (g^{ij})$ results in the (interpretable) weight coefficients:

$$\overline{(X_{(h)}^R)}, \overline{(X_{(h)}^S)^*} = (H^{Rj})(\bar{h}_{jv})(g^{ij})(\bar{h}_{ji})(H^{iS}) = \text{in general } \neq (H^{RS}) \quad (3.3.9)$$

$$\left. \begin{aligned} \overline{F}, \overline{F} &= (A_R)(G^{RS})(A_S)^* \\ \overline{F}_{(h)}, \overline{F}_{(h)} &= (A_R)(X_{(h)}^R, X_{(h)}^S)^* (A_S)^* \geq \overline{F}, \overline{F} \end{aligned} \right\} \dots \dots \dots (3.3.10)$$

From (3.3.8'') follows with (3.3.5'') the pseudo weight coefficient of $\underline{F}_{(h)}$:

$$(A_R)(H^{RS})(A_S)^* \dots \dots \dots (3.3.11)$$

Also taking into consideration \underline{E} from (2.3.16) and $\underline{E}_{(h)}$ from (3.3.5'''), the following general result can be obtained:

$$\boxed{\begin{aligned} (A_R)(H^{RS})(A_S)^* &\geq \overline{F}_{(h)}, \overline{F}_{(h)} \geq \overline{F}, \overline{F} \\ \underline{E}_{(h)} &\leq \underline{E} \end{aligned}} \dots \dots \dots (3.3.12)$$

The indefiniteness of the inequalities in (3.3.12) is the reason that without further analysis of (H^{RS}) it is not known whether its use in the planning of networks leads to an under- or an overestimation of the precision of $(X_{(h)}^R)$, whereas with respect to $\underline{E}_{(h)}$ it is not known whether its use in tests increases or decreases the probability of rejecting H_0 . An examination of the the results of earlier adjustments has in this respect led to very remarkable conclusions, and this may be the cause that the theory of observations is so unpopular with many men from practice .

A further analysis shows, however, that, if used with prudence, the introduction of (h^{ij}) which in practice is almost unavoidable, can in certain cases give interpretable results.

For, if two matrices of pseudo weight coefficients (h_{\max}^{ij}) and (h_{\min}^{ij}) can be found satisfying:

$$\left. \begin{aligned} (h_{\max}^{ij} - g^{ij}) &\text{ is positive (semi-) definite} \\ (h_{\min}^{ij} - g^{ij}) &\text{ is negative (semi-) definite} \end{aligned} \right\} \dots \dots \dots (3.3.13)$$

and (h_{\max}^{ij}) is used in the 2nd part, (h_{\min}^{ij}) in the 3rd part of the adjustment, so that in fact the adjustment algorithm is computed twice, then:

$$\boxed{\begin{aligned} (A_R)(H_{\max}^{RS})(A_S)^* &\geq \overline{F}_{(h_{\max})}, \overline{F}_{(h_{\max})} \geq \overline{F}, \overline{F} \\ \underline{E}_{(h_{\min})} &\geq \underline{E} \end{aligned}} \dots \dots \dots (3.3.14)$$

The consequence is that when (H_{\max}^{RS}) is used an overestimation of variances occurs, and that when $\underline{E}_{(h_{\min})}$ is used in a test, H_0 will be rejected sooner. This means that one is actually using more stringent requirements, which, as practice in the Netherlands has shown, need not cause difficulties.

The disadvantage of (3.3.13), viz. the double computation, can be eliminated by making the two (h^{ij}) -matrices fulfill the relation:

$$(h_{\max}^{ij}) = \tau^2 \cdot (h_{\min}^{ij}), \quad \tau^2 > 1 \quad \dots \dots \dots (3.3.15)$$

If now with a view to the planning of the precision of geodetic networks the adjustment computation is done with (h_{\max}^{ij}) , then it can be shown that:

$$\boxed{\begin{matrix} (A_R)(H_{\max}^{RS})(A^S)^* \geq F_{(h_{\max})}, F_{(h_{\max})} \geq F, F \\ \tau^2 \underline{E}_{(h_{\max})} \geq \underline{E} \end{matrix}} \dots \dots \dots (3.3.16)$$

(3.3.16) makes it possible to answer the question whether the elements of the main diagonal of (g^{ij}) should be increased or decreased when rounding them off, in the case of uncertainty in the assumption of (g^{ij}) and absence of correlation. According to (3.3.15) and (3.3.16) one should do *both*, to get a possibility for finding the size of the factor τ^2 .

Here too, the practical application of the theory is not simple. It is possible to establish a practically applicable theory [BAARDA 1964] for distance- and direction measurements in such a way that uncorrelatedness can be introduced, but the same theory shows that in principle for given coordinates the introduction of correlation is necessary, even inevitable. One will therefore have to revert to the general requirements (3.3.13), possibly together with (3.3.15).

The theory mentioned seems to lead to the consequence that given coordinates must never get a correction, i.e. in an adjustment problem the part of the covariance matrix pertaining to given coordinates must be assumed to be zero. This is in agreement with geodetic practice, where the same thing is done to prevent confusion about the date when the coordinates of monuments have been fixed.

Here one is faced with a very particular application of (h_{\min}^{ij}) , for which (3.3.15) is not valid. Because, in particular in small networks, the influence of given coordinates on the results of an adjustment can be important, it has in practice been found that in this case:

$$\underline{E}_{(h_{\min})} \gg \underline{E} \dots \dots \dots (3.3.17')$$

and consequently, see (2.2.17):

$$E\{\hat{g}_{(h_{\min})}^2 | H_0\} = E\left\{\frac{1}{b} \underline{E}_{(h_{\min})} | H_0\right\} \gg \sigma^2 \dots \dots \dots (3.3.17'')$$

so that the use of $\underline{E}_{(h_{\min})}$ for testing purposes leads too often to rejection of H_0 if only it is because of (3.3.17'').

For small networks a solution of this problem has been found by executing mainly acceptable tests in the steps I-III as indicated in (3.2.23) ff.; after this, step IV is in principle computed twice, the first time with the special (h_{\min}^{ij}) ignoring the probability distribution of the given coordinates in order to find estimates $X_{(h_{\min})}^R$, the second time with a (h_{\max}^{ij}) in which a rather rough estimation of the covariance matrix of given coordinates has been used to complement (h_{\min}^{ij}) , whereas also the factor τ^2 is estimated for computing $\tau^2 \cdot \underline{E}_{(h_{\max})}$.

The same (h_{\max}^{ij}) is also used to find a practically usable pseudo covariance matrix of $(X_{(h_{\min})}^R)$, but this method which since about ten years is used in the Netherlands will not be further discussed here.

It should be mentioned, however, that the solution indicated has led to quite different requirements for the establishment of small geodetic networks. These

requirements will in the years to come be tested in practice on their merits regarding economy.

The attempts to establish a (h_{\max}^{ij}) -matrix for given coordinates in a certain adjustment problem, have clearly revealed the fundamental difficulty of the exact definition of the concept “coordinate”. This is already the case with computations in the plane. An exposition of these difficulties is found in [BAARDA 1962], where a first version was given of the theory now called “Polygon theory in the complex plane” [BAARDA 1964]. If one wishes to find such a (h_{\max}^{ij}) for coordinates, one must be able to detach oneself in the first instance from the two base points (generalized or not), as are defined in par. 6 of [BAARDA 1962]. In par. 7 of the same paper a possible solution has been suggested, which might also provide a connection with the purpose of measurement.

In spite of intensive research by ALBERDA and myself we have not yet succeeded in constructing a satisfactory positive (semi-) definite matrix which also has the relative invariance properties required by the theory. Should this construction succeed, then a generalization for spatial computations is certainly possible because the basic theory for a spatial polygon theory has already been developed.

The planning of geodetic networks with respect to precision as well as the analysis of statistical tests and error control considerations can actually only be exactly formulated if this matrix can be constructed.

3.4 Effects of non-linearity in functional relationships

Generally the different functional relationships mentioned in par. 2.2 will not be linear.

Consequently (2.2.6), (2.2.5) and (2.2.9) will have to be written as, successively:

$$(0) = (Y^a\{\dots, \tilde{x}^i, \dots\}) \quad \begin{matrix} i = 1, \dots, m \\ a = 1, \dots, b \end{matrix} \quad \dots \dots \dots (3.4.1)$$

$$(\tilde{x}^i) = (X^i\{\dots, \tilde{y}^a, \dots\}) \quad a = b+1, \dots, m \quad \dots \dots \dots (3.4.2)$$

$$(\tilde{x}^r) = (X^r\{\dots, \tilde{x}^i, \dots\}) \quad r = \dots \quad \dots \dots \dots (3.4.3)$$

For a comparison with the linear relationships from par. 2.2, (3.4.1)–(3.4.3) are expanded in Taylor’s series omitting third- and higher order terms. This expansion must be done operationally, i.e. with respect to a vector of actually assignable approximate values, not with respect to (\tilde{x}^i) or (\tilde{y}^a) , because the latter are just the vectors to be estimated.

Because we have to make a connection with adjustment problems anyway, we shall follow the usual way of computing the 2nd standard problem, viz. the assumption of a vector of approximate values for the unknowns (Y^a) :

$$(Y_0^a) \quad \dots \dots \dots (3.4.4)$$

from which, following in principle (3.4.2) and (3.4.3), the vectors of derived approximate values for (X^i) and (X^r) are computed:

$$(X_0^i) = (X^i\{\dots, Y_0^\alpha, \dots\}) \dots \dots \dots (3.4.5)$$

$$(X_0^j) = (X^j\{\dots, X_0^i, \dots\}) \dots \dots \dots (3.4.6)$$

The choice (3.4.4)–(3.4.6) is made in order to get expansions in the same “point” of the sample space of (x^i) .

For partial derivatives the following notation is introduced:

$$\left. \begin{aligned} \left\{ \frac{\partial Y^q}{\partial x^i} \right\}_{(X_0^i)} &\equiv u_i^q, & \frac{1}{2} \left\{ \frac{\partial^2 Y^q}{\partial x^j \partial x^i} \right\}_{(X_0^i)} &\equiv u_{j,i}^q = u_{i,j}^q \\ \left\{ \frac{\partial X^j}{\partial y^\alpha} \right\}_{(Y_0^\alpha)} &\equiv a_\alpha^j, & \frac{1}{2} \left\{ \frac{\partial^2 X^j}{\partial y^\beta \partial y^\alpha} \right\}_{(Y_0^\alpha)} &\equiv a_{\beta,\alpha}^j = a_{\alpha,\beta}^j \\ \left\{ \frac{\partial X^r}{\partial x^i} \right\}_{(X_0^i)} &\equiv A_i^r, & \frac{1}{2} \left\{ \frac{\partial^2 X^r}{\partial x^j \partial x^i} \right\}_{(X_0^i)} &\equiv A_{j,i}^r = A_{i,j}^r \end{aligned} \right\} \dots \dots \dots (3.4.7)$$

The choice (3.4.4)–(3.4.6) then gives the relation, compare (2.2.3):

$$(u_i^q)(a_\alpha^j) = (0) \dots \dots \dots (3.4.8)$$

The expansions sought can then, omitting details, be written in the form:

(3.4.1)	$(0) = (u_i^q) (\tilde{x}^i - X_0^i) + (\tilde{x}^j - X_0^j) * (u_{j,i}^q) (\tilde{x}^i - X_0^i)$	(3.4.9)
(3.4.2)	$(\tilde{x}^i - X_0^i) = (a_\alpha^i) (\tilde{y}^\alpha - Y_0^\alpha) + (\tilde{y}^\beta - Y_0^\beta) * (a_{\beta,\alpha}^i) (\tilde{y}^\alpha - Y_0^\alpha)$	
(3.4.3)	$(\tilde{x}^r - X_0^r) = (A_i^r) (\tilde{x}^i - X_0^i) + (\tilde{x}^j - X_0^j) * (A_{j,i}^r) (\tilde{x}^i - X_0^i)$	

Up to and including the second order terms, we have then again obtained a consistent system of equations.

If (3.4.9) is replaced by difference equations omitting the second order terms, and these are opposed to (2.2.6), (2.2.5) and (2.2.9), one gets:

$(0) = (u_i^q) (\tilde{x}^i - a_0^i)$	$(0) = (u_i^q) (\tilde{x}^i - X_0^i)$. . . (3.4.10)
$(\tilde{x}^i - a_0^i) = (a_\alpha^i) (\tilde{y}^\alpha)$	$(\tilde{x}^i - X_0^i) = (a_\alpha^i) (\tilde{y}^\alpha - Y_0^\alpha)$	
$(\tilde{x}^r - a_0^r) = (A_i^r) (\tilde{x}^i - a_0^i)$	$(\tilde{x}^r - X_0^r) = (A_i^r) (\tilde{x}^i - X_0^i)$	

From (3.4.8) and (3.4.10) it follows how, with some unimportant changes, the adjustment algorithm indicated in par. 2.2. can be applied to non-linear relationships (3.4.1)–(3.4.3). Here, too, we leave out the many details.

The effect of adjusting on the basis of the difference equations (3.4.10) is twofold:

- a. Substitution of the obtained estimates into (3.4.1)–(3.4.3) will show theoretical misclosures (remainders).
- b. All derived variates generated by the adjustment algorithm will have a bias, depending on the choice (3.4.4.)

Schematically we can indicate item a. as:

$$\begin{array}{l}
 (\underline{R}^0), (\underline{R}^i), (\underline{R}^r) \text{ not specified here} \\
 (0) = (Y^0\{\dots, \underline{X}^i, \dots\}) + (\underline{R}^0) \\
 (\underline{X}^i) = (X^i\{\dots, Y^a, \dots\}) + (\underline{R}^i) \\
 (\underline{X}^r) = (X^r\{\dots, \underline{X}^i, \dots\}) + (\underline{R}^r)
 \end{array} \dots \dots \dots (3.4.11)$$

Some formulas concerning item b. will be given, but with some reserve because the theory has been developed only recently.

Substitution of (3.4.2) into (3.4.3) makes it possible to compute the partial derivative:

$$\left\{ \frac{\partial X^r}{\partial y^\beta \partial y^\alpha} \right\}_{(X_0^i)} = A_{\beta, \alpha}^r = A_{\alpha, \beta}^r \dots \dots \dots (3.4.12')$$

Then the following is valid, compare (2.2.15):

$$\begin{array}{l}
 (R_0^i) = (\hat{y}^\beta - Y_0^\beta) * (a_{\beta, \alpha}^i) (\hat{y}^\alpha - Y_0^\alpha) \\
 (R_0^r) = (\hat{y}^\beta - Y_0^\beta) * (A_{\beta, \alpha}^r) (\hat{y}^\alpha - Y_0^\alpha) \\
 (E\{\underline{y}^0\} - 0) = (u_0^0) (R_0^i) \\
 (E\{\underline{X}^i\} - \hat{x}^i) = (G^{ij}) (\tilde{g}_{jr}) (R_0^r) - (R_0^i) \\
 (E\{\underline{Y}^a\} - \hat{y}^a) = (G^{aj}) (\tilde{g}_{ji}) (R_0^i) \\
 (E\{\underline{X}^r\} - \hat{x}^r) = (G^{rj}) (\tilde{g}_{ji}) (R_0^i) - (R_0^r)
 \end{array} \dots \dots \dots (3.4.12'')$$

$$\begin{array}{l}
 (E\{\underline{y}^0\} - 0) = (u_0^0) (R_0^i) \\
 (E\{\underline{X}^i\} - \hat{x}^i) = (G^{ij}) (\tilde{g}_{jr}) (R_0^r) - (R_0^i) \\
 (E\{\underline{Y}^a\} - \hat{y}^a) = (G^{aj}) (\tilde{g}_{ji}) (R_0^i) \\
 (E\{\underline{X}^r\} - \hat{x}^r) = (G^{rj}) (\tilde{g}_{ji}) (R_0^i) - (R_0^r)
 \end{array} \dots \dots \dots (3.4.12''')$$

From:

$$(E\{\underline{y}^0\}) \neq (0)$$

follows a similar effect on \underline{E} and consequently on $\hat{\sigma}^2$ as has been indicated in (3.1.5) ff. for an alternative hypothesis. Hence, writing $\bar{\lambda}$ instead of λ :

$$\begin{array}{l}
 \left\{ \frac{\hat{\sigma}^2}{\sigma^2} \middle| H_0 \right\} = F'_{b, \infty, \bar{\lambda}} \\
 E \left\{ \frac{\hat{\sigma}^2}{\sigma^2} \middle| H_0 \right\} = 1 + \frac{\bar{\lambda}}{b} \\
 \bar{\lambda} = \frac{1}{\sigma^2} (E\{\underline{y}^r\}) * (\tilde{g}_{r0}) (E\{\underline{y}^0\})
 \end{array} \dots \dots \dots (3.4.13)$$

Because when decomposing into steps the same is valid for $\bar{\lambda}$ as for λ in (3.1.12), the phenomenon that the estimate $\hat{\sigma}^2$ for σ^2 has a tendency to increase as b (= number of condition equations) increases, might possibly be ascribed to the influence of non-linearity of (3.4.1) and (3.4.2). In geodetic practice this situation is repeatedly noticed.

If an adjustment shows appreciable remainders, i.e. if the least squares estimates do not fulfill the condition equations to a sufficient degree, iterative procedures are often used. Usually the procedure converges well enough, but it can be shown that the bias caused by non-linearity persists, even if the resulting estimates fulfill the equations exactly. In the 2nd standard problem it also remains to be shown that the result of an iterative procedure is independent of the assumed approximate values for the unknowns.

Note on the choice of (3.4.4)–(3.4.6)

This is indeed a choice. The practice of adjustment technique shows that other choices can be made. The most important one is, that in the 1st standard problem one tries to evade the assumption of the vector (X_0^i) . Instead of computing, according to (3.4.10):

$$(y^e) = (u_i^e) (x^i - X_0^i)$$

one follows (3.4.1) directly, so that with (3.4.9):

$$(y^e) = (Y^e\{\dots, x^i, \dots\}) \approx (u_i^e)(x^i - X_0^i) + (x^j - X_0^j) * (u_{j,i}^e)(x^i - X_0^i) \quad (3.4.14)$$

Further one follows the adjustment algorithm of the 1st standard problem in (2.2.11) ff.

This means that the solutions according to the 1st and the 2nd standard problem do not give entirely identical results.

The method of (3.4.14) has a very favourable effect, the remainders R are in general smaller and their expectation can always be computed. For example, instead of (3.4.11) and (3.4.12) one gets:

1st standard problem with (3.4.14)	
$(E\{\underline{R}^e\}) = \left(\sum_{i,j} u_{j,i}^e \cdot \sigma_{x^i x^j} \right)$	
$(0) = (Y^e\{\dots, \underline{X}^i, \dots\}) + (\underline{R}^e)$ (3.4.15)
$(E\{y^e\} - 0) = \left(\sum_{i,j} u_{j,i}^e \cdot \sigma_{x^i x^j} \right)$	
$(E\{\underline{X}^i\} - \hat{x}^i) = -(g^{ir})(\bar{g}_{r\varrho})(E\{y^e\})$	

It follows from (3.4.15) that the influence of the vector (X_0^i) has been practically eliminated by the choice (3.4.14). Only the small influence of the partial derivatives according to (3.4.7) remains.

Because in many problems the number of digits which must be used for the elements of (u_i^e) is limited, u_i^e in (3.4.7) is often replaced by:

$$u_i^e = \left\{ \frac{\partial Y^e}{\partial \hat{x}^i} \right\}_{(x_s^i)} \quad \dots \dots \dots (3.4.16')$$

Actually one gets then stochastic coefficients:

$$\underline{u}'_i = \left\{ \frac{\partial Y^e}{\partial \bar{x}^i} \right\}_{(x^i)} \dots \dots \dots (3.4.16'')$$

Besides, if there are redundant observations ($b < m$) one cannot get a unique result, in contra-distinction to (3.4.7) and (3.4.5). This problem is becoming more and more important as further consequences are drawn of the application of large computers. Actually, a solution must be sought in the much wider context of the theory of rounding-off, but here we cannot go further into this question.

Final remark

The assumption of functional relationships like (3.4.1)–(3.4.3) in the formulation of the null hypothesis will only lead to unambiguous results if these relationships are taken from a unique model of functional relations, e.g. from plane or spatial Euclidean geometry, etc.

In general, there will then be a possibility of choice, the only requirement being that certain conditions of functional independence are fulfilled. This means that systems of equations can be submitted to certain transformations admitted by the theory.

The theory of rounding-off errors, as well as the theory of non-linearity sketched here, show that one choice of a system of equations results in smaller remainders than another. The systems may be isomorphic from a theoretical mathematical point of view, but numerical computations may nevertheless lead to results that are sometimes appreciably different. This, together with the subject-matter of previous paragraphs, makes *error control* to one of the most difficult subjects in geodesy.

One of the most remarkable outcomes of the theory developed is that effects of non-linearity may be made smaller by a particular choice for the relation between the precision of distance- to the precision of direction measurements, together with a particular choice for the systems (3.4.1)–(3.4.2). The result seems to imply that from a theoretical point of view the measurement of distances *and* directions in all points of a geodetic network should be preferred to measurement of only distances (trilateration) or only directions (triangulation).

Here it consequently appears that there is a strong influence of the probability distribution of the different observed or derived variates, an influence which is also known from the theory of rounding-off errors.

3.5 Effects of the assumption of a model of functional relationships

The purposeful use of the word “model” indicates that the expression “laws of nature” in (2.2.2) ff. can only have a relative value. As a result of the clear and didactic teaching in high schools or universities, almost everyone is convinced of the acceptability (if not the truth) of the laws of physics, and only rarely such “laws of nature” are seen as parts of a model consisting of mathematical relations, where quantities have been labeled with a physical name (often in a rather careless and arbitrary manner). But the more one tries to generalize in his own field and to seek connections with generalizing trends in other fields, the more it becomes evident

that there is a *possibility of choice* with respect to the mathematical framework describing physical events.

For example, one may beside (3.4.1)–(3.4.3) arrive at a possible alternative assumption:

$$\left. \begin{aligned} (0) &= (\bar{Y}^a\{\dots, \tilde{x}^i, \dots\}) & i &= 1, \dots, \bar{m} \\ & & \bar{q} &= 1, \dots, \bar{b} \\ (\tilde{x}^i) &= (\bar{X}^i\{\dots, \tilde{y}^a, \dots\}) & \bar{a} &= \bar{b}+1, \dots, \bar{m} \\ (\tilde{x}^r) &= (\bar{X}^r\{\dots, \tilde{x}^i, \dots\}) & r &= \dots \end{aligned} \right\} \dots \dots \dots (3.5.1)$$

If we oppose to each other two models which are not isomorphic, such as e.g. the classical ellipsoidal computing model and a really spatial computing model, we see that e.g. the measurement of vertical angles in the second model is a necessity, whereas in the first model the measurement of vertical angles can often be omitted. This means, consequently, that:

$$\bar{m} \neq m \dots \dots \dots (3.5.2')$$

But this example shows also that the number of condition equations and/ or parameters can be different, or:

$$\bar{b} \neq b \dots \dots \dots (3.5.2'')$$

From (3.5.1) and (3.5.2) it follows that the statistic:

$$\hat{\sigma}^2 \dots \dots \dots (3.5.3)$$

will certainly be different in the two models, so that in principle a test might be used to decide in favour of one or the other.

In principle ..., because the following should not be forgotten:

- a. The difficulties inherent in every method of testing (par. 3.2);
- b. The effects treated in par. 3.3. and 3.4;
- c. The innumerable corrections to measurements which have to be applied to establish correspondence with mathematical quantities, and which are often assumed to be constants (errorless quantities);
- d. The insufficiency of *any* mathematical model to describe physical events completely, not to mention the fact that the recognition and definition of “physical events” has a strongly subjective aspect, and is usually directed towards the mathematical model assumed.

Therefore it need not be surprising that testing by (3.5.3) is not conclusive. This might be an explanation of the negative results of the test computations by HOTINE mentioned in par. 1.

It therefore appears to me that actually the only criterion for the choice of a mathematical model of functional relationships is given by the results of predictions based on the model. Of course one is here again faced with a new difficulty, viz. the assumption concerning the registration *R* (see (2.2.1)) of future measurements. In this respect the execution of experiments (unfortunately always too limited) may give some likelihood of success, although as a warning we must refer to the base extension net in the example treated at the end of par. 3.2.

The question how one arrives at the construction of an alternative model when predictions based on a previously used model do not work satisfactorily, seems to me to contain elements that are too strongly personal to permit a general answer. It is done on the basis of experience, experiments, influences from literature, but above all by intuition. Modern computer technique with its urge towards generality will often be a stimulus; sometimes a set-up for a modern computing technique brings out that the consistency of the model is not satisfactory. The latter is e.g. the case with classical ellipsoidal computation methods, at least in the form as was analysed in [BAARDA 1954, 57]. Curiously, this may be masked by choosing the adjustment algorithm of the 2nd standard problem, using coordinates as parameters.

4 MODEL THEORY. A TENTATIVE APPROACH

4.1 Experiment and model

This paragraph is devoted to a sketchy survey of the significance of (2.2.6), (2.2.5), (2.2.9) or (3.4.1)–(3.4.3), or (3.5.1). From par. 3.5 it follows that these relationships are supposed to be derived from a consistent functional model. In par. 2.2 it was stated that a reasoning could be given, leading to the notation of these relationships as being valid between quantities with the tilde sign (\sim).

In view of the establishment of a link in (2.2.1) ff. between a vector of observational numbers and the results of an (earlier) experiment, the notation indicated must have a connection with the analysis of an experiment.

From the frequentist background, we consider the experiment to be conceived as an N -fold “repetition” of a vector (x^i) , each repetition consisting of one measurement of a number of directions, distances etc., to which the registration R is assigned; compare [BAARDA 1960, 62]. We obtain:

$$\{(x_1^i); R_1\}, \dots, \{(x_s^i); R_s\}, \dots, \{(x_N^i); R_N\} \dots \dots \dots (4.1.1)$$

Now we leave out of consideration such differences in registration as, according to “personal” judgement, are irrelevant, resulting in:

$$R_1 = \dots = R_s = \dots = R_N \rightarrow R \dots \dots \dots (4.1.2)$$

Then (4.1.1) gives “repeated measurements under the same circumstances”.

In many physical experiments, this set-up can be followed by computing (algebraic) means (\bar{x}^i) , the “first moments”. These quantities have, in the probability model, equivalents with particularly welcome mathematical properties. One obtains:

$$\{(\bar{x}^i); R\} = \left(\frac{1}{N} \sum_{s=1}^N x_s^i \right) \dots \dots \dots (4.1.3)$$

The further description of (4.1.1) with (4.1.2) can if necessary be done with higher-order moments, of which we shall only mention the second reduced moments:

$$\{(s_{x^i x^j}); R\} = \left(\frac{1}{N-1} \sum_{s=1}^N \{x_s^i - \bar{x}^i\} \{x_s^j - \bar{x}^j\} \right) \dots \dots \dots (4.1.4)$$

If now “repetition” of this experiment with sufficiently large N gives “almost” (personal judgement) the same results (4.1.3) and (4.1.4) etc., then (4.1.1) with (4.1.2) can be considered as the frequency distribution of a vector of stochastic quantities (variates). Ignoring many details, we can then oppose to each other equivalent quantities in experiment and probability model:

Experiment	Probability model
(4.1.1), (4.1.2)	vector of variates: (x^i)
(4.1.3)	vector of means: $(\bar{x}^i) = (E\{x^i\})$
(4.1.4)	covariance matrix: $(\sigma_{x^i x^j}) = (E\{x^i - \bar{x}^i\}\{x^j - \bar{x}^j\})$
.....

. . . (4.1.5)

If the number of variates x^i increases, then N must be many times larger than 100, whereas the experiment must certainly be repeated several times. This means that the execution of all measurements will take a long time, so that (4.1.2) will no doubt be a too strong simplification in most cases. In view of the costs (in different respects) of such experiments in the field of geodesy, the experiment will usually be restricted to some x^i 's. In other words: (4.1.1)–(4.1.5) will mainly be a hypothetical experiment, executed and elaborated in thought. Or in still other words: the subjective approach to the whole theory is inevitable.

Nevertheless this approach can be directive for a further elaboration of the theory.

4.2 Considerations concerning the linking-up of a model of functional relationships

Consider in an extensive experiment the measurement of one horizontal direction from P_i to P_k : r_{ik} . Can this measurement be repeated over a longer period with the same registration? From experience we know that this is not possible because variations in the measuring instrument are unavoidable. It is possible to make each time a reference pointing to a point P_j , but in doing so one actually introduces an angle:

$$a_{jik} = r_{ik} - r_{ij} \quad (4.2.1)$$

Concerning a_{jik} we can reasonably say that repeated measurement with the same registration is possible. Consequently, the execution and evaluation of an experiment according to par. 4.1 is only possible for a_{jik} .

If one now successively measures a series of directions, e.g.:

$$r_{ij}, r_{ik}, r_{il} \quad (4.2.2)$$

then (4.1.1)–(4.1.4) can be executed for the pairs of angles:

α	jik	kil	lij	kij	lik	jil
	jil	kij	lik	lij	jik	kil

but it is clear that only one of the pairs may be introduced as functionally independent, such as:

$$\left. \begin{aligned} \alpha_{jik} &= r_{ik} - r_{ij} \\ \alpha_{kil} &= r_{il} - r_{ik} \end{aligned} \right\} \dots \dots \dots (4.2.3)$$

In order to be independent of this choice (4.2.3) one introduces according to (4.1.5), beside

$$\alpha_{jik}, \alpha_{kil} \dots \dots \dots (4.2.4)$$

the (theoretical) variates:

$$r_{ij}, r_{ik}, r_{il} \dots \dots \dots (4.2.5')$$

under the strict *requirement*:

$$\left. \begin{aligned} \alpha_{jik} &= r_{ik} - r_{ij} & \text{hence: } E\{\alpha_{jik}\} &= E\{r_{ik}\} - E\{r_{ij}\} \\ \alpha_{kil} &= r_{il} - r_{ik} & E\{\alpha_{kil}\} &= E\{r_{il}\} - E\{r_{ik}\} \end{aligned} \right\} \dots \dots \dots (4.2.5'')$$

so that in the model the variates r can be *considered* as theoretical "observed" variates.

Along with (4.2.5) one further has to establish a theoretical probability distribution for the variates r , in such a way that from (4.2.5'') follows the probability distribution of (4.2.4) derived from the experiment. It would seem that in practice this is only possible if a restriction to the Laplace-Gauss distribution is made.

The introduction of the model now proceeds one step further: we assume a method of measurement and computation which fits a description in a two-dimensional Euclidean space (plane).

Then the variates r in (4.2.5') can be interpreted as bearing-variates in a local (i)-system:

$$\left. \begin{aligned} \underline{A}_{ij}^{(i)} &\equiv r_{ij} \\ \underline{A}_{ik}^{(i)} &\equiv r_{ik} \\ \underline{A}_{il}^{(i)} &\equiv r_{il} \end{aligned} \right\} \dots \dots \dots (4.2.6)$$

The notation is in agreement with [BAARDA 1962].

The next step, which fits completely in the assumed model, is the introduction of the orientation of this (i)-system with respect to a more general (regional) (a)-system, with orientation-variate ϱ_i^a :

$$\left. \begin{aligned} \underline{A}_{ij}^{(a)} &= \varrho_i^a + \underline{A}_{ij}^{(i)} \\ \underline{A}_{ik}^{(a)} &= \varrho_i^a + \underline{A}_{ik}^{(i)} \\ \underline{A}_{il}^{(a)} &= \varrho_i^a + \underline{A}_{il}^{(i)} \end{aligned} \right\} \dots \dots \dots (4.2.7)$$

If one goes back from (4.2.7) to (4.2.4), than all these theoretical entities disappear, e.g.:

$$\underline{A}_{ik}^{(a)} - \underline{A}_{ij}^{(a)} = \underline{A}_{ik}^{(i)} - \underline{A}_{ij}^{(i)} = r_{ik} - r_{ij} = \alpha_{jik} \dots \dots \dots (4.2.8)$$

Consider now a triangulation in this geometric model. It appears that measurement of angles suffices to fix the relative position of points. In these computations it is remarkable that by using the law of sines in a triangle, no distances but distance ratios v are derived. Hence, the v 's seem to play a part analogous to the α 's. Therefore we shall search for the development which is parallel to (4.2.1)–(4.2.8).

Consider, then, the measurement of the distance from P_i to P_k : d_{ik} . The meaning of d_{ik} is the number of times an instrumental unit of length is comprised in $\overline{P_i P_k}$.

In an experiment we are again faced with the difficulty of “repeating” the measurement with the same registration. Experience shows that the instrumental unit of length is not constant over a period of some duration, whereas the principle of standardizing the instrument is left out of consideration at this early stage of experimentation. If the unit of length of the instrument changes slowly, we can meet this difficulty by measuring each time also the distance from P_i to a point P_j , and by introducing the distance ratio:

$$v_{jik} = \frac{d_{ik}}{d_{ij}} \dots \dots \dots (4.2.9)$$

Also, a series of distance measurements can be measured in P_i :

$$d_{ij}, d_{ik}, d_{il} \dots \dots \dots (4.2.10)$$

and we can e.g. introduce the distance ratios:

$$\left. \begin{aligned} v_{jik} &= \frac{d_{ik}}{d_{ij}} \\ v_{kil} &= \frac{d_{il}}{d_{ik}} \end{aligned} \right\} \dots \dots \dots (4.2.11)$$

If now, in analogy to (4.2.4) and (4.2.5), one wishes to introduce, beside v_{jik} and v_{kil} , the variates

$$\underline{d}_{ij}, \underline{d}_{ik}, \underline{d}_{il}$$

then, for the theoretical model, two difficulties arise:

- a. $E\{v_{jik}\} \neq \frac{E\{\underline{d}_{ik}\}}{E\{\underline{d}_{ij}\}}$, etc.
- b. For a Laplace-Gauss distribution we have

$$-\infty < \text{range} < +\infty$$

whereas for a distance d we have

$$d > 0$$

hence

$$v > 0$$

(for quantities r and a this difficulty does not arise because of the theoretical multiple-valuedness).

Both difficulties can be met by introducing in the model the natural logarithm of quantities:

$$\underline{\ln} v_{jik}, \underline{\ln} v_{kil} \dots \dots \dots (4.2.12)$$

and also the (theoretical) variates:

$$\underline{\ln d_{ij}}, \underline{\ln d_{ik}}, \underline{\ln d_{il}} \dots \dots \dots (4.2.13')$$

under the strict requirement:

$$\left. \begin{aligned} \underline{\ln v_{jik}} &= \underline{\ln d_{ik}} - \underline{\ln d_{ij}} \\ \underline{\ln v_{kil}} &= \underline{\ln d_{il}} - \underline{\ln d_{ik}} \end{aligned} \right\} \dots \dots \dots (4.2.13'')$$

hence:

$$\left. \begin{aligned} E\{\underline{\ln v_{jik}}\} &= E\{\underline{\ln d_{ik}}\} - E\{\underline{\ln d_{ij}}\} \\ E\{\underline{\ln v_{kil}}\} &= E\{\underline{\ln d_{il}}\} - E\{\underline{\ln d_{ik}}\} \end{aligned} \right\} \dots \dots \dots (4.2.13''')$$

with considerations analogous to those given with reference to (4.2.5).

Now, (4.2.13') can be interpreted as the ln of distance-variates in a local (i)-system:

$$\left. \begin{aligned} \underline{\ln s_{ij}^{(i)}} &= \underline{\ln d_{ij}} \\ \underline{\ln s_{ik}^{(i)}} &= \underline{\ln d_{ik}} \\ \underline{\ln s_{il}^{(i)}} &= \underline{\ln d_{il}} \end{aligned} \right\} \dots \dots \dots (4.2.14)$$

The notation is in agreement with [BAARDA 1962].

After this, the connection with a more general (regional) (a)-system can be made, with the scale factor λ_i^a :

$$\left. \begin{aligned} \underline{\ln s_{ij}^{(a)}} &= \underline{\ln \lambda_i^a} + \underline{\ln s_{ij}^{(i)}} \\ \underline{\ln s_{ik}^{(a)}} &= \underline{\ln \lambda_i^a} + \underline{\ln s_{ik}^{(i)}} \\ \underline{\ln s_{il}^{(a)}} &= \underline{\ln \lambda_i^a} + \underline{\ln s_{il}^{(i)}} \end{aligned} \right\} \dots \dots \dots (4.2.15)$$

A return from (4.2.15) to (4.2.12) causes the disappearance of all theoretical entities:

$$\underline{\ln s_{ik}^{(a)}} - \underline{\ln s_{ij}^{(a)}} = \underline{\ln s_{ik}^{(i)}} - \underline{\ln s_{ij}^{(i)}} = \underline{\ln d_{ik}} - \underline{\ln d_{ij}} = \underline{\ln v_{jik}} \quad (4.2.16)$$

The variate $\underline{\ln \lambda_i^a}$ makes it possible to include in the model the influence of standardization of measuring apparatus, but for brevity we shall leave this out of consideration.

In view of the remarkable parallelism between (4.2.4)–(4.2.8) and (4.2.12)–(4.2.16), we introduce complex variates, such as:

$\underline{\Pi}_{jik} = \underline{\ln v_{jik}} + i \underline{a}_{jik}$	$\underline{\ln \gamma_i^a} = \underline{\ln \lambda_i^a} + i \underline{o_i^a}$	(4.2.17)
$\underline{A}_{ij}^{(i)} = \underline{\ln s_{ij}^{(i)}} + i \underline{A}_{ij}^{(i)}$	$\underline{A}_{ij}^{(a)} = \underline{\ln s_{ij}^{(a)}} + i \underline{A}_{ij}^{(a)}$	
$\underline{A}_{ik}^{(i)} = \underline{\ln s_{ik}^{(i)}} + i \underline{A}_{ik}^{(i)}$	$\underline{A}_{ik}^{(a)} = \underline{\ln s_{ik}^{(a)}} + i \underline{A}_{ik}^{(a)}$	

Hence:

$\underline{U}_{jik} = \underline{A}_{ik}^{(i)} - \underline{A}_{ij}^{(i)} = \underline{A}_{ik}^{(a)} - \underline{A}_{ij}^{(a)}$	(4.2.18)
$\underline{A}_{ij}^{(a)} = \underline{\ln \gamma_i^a} + \underline{A}_{ij}^{(i)}$	
$\underline{A}_{ik}^{(a)} = \underline{\ln \gamma_i^a} + \underline{A}_{ik}^{(i)}$	

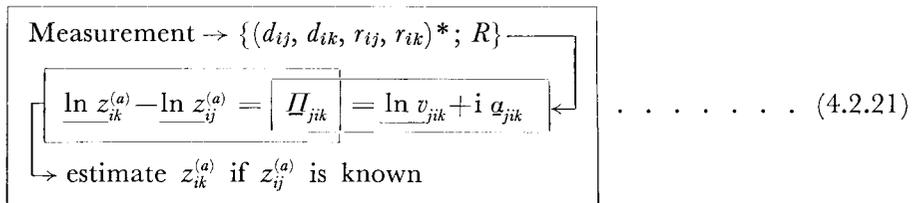
Our mathematical model contains two-dimensional vectors z whose modulus is s and whose argument (bearing) is A , see [BAARDA 1962]:

$$x + iy = z = s \cdot e^{iA} = e^{\ln s + iA} \dots \dots \dots (4.2.19)$$

so that the following interpretation follows from (4.2.17) and (4.2.18):

$A_{ij}^{(i)} \equiv \ln z_{ij}^{(i)}$	$A_{ij}^{(a)} \equiv \ln z_{ij}^{(a)}$ (4.2.20)
$A_{ik}^{(i)} \equiv \ln z_{ik}^{(i)}$	$A_{ik}^{(a)} \equiv \ln z_{ik}^{(a)}$	
$\ln z_{ij}^{(a)} = \ln \gamma_i^a + \ln z_{ij}^{(i)}$		
$\ln z_{ik}^{(a)} = \ln \gamma_i^a + \ln z_{ik}^{(i)}$		
$\Pi_{jik} = \ln z_{ik}^{(i)} - \ln z_{ij}^{(i)} = \ln z_{ik}^{(a)} - \ln z_{ij}^{(a)}$		

(4.2.17) and (4.2.20) provide the possibility to obtain coordinates from measurements of distances and directions, or rather to obtain coordinate differences from them. The Π -variate acts as a connecting link. The following diagram illustrates this:



Here we also meet an example of the subject-matter of par. 3.4, because $z_{ik}^{(a)}$ is a derived variate:

$$z_{ik}^{(a)} = e^{\ln z_{ik}^{(a)}} = e^{\ln z_{ij}^{(a)} + \Pi_{jik}} \dots \dots \dots (4.2.22)$$

Suppose now that we can ignore the probability distribution of $z_{ij}^{(a)}$:

$$z_{ij}^{(a)} \rightarrow z_{ij}^{(a)} \dots \dots \dots (4.2.23)$$

then it follows from (4.2.21) with (4.2.23) that:

$$E\{\ln z_{ik}^{(a)}\} \equiv \ln \tilde{z}_{ik}^{(a)} = \ln z_{ij}^{(a)} + \ln \tilde{v}_{jik} + \tilde{a}_{jik} \dots \dots \dots (4.2.24)$$

From (4.2.22) with (4.2.24) it follows that:

$$z_{ik}^{(a)} = e^{\ln z_{ij}^{(a)} + \ln v_{jik} + i a_{jik}} = e^{\ln \tilde{z}_{ik}^{(a)} + \{\ln v_{jik} - \ln \tilde{v}_{jik}\} + i\{a_{jik} - \tilde{a}_{jik}\}}$$

With the expansion:

$$e^\omega = 1 + \omega + \frac{1}{2}\omega^2 + \dots$$

one obtains for the expectation under (4.2.23):

$$E\{z_{ik}^{(a)}\} = e^{\ln \tilde{z}_{ik}^{(a)}} [1 + \frac{1}{2}\{\sigma_{\ln v_{jik}}^2 - \sigma_{a_{jik}}^2\} + i \sigma_{\ln v_{jik}, a_{jik}}] \dots \dots \dots (4.2.25')$$

In general we consequently have:

$$E\{z_{ik}^{(a)}\} \neq e^{E\{\ln z_{ik}^{(a)}\}} \dots \dots \dots (4.2.25'')$$

Up to and including second order terms, this bias is absent if:

$\sigma_{\ln v_{jik}}^2 = \sigma_{\alpha_{jik}}^2$	$\sigma_{\ln v_{jik}, \alpha_{jik}} = 0$ (4.2.26')
$E\{z_{ik}^{(a)} (4.2.23)\} = e^{E\{\ln z_{ik}^{(a)}\}}$		

(4.2.26') can be interpreted as being generated by the covariance matrix:

cov.	$\ln d_{ij}$	$\ln d_{ik}$	r_{ij}	r_{ik} (4.2.26'')
$\ln d_{ij}$	σ^2	0	0	0	
$\ln d_{ik}$	0	σ^2	0	0	
r_{ij}	0	0	σ^2	0	
r_{ik}	0	0	0	σ^2	

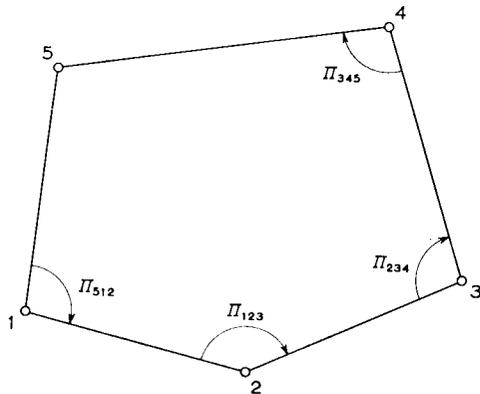


Fig. 4.2-1

Let us now look at Fig 4.2-1. The mathematical model requires the closure of the polygon

$$0 = z_{51} + z_{12} + z_{23} + z_{34} + z_{45} \dots \dots \dots (4.2.27)$$

Start from $z_{15}^{(a)}$ without probability distribution and apply (4.2.22), with successively:

j	i	k	Π (4.2.28)
5	1	2	Π_{512}	
1	2	3	Π_{123}	
2	3	4	Π_{234}	
3	4	5	Π_{345}	

Then from (4.2.27) follows the complex misclosure-variate:

$$\begin{aligned} \eta &= \frac{1}{z_{51}^{(a)}} \{z_{51}^{(a)} + z_{12}^{(a)} + z_{23}^{(a)} + z_{34}^{(a)} + z_{45}^{(a)}\} = \\ &= 1 - e^{U_{512}} \{1 - e^{U_{123}} \{1 - e^{U_{234}} \{1 - e^{U_{345}}\} \dots\} \dots\} \dots \dots \dots (4.2.29) \end{aligned}$$

If \underline{U}_{451} has also been measured, then one can establish in a similar way e.g.:

$$\eta' = 1 - e^{U_{123}} \{1 - e^{U_{234}} \{1 - e^{U_{345}} \{1 - e^{U_{451}}\} \dots\} \dots\} \dots \dots \dots (4.2.30)$$

(4.2.30) may also be replaced by:

$$\eta'' = U_{512} + U_{123} + U_{234} + U_{345} + U_{451} - 5 \ln(-1) \dots \dots \dots (4.2.31)$$

because (4.2.31) turns out to be dependent on (4.2.29) and (4.2.30). In all these relationships one must carefully take account of the multi-valuedness of complex quantities.

From (4.2.25) it follows that:

$$E\{\eta\} \neq 0; \quad E\{\eta'\} \neq 0; \quad E\{\eta''\} = 0 \dots \dots \dots (4.2.32)$$

which is again an illustration of what was said in par. 3.4.

(4.2.29) and, e.g., (4.2.31) may be thought to reflect "laws of nature" which can be verified by means of an experiment. Putting $\tilde{U} = \ln v + i \tilde{a}$:

$$\left[\begin{array}{l} 0 = 1 - e^{\tilde{U}_{512}} \{1 - e^{\tilde{U}_{123}} \{1 - e^{\tilde{U}_{234}} \{1 - e^{\tilde{U}_{345}}\} \dots\} \dots\} \dots\} \\ 0 = \tilde{U}_{512} + \tilde{U}_{123} + \tilde{U}_{234} + \tilde{U}_{345} + \tilde{U}_{451} - 5 \ln(-1) \end{array} \right] \dots \dots \dots (4.2.33)$$

The computation of $z_{ik}^{(a)}$, on the contrary, makes use of derived functions.

Naturally, all these considerations are only valid within the framework of the function model chosen, which is here the two-dimensional Euclidean geometry.

In the situation (4.2.26) one obtains instead of (4.2.32):

$$(4.2.26) \quad \left| E\{\eta\} = E\{\eta'\} = E\{\eta''\} = 0 \right| \dots \dots \dots (4.2.34)$$

By the exposition given, the "Polygon theory in the complex plane" is essentially established; the name chosen is characterized by figure 4.2-1. The paper by KRIJGER [1966] deals with computational aspects of the theory (the notation is here chosen in agreement with the one which is customary in the Netherlands). A complete elaboration of the theory would of course require more space than was available in these papers.

Some very remarkable stochastic aspects, which are important for the planning of geodetic networks, are indicated in [BAARDA 1962]. In this publication the point of departure is given by mathematical relationships in the two-dimensional Euclidean geometry. After certain transformations of these relationships and applying the tilde sign to certain quantities, we arrive at the same "laws of nature" and other relationships as were given in the preceding pages. However, the line of thought given here has been directive for the development in the previous paper.

The line of thought developed in linking-up the chosen mathematical model demonstrates that this model is required to be invariant with respect to a similarity

transformation. This implies that the choice of the classical ellipsoidal computation model (a two-dimensional curved space) must in principle be deemed incorrect. For in this model, with a reasonable approximation, a rotation is permissible, but *not* a change in the scale of distances. This explains the inconsistency of this computation model in combination with the measuring procedure, which inconsistency was pointed out at the end of par. 3.5. Probably the effect of this will only be appreciable in geodetic networks over distances larger than 1000 km, so that it can be ignored in smaller networks.

The establishment of “laws of nature” in the form of relations between dimensionless quantities, as illustrated by (4.2.33), seems to be a *characteristic* of this development of the theory of linking-up a mathematical model.

This provides a connection with the *dimensional analysis* of mathematical physics. Many conclusions of this analysis can now again be used as a directive for further developments. On the other hand the theory developed in the foregoing appears to lead to sharper conclusions on some particular points.

4.3 Further considerations

From experience it has already for a long time been known that the two-dimensional theory as developed in par. 4.2 leads to insufficient realization of predictions in more extensive geodetic networks. After the intermediate phase of the now already classical ellipsoidal geometric and gravimetric geodesy, an increasing number of theories are being advanced which are based on a truly three-dimensional space. The speculative element in the construction of these theories increases accordingly, because the actual execution of conclusive experiments with geodetic measurements stretching over extensive areas of the earth, is practically and economically almost impossible.

Every scientist in this field therefore chooses again directives for a further development of theories.

From a theoretical point of view it is interesting that the line of thought developed in par. 4.2 can be generalized to a spatial polygon theory, in which the model is the three-dimensional Euclidean space. The place of complex numbers is then taken by quaternions; the algebraic elaboration, however, is considerably complicated by the non-commutativity of the latter quantities. There are also difficulties in the interpretation of the quaternion-equations.

Although it proved possible to bring methods of geodetic astronomy, photogrammetry and satellite geodesy (Väisälä methods) under this theory, the problems of the linking-up of the mathematical model are very difficult. The knowledge required exceeds in different respects the limits of geodesy proper, so that such a theory can only be considered as a frame which must be complemented or extended by experts. For example, the application of methods of geodetic astronomy and satellite geodesy involves the difficulty of the interpretation of stellar coordinates, of the direction of the rotational axis of the earth, etc.

To eliminate from the theory the (physical) vertical in each theodolite station, is as impossible as to eliminate the direction of the rotational axis of the earth. These

verticals provide the connection between the previously described geometric theory and the methods of gravimetric geodesy; they indicate the direction of the acceleration of gravity in distinct points on the earth, the magnitude of these acceleration vectors being measured by gravimeters. Whereas by methods of geodetic astronomy the directions of the vertical in different stations can be interconnected with a reasonable accuracy, the relative positions of points on the earth in the direction perpendicular to the surface of the earth are not reliably determined by terrestrial theodolite measurements because of the influence of refraction on vertical angle measurements. It remains to be seen if methods of satellite geodesy will give a great improvement in this respect.

It is an interesting experiment now to transform tentatively the mathematical model theory of gravimetric geodesy, as developed by others, according to the directives of par. 2.4. Then indeed this theory proves in principle to fill this weak spot of geometric geodesy. Beside dimensionless quantities like distance ratios, angles and ratios of gravity in distinct points, there appear in the transformed integral equations a number of dimensionless coefficients or numbers, in which also the results of spirit levelling find their place. In these integral equations which can be interpreted as “laws of nature”, quantities like the mass of the earth, “geocentric” coordinates and potential do not appear as essential quantities; they can be considered as derived quantities.

Of course the mathematical analysis and the connection with classical methods remain as difficult as they are in other approaches to this field. Consequently the mathematical treatment need not be original, since the primary goal aimed at is to increase the possibility of a sharper insight based on better interpretability.

Whereas for the questions so far discussed a reasonable connection could be established between the measuring procedure and mathematical theory, time t is a quantity which is much more difficult to understand.

Thinking of modern methods of distance measurement, let t_{ij} be a time interval and v , e.g., the velocity of light in some medium, then the following mathematical relationship exists:

$$s_{ij} = v \cdot t_{ij} \dots \dots \dots (4.3.1)$$

If we follow the line of thought of par. 4.2, then the question can be posed which of the following relationships must be used:

$$\underline{\ln s_{ij}^{(i)}} = \underline{\ln v} + \underline{\ln t_{ij}^{(i)}} \dots \dots \dots (4.3.2)$$

or:

$$\underline{\ln s_{ij}^{(i)}} = \underline{\ln v^{(i)}} + \underline{\ln t_{ij}} \dots \dots \dots (4.3.3)$$

or:

$$\underline{\ln s_{ij}^{(i)}} = \underline{\ln v^{(i)}} + \underline{\ln t_{ij}^{(i)}} \dots \dots \dots (4.3.4)$$

Or to put it in another way: when is it sufficient to use local time, when do we need „absolute” time?

And then: where do ratios of time intervals occur in geodetic astronomy?

The conclusion is that the line of thought developed in chapter 4 for the time being raises more questions than it answers.

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After this publication was set in type the following reference related to the problems mentioned in (2.4.6) – (2.4.12) and (3.2.10) – (3.2.15) came to the author's knowledge:

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APPENDIX

A GENERALIZATION OF THE CONCEPT
STRENGTH OF FIGURE

by W. BAARDA

REPORT SUBMITTED TO THE MEMBERS OF
I.A.G. SPECIAL STUDY GROUP No. 1 : 14
1962

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A GENERALIZATION OF THE CONCEPT STRENGTH OF FIGURE

1 Introduction

One practical problem in geodesy is the measurement of a geodetic network, resulting in the numerical values of the observations and their registration, i.e. the description of the procedure and the circumstances of measurement [BAARDA 1960, 62].

A second problem in geodesy, even more important from a practical point of view because of the financial and other consequences, is the question, *what, how and how much* has to be measured. In answering this question one will first of all have to investigate what use will be made of the network; this includes in particular the use by those who are not geodesists. This means that a *purpose* has to be formulated – a very difficult task indeed.

The *prediction of the registration that can be expected* for future measurements is a fundamental part of the second problem, and it is as difficult as the formulation of the purpose.

For only by way of the registration it is possible to link up the adjustment model with the measuring process. This model consists of the probability model of the observational quantities or variates and the functional model with the condition equations; what these models express can be summarized under the name “null hypothesis”.

Experience concerning possible disturbances and mistakes in the measuring process leads here to the formulation of “alternative hypotheses” [BAARDA 1960, 62].

A good geodesist is characterized by the way he constructs a model, by well-balanced simplifications in the null hypothesis and the alternative hypotheses. He also deliberates upon the function of the observational quantities which has to be introduced as the estimator in the adjustment procedure; a least-squares estimator, a pseudo least-squares estimator, or other types of estimators, known from the theory of mathematical statistics.

When these problems have been decided upon, the probability distribution of the estimators is theoretically known, and a comparison with the precision requirements resulting from the purpose can be made. The *reliability* of this probability distribution is dependent, firstly, on the correctness of the predicted registration, and secondly on the influence of simplifications made in the null hypothesis.

But a second requirement resulting from the purpose is that, with a reasonable certainty, the test procedure connected with the adjustment shows up certain situations corresponding to alternative hypotheses. This implies that the user of the geodetic network will have no appreciable trouble resulting from disturbances and mistakes in the geodetic measurement and computing procedures. For this purpose,

extra checks must be built into the geodetic network, which in themselves would not be necessary to obtain the precision required.

The precision and the checks, together with consequences of personnel, instruments and finance, will have to be weighted in the final and total *decision process*. Here, the purpose is fundamental; therefore the geodesist can only give a partial contribution to the final decision. After this, he has the responsibility for the execution of the measurements and the computations.

2 Scope of the paper

The main subject is: how to find a basis for the comparison of probability distributions of estimators from a geodetic adjustment, with precision requirements resulting from the purpose.

Wherever in the sequel mention is made of a covariance matrix, a Laplace-Gauss probability distribution is suggested; of course this is not essential.

The theory has been developed for survey systems in the plane. This is not essential either, but it has the advantage that the theoretical model is consistent. The material treated here is only a part of a more comprehensive theory whose merits are now being investigated in Delft.

A generalization of the theory to three dimensional space has been framed, making use of certain hypercomplex numbers.

The possibility of an application to ellipsoidal computations is also being studied. The fundamental difficulty here is that research has given strong evidence that the classical model of ellipsoidal computations is not consistent.

Nevertheless it seems to be desirable to include ellipsoidal computations in the projected theory, because the spatial computations by HOTINE could not demonstrate significant differences in φ , λ coordinates with ellipsoidal computations. Against the greater complexity of spatial computations we have therefore only the advantage of a better theoretical insight in the model, but so far no practical advantage.

Conclusions can be found in section 7. The reference that is made, in the title of this paper and in section 8, to the concept "strength of figure", is not essential for the theory developed here. But it has the advantage that a connection can be made with a complex of ideas familiar to practically all geodesists.

Fundamental thoughts of the theory date from 1944. In a first stage, the theory provided a basis for the cadastral manual [BAARDA et al. 1956]. In this manual, particular attention was paid to the requirements made by the users of a network with respect to the juridical status of landed property in the Netherlands.

3 Essential quantities

Coordinates of points are not directly measurable; they are artificial and not concrete. Only the points themselves can be concretely indicated; for the characterization of the mutual position of points one must therefore try to find quantities that are invariant with respect to the arbitrary adoption of a coordinate system.

If we consider a rectangular x, y coordinate system with equal units of measurement on the axes, the only transformation admissible within Euclidean geometry is the general similarity transformation.

The arbitrary choice of a coordinate system can therefore be expressed by the introduction of a coordinate system which is not uniquely determined but can be subjected to similarity transformations.

If we introduce the complex quantity

$$z = x + iy \quad \dots \dots \dots (3.1)$$

this means that the following z' system is equally admissible:

$$z' = \gamma z + \delta \quad \dots \dots \dots (3.2)$$

in which γ and δ are complex transformation constants.

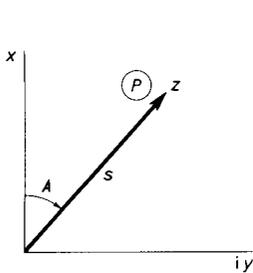


Fig. 3-1



Fig. 3-2

For the theory to be developed, the essential quantities are those that are invariant with respect to the transformation (3.2.)

Consider the points P_i, P_j, P_k, P_l and define relative coordinates or coordinate differences as follows:

$$x_{ij} = x_j - x_i$$

$$y_{ij} = y_j - y_i$$

and consequently

$$z_{ij} = x_{ij} + iy_{ij} = z_j - z_i \quad \dots \dots \dots (3.3)$$

For point coordinates we have, by (3.2):

$$\left. \begin{aligned} z'_i &= \gamma z_i + \delta \\ z'_j &= \gamma z_j + \delta \\ z'_k &= \gamma z_k + \delta \\ z'_l &= \gamma z_l + \delta \end{aligned} \right\} \dots \dots \dots (3.4)$$

and for relative coordinates, by (3.3):

$$\left. \begin{aligned} z'_{ij} &= \gamma z_{ij} \\ z'_{kl} &= \gamma z_{kl} \end{aligned} \right\} \dots \dots \dots (3.5)$$

The introduction of relative coordinates results in quantities that are invariant with respect to changes in δ , but not with respect to changes in γ .

Quantities invariant with respect to changes in both δ and γ follow from (3.5):

$$\frac{z'_{kl}}{z'_{ij}} = \frac{z_{kl}}{z_{ij}} \dots \dots \dots (3.6)$$

Let

$$\frac{z_{kl}}{z_{ij}} = e^{(A_{kl}-A_{ij})} \dots \dots \dots (3.7a)$$

with (\ln denotes the natural logarithm):

$$\left. \begin{aligned} A_{ij} &= \ln z_{ij} = \ln s_{ij} + iA_{ij} \\ A_{kl} &= \ln z_{kl} = \ln s_{kl} + iA_{kl} \end{aligned} \right\} \dots \dots \dots (3.7b)$$

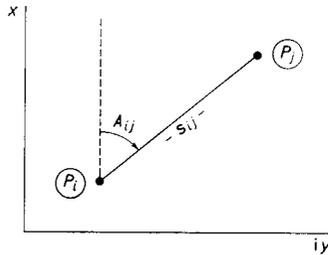


Fig. 3-3

Separating the real and imaginary parts results in:

$$\left. \begin{aligned} R\{A_{kl}-A_{ij}\} &= \ln s_{kl} - \ln s_{ij} = \ln \frac{s_{kl}}{s_{ij}} \\ I\{A_{kl}-A_{ij}\} &= A_{kl}-A_{ij} \end{aligned} \right\} \dots \dots \dots (3.7c)$$

It will turn out to be more important to consider (3.7) for three points instead of four, and we will consequently introduce a new symbol for this case.

For the points P_i, P_j and P_i, P_k , (3.7) results in the quantity:

$$e^{\Pi_{jik}} = e^{(A_{ik}-A_{ij})} = \frac{z_{ik}}{z_{ij}} \dots \dots \dots (3.8a)$$

with:

$$\Pi_{jik} = \ln \frac{z_{ik}}{z_{ij}} = \ln \frac{s_{ik}}{s_{ij}} + i(A_{ik}-A_{ij}) \dots \dots \dots (3.8b)$$

If we put

$$\left. \begin{aligned} v_{jik} &= \frac{s_{ik}}{s_{ij}} \\ \alpha_{jik} &= A_{ik}-A_{ij} \end{aligned} \right\} \dots \dots \dots (3.8c)$$

we obtain:

$$\boxed{\Pi_{jik} = \ln v_{jik} + i \alpha_{jik}} \dots \dots \dots (3.8d)$$

In the complex plane we can represent Π_{jik} in the same way as α_{jik} in the real plane, i.e. *with* the direction of rotation. The positive direction is chosen as in Fig. 3-1 and Fig. 3-3, i.e. clockwise. See Fig. 3-4.

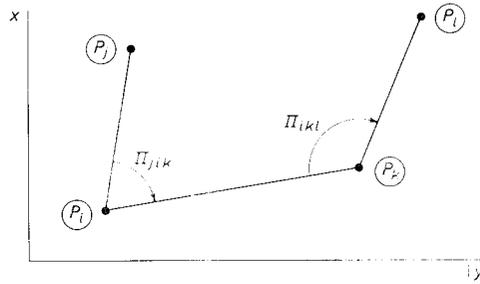


Fig. 3-4

The quantities $(A_{kl}-A_{ij})$, which are in a certain sense more general, can be constructed from the II -quantities, see Fig. 3-4. From (3.7) we have:

$$A_{kl}-A_{ij} = \ln z_{kl} - \ln z_{ij} = \ln z_{kl} - \ln z_{ki} + \ln z_{ik} + \ln(-1) - \ln z_{ij}$$

Or, with (3.8) and with $\ln(-1) = i\pi$:

$$\boxed{(A_{kl}-A_{ij}) = II_{jik} + II_{ikl} + \ln(-1)} \dots \dots \dots (3.9a)$$

$$e^{A_{kl}-A_{ij}} = -e^{II_{jik} + II_{ikl}} \dots \dots \dots (3.9b)$$

If we compare A and II in the complex plane with, respectively, the bearing A and the angle α in the real plane, (3.9a) illustrates the way in which calculations can be made with the quantities A and II . Of course this parallelism is only partially valid.

4 The mutual position of points

If we leave out of consideration translations and changes in scale and orientation, the mutual position of three points is entirely determined by one II -quantity.

A third point is thereby fixed with respect to two other points, which we shall call base points. For the latter we choose the points P_i and P_j , and (3.8a) gives us

$$\boxed{z_{ik} = z_{ij}e^{II_{jik}}} \dots \dots \dots (4.1)$$

by which P_k is fixed with respect to P_i, P_j .

If we consider a system of more than three points, this relation is valid when k runs through the sequence of point numbers other than i and j .

For a system formed by n points we need consequently $(n-2)$ II -quantities in order to determine the mutual positions.

However, we are free to use another way of expressing the relative positions, by introducing other II -quantities.

Let us first consider four points, as indicated in Fig. 3-4. Then it follows from (3.7a):

$$z_{kl} = z_{ij}e^{(A_{kl}-A_{ij})} \dots \dots \dots (4.2a)$$

or, with (3.9b):

$$\boxed{z_{kl} = -z_{ij}e^{II_{jik}+II_{ikl}}} \dots \dots \dots (4.2b)$$

Besides, we have:

$$\boxed{z_{il} = z_{ik} + z_{kl}} \dots \dots \dots (4.3)$$

so that from (4.1), (4.2) and (4.3) it follows that:

$$z_{il} = z_{ij}(e^{H_{jik}} - e^{H_{jik} + H_{ikl}}) \dots \dots \dots (4.4)$$

Simplification of (4.4) leads back to the form of (4.1) because:

$$z_{il} = z_{ij}e^{H_{jik}}(1 - e^{H_{ikl}}) = z_{ij}e^{H_{jik} - H_{ikl}} = z_{ij}e^{H_{jil}}$$

The technique of such reductions will not be further considered here, since it is not essential for the problem to be formulated.

According to (4.1), (4.2b) and (4.3), the mutual position of P_i , P_j and P_k , P_l are now in principle determined by the quantities

$$H_{jik} \text{ and } H_{ikl}$$

(4.1) and (4.2b) illustrate furthermore the necessity of the introduction of a pair of base points if a calculation in coordinates is to be made.

Adoption of z_i and z_j , or of z_i and z_{ij} determines the two translation constants, the scale constant and the orientation constant, since

$$\left. \begin{aligned} z_i &= x_i + iy_i \\ \ln z_{ij} &= \ln s_{ij} + iA_{ij} = A_{ij} \end{aligned} \right\} \dots \dots \dots (4.5)$$

5 Differential relations *)

Since the purpose of our study is an analysis of the probability model of computed coordinates, we will deduct in this section the necessary differential relations. At the same time we obtain the linearization of the relationships, which is extremely valuable for obtaining an insight in and a survey of the results.

From (3.1): $dz_i = dx_i + idy_i \dots \dots \dots (5.1)$

From (3.3): $dz_{ij} = dx_{ij} + idy_{ij} = dz_j - dz_i \dots \dots \dots (5.2)$

From (3.7b) $dA_{ij} = \frac{1}{z_{ij}} dz_{ij} \dots \dots \dots (5.3a)$

$$dz_{ij} = z_{ij} dA_{ij} \dots \dots \dots (5.3b)$$

$$dA_{ij} = d \ln s_{ij} + i dA_{ij} = dA_{ji} \dots \dots \dots (5.3c)$$

From (3.7a): $d(A_{kl} - A_{ij}) = \frac{1}{z_{kl}} dz_{kl} - \frac{1}{z_{ij}} dz_{ij} \dots \dots \dots (5.4)$

From (3.8a): $dH_{jik} = d(A_{ik} - A_{ij}) \dots \dots \dots (5.5a)$

$$dH_{jik} = \frac{1}{z_{ik}} dz_{ik} - \frac{1}{z_{ij}} dz_{ij} \dots \dots \dots (5.5b)$$

*) All relations may be replaced by difference relations, the approximation error is practically negligible.

From (3.8d): $dII_{jik} = d \ln v_{jik} + i d\alpha_{jik} \dots \dots \dots (5.5c)$

From (3.9a): $d(\Lambda_{kl} - \Lambda_{ij}) = dII_{jik} + dII_{ikl} \dots \dots \dots (5.6)$

Taking logarithms in (4.1) and (4.2) results, with (3.7b) in (compare also (3.8a) and (3.9b)):

$$\Lambda_{ik} = \Lambda_{ij} + II_{jik}$$

$$\Lambda_{kl} = \ln(-1) + \Lambda_{ij} + II_{jik} + II_{ikl}$$

Or, with (5.3a), (5.5a), (3.7a) and (3.8a), we have on account of (5.6):

From (4.1):	$dz_{ik} = e^{II_{jik}} dz_{ij} + z_{ik} dII_{jik}$	(5.7)
-------------	---	-------

From (4.2):	$dz_{kl} = e^{(\Lambda_{kl} - \Lambda_{ij})} dz_{ij} + z_{kl} (dII_{jik} + dII_{ikl})$	(5.8)
	$= e^{(\Lambda_{kl} - \Lambda_{ij})} dz_{ij} + z_{kl} d(\Lambda_{kl} - \Lambda_{ij})$	

From (4.3):	$dz_{il} = dz_{ik} + dz_{kl}$	(5.9)
-------------	-------------------------------	-------

In this section, all differential quantities will be considered as stochastic quantities (variates). In the following sections variates will be characterized by an underscore. A superscript 0 indicates, on the contrary, a quantity for which an adopted value can be inserted.

A rough approximation suffices for the numerical values of the coefficients of differential quantities.

6 The significance of the choice of computational base points for the computation of variances of coordinates

Suppose that a geodetic network (triangulation, trilateration or traverse net) has been adjusted and computed in coordinates, starting from two computational base points P_1, P_2 , for which coordinates have been adopted, i.e.:

	z_1^0, z_2^0	}		(6.1)
or	z_1^0, z_{12}^0			
or	z_1^0, Λ_{12}^0			

One can then, in principle, assume too that Λ - and II -quantities have been computed, e.g. by (3.7b) and (3.8d).

Consider now two arbitrary points of the net P_i, P_j . In (5.7), (5.8) and (5.9) we replace

i by 1	}		(6.2a)
j by 2			
k by i			
l by j			

For the application of the law of propagation of variances we put, in view of (6.1):

$dz_1^0 = 0$	}		(6.2b)
$dz_{12}^0 = 0$			

then it follows from (5.2):

$$\begin{aligned} \underset{12}{d} z_{1i} &= \underset{12}{d} z_i \\ \boxed{\underset{12}{d} z_i} &= z_{1i} \underset{12}{d} \Pi_{21i} \dots \dots \dots (6.3a) \\ \boxed{\underset{12}{d} z_{ij}} &= z_{ij} \underset{12}{d} (A_{ij} - A_{12}) \dots \dots \dots (6.3b) \\ \boxed{\underset{12}{d} z_j} &= \underset{12}{d} z_i + \underset{12}{d} z_{ij} \dots \dots \dots (6.3c) \end{aligned}$$

in which for d has been written $\underset{12}{d}$ in order to indicate that the computations have been made in the system which is based on the base points P_1, P_2 .

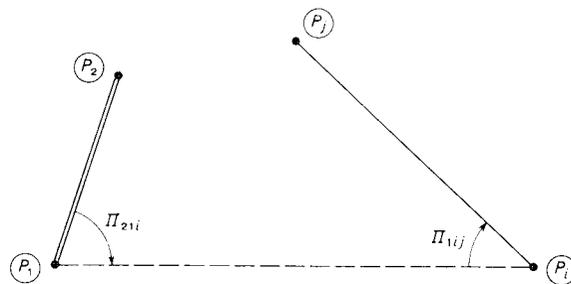


Fig. 6-1

The formulation of (6.3) has been chosen so as to make possible a comparison with existing literature on the subject.

The application of the law of propagation of variances on (6.3) furnishes then expressions for the precision of the position of P_i and P_j with respect to P_1, P_2 – which in the sequel we shall call “*point-precision*” – and for the precision of the relative position of P_i, P_j , again with respect to P_1, P_2 – which we shall call “*relative precision*”.

From American geodetic literature ideas have spread about a part of these concepts of precision, namely on what is called the “*strength of figure*”. This concept concerns only a small part of (6.3), namely:

$$\underset{12}{d}(A_{ij} - A_{12}) = \underset{12}{d}(\ln s_{ij} - \ln s_{12}) + i \underset{12}{d}(A_{ij} - A_{12}) \dots \dots \dots (6.4)$$

In many respects, the point precision is much more important than the relative precision. Therefore, considerations on the “*strength of figure*” should also be concerned with (6.3a):

$$\underset{12}{d} \Pi_{21i} \dots \dots \dots (6.5)$$

Consider now two other points P_r, P_t .

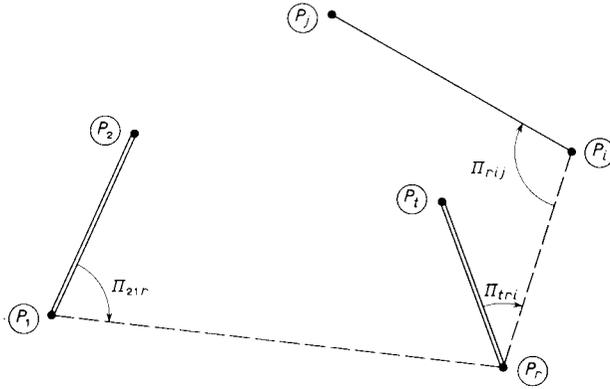


Fig. 6-2

Then it follows:

$$\text{from (5.7): } \quad d z_{ri} = e^{I_{tri}} d z_{rt} + z_{ri} d I_{tri} \quad \dots \quad (6.6a)$$

$$\text{from (5.8): } \quad d z_{ij} = e^{(A_{ij}-A_{rt})} d z_{rt} + z_{ij} d(A_{ij}-A_{rt}) \quad \dots \quad (6.6b)$$

Or, in view of (5.2) and (6.3b), and with

$$d z_{ri} = d z_i - d z_r$$

$$d z_{rt} = z_{rt} d(A_{rt}-A_{12})$$

it follows that:

$$\text{from (6.6a): } \quad d z_i = z_{ri} d I_{tri} + \left\{ d z_r + z_{ri} d(A_{rt}-A_{12}) \right\} \quad \dots \quad (6.7a)$$

$$\text{from (6.6b): } \quad d z_{ij} = z_{ij} d(A_{ij}-A_{rt}) + z_{ij} d(A_{rt}-A_{12}) \quad \dots \quad (6.7b)$$

If, for the computation of variances of coordinates, one would have started from the computational base points P_r, P_t , see Fig. 6-2, with

$$\text{or } \left. \begin{matrix} z_r^0, z_t^0 \\ z_r^0, z_{rt}^0 \\ z_r^0, A_{rt}^0 \end{matrix} \right\} \quad \dots \quad (6.8)$$

so that as far as the stochastic properties are concerned one would have worked in a transformed coordinate system, then we would have according to (6.3):

$$d z_i = z_{ri} d I_{tri} \quad \dots \quad (6.9a)$$

$$d z_{ij} = z_{ij} d(A_{ij}-A_{rt}) \quad \dots \quad (6.9b)$$

Substituting (6.7) into (6.9) gives finally:

$$\frac{d z_i}{rt} = \frac{d z_i}{12} - \left\{ \frac{d z_r + z_{ri} d(A_{rt} - A_{21})}{12} \right\} \dots \dots \dots (6.10a)$$

$$\frac{d z_{ij}}{rt} = \frac{d z_{ij}}{12} - \{ z_{ij} d(A_{rt} - A_{21}) \} \dots \dots \dots (6.10b)$$

If we consider (6.10) as the base for the application of the law of propagation of variances, then (6.10) expresses in symbolic form that the concepts *point precision* and *relative precision* are dependent on the choice of a computational base. Therefore, these concepts are unsuitable for the formulation of requirements concerning the precision of geodetic networks.

7 The fundamental problem of discussions on the precision of geodetic networks

The negative conclusion at the end of section 6 can easily be made into a positive one.

Let us for this purpose continue the subject of sections 3 and 4 by considering four points P_i, P_j, P_k and P_l .

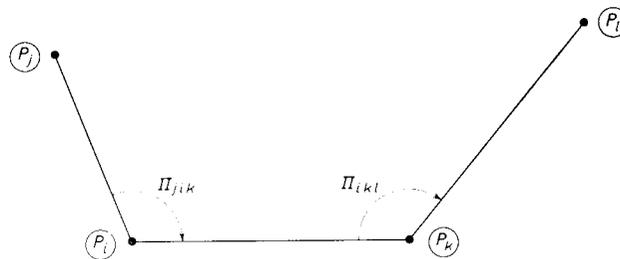


Fig. 7-1

We introduce P_i, P_j as computational base points with

$$z_i^0 = \text{known } z_i$$

$$z_j^0 = \text{known } z_j$$

By a change of indices we have from (6.3) :

$$\frac{d z_k}{ij} = z_{ik} d II_{jik} \dots \dots \dots (7.1a)$$

$$\frac{d z_{kl}}{ij} = z_{kl} d(A_{kl} - A_{ij}) = z_{kl} (d II_{jik} + d II_{ikl}) \dots \dots \dots (7.1b)$$

From (7.1) it follows, entirely in agreement with section 4, that formulations concerning the precision of the mutual positions of the points considered are possible if the covariance matrix of the quantities

$$II_{jik} \text{ and } II_{ikl}$$

is known.

Conversely, one will no doubt be able to deduct this covariance matrix from the relations resulting from (7.1):

$$\left. \begin{aligned} d\Pi_{jik} &= \frac{1}{z_{ik} \ ij} d z_k \\ d\Pi_{jik} + d\Pi_{ikl} &= \frac{1}{z_{kl} \ ij} d z_{kl} \end{aligned} \right\} \dots \dots \dots (7.2)$$

The *fundamental problem* is caused by the fact that, in general, no precise information on the covariance matrix of the quantities z is available, and that consequently one has to manage with an appraisal of this matrix (not an estimate in the statistical sense). This means that an artificial covariance matrix must be constructed. The solution of this problem will not be easy, for on the one hand a matrix will have to be constructed which is positive definite, and, on the other hand, simple and practically manageable rules for establishing such a matrix will have to be given.

From (7.2) it follows that the artificial covariance matrix to be established will have to be deduced in the first instance from an analysis of the precision of geodetic networks. This is understandable, because the attainable lower limit for variances will always be determined by the geodetic possibilities.

This looks suspiciously like a reversal of the problem, because the purpose of the establishment of the covariance matrix involved is to formulate rules for a geodetic network whose precision fulfills certain requirements.

However, from a more or less idealized form of the geodetic network we can deduct the covariance matrix, and use the thus obtained matrix for testing the precision of nets that are more adopted to actual practice.

Whereas the lower limit of variances is determined by geodetic possibilities, an upper limit will have to be deduced from requirements finding their source in essentially non-geodetic practical problems. Perhaps the lower limit will also have to be made dependent on these requirements, if this is possible.

One of the very important *tasks* for the members of Special Studygroup No. 1 : 14 will therefore be an investigation into and an analysis of the, mostly non-geodetic, uses to which geodetic coordinates are put. In this way one arrives at a formulation of the purpose of geodetic networks; in this formulation the concept of precision will be of paramount importance.

A clear illustration of the significance of Π -quantities is the *setting out of technical projects*. When the measures of these have been computed in coordinates, the different corner points must be set out in the terrain. If in the terrain we have the points P_i and P_j , already known in coordinates, we may obtain any other point P_k by setting out the quantities:

$$\Pi_{jik}$$

Another application is possible if $P_i, P_j, P_k,$ and P_l are corners of a technical project, in which case (7.1) can be used to develop a study on the precision.

If one is concerned with a project where certain constructional units are built in another place and later moved to the site, and where consequently the scale of the

distances is important, one must add to (7.1) the effect of a scale difference between the units of distance measurement of survey and of construction. In the set-up developed here, this proves to be possible.

Another problem presents itself in *navigation*, such as navigation with Decca-equipment

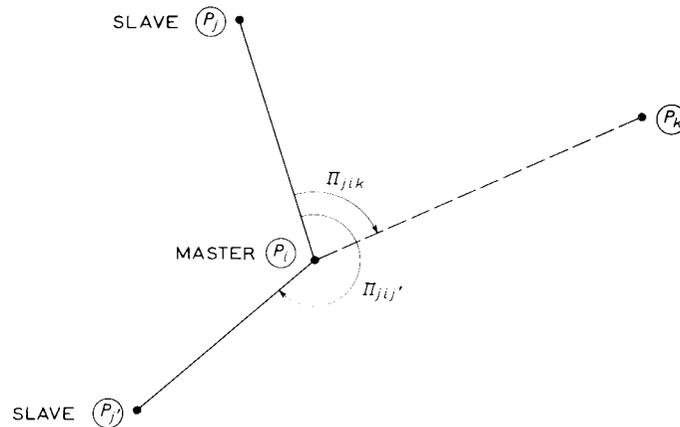


Fig. 7-2

If, in Fig. 7-2, P_k is an arbitrary point in the Decca pattern, the precision of

$$\underline{\Pi}_{ji'}$$
 and $\underline{\Pi}_{jik}$

is important for the navigation.

The quantity $\underline{\Pi}_{kij'}$ is not independent, because

$$\underline{\Pi}_{kij'} = \underline{\Pi}_{ji'} - \underline{\Pi}_{jik}$$

The precision of the Decca pattern is determined by the coordinate precision of P_i , P_j , $P_{j'}$, i.e. by

$$\underline{\Pi}_{ji'}$$

Then $\underline{\Pi}_{jik}$ determines the possibilities of navigation in P_k .

Analogous problems are met in aerial navigation, in the determination of direction and distance for radio- and television communications, etc. One may have to deal with distances of hundreds of kilometres, but the points already known in coordinates may be only 20 km apart on account of the possibilities of direct vision.

Of special importance are navigational problems in the overlapping areas of two or more different navigation systems. In Fig. 7-3 is given a situation with two Decca systems.

The terrain point P_m is now determined by:

$$\underline{\Pi}_{jim} \text{ in system I}$$

$$\underline{\Pi}_{ikm} \text{ in system II}$$

In this case it is essential for a sufficient identity in the determination of P_m that one obtains a reasonable precision in

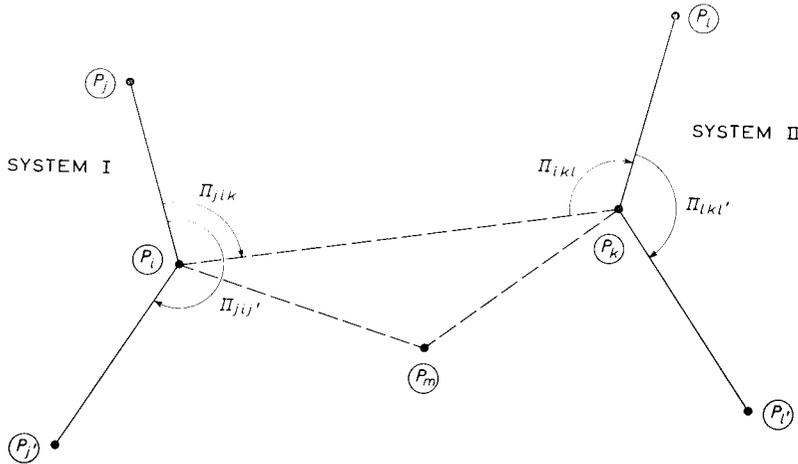


Fig. 7-3

$\Pi_{jij'}$ for system I

$\Pi_{ikl'}$ for system II

Π_{jik} and Π_{ikl} for the mutual consistency of the systems I and II.

8 Strength of figure

From the preceding – see in particular (6.4) and the related text – it follows, that for a figure formed by n points the customary concept “strength of figure” can be generalized by establishing a theoretical limit for the covariance matrix of $(n-2)$ functionally independent Π -quantities.

The expression “strength of figure”, however, may cause confusion; it would be better to speak about “precision of figure”. For the formulas are only related to the probability distribution of a group of coordinates.

An entirely different approach to the concept “strength of figure” is made if one requires the network to be such that it gives certain guarantees concerning the signalization of errors made in the measurement or the computation.

From geodetic literature, the paper by REICHENEDER [1941] must be honorably mentioned here. His considerations are, however, limited to a very simple case of a geodetic network and they cannot be generalized.

A general theory can be established with the aid of the power function of a statistical test. For this subject, reference is made to [BAARDA 1960, 62] and for some geodetic applications to [BAARDA 1960] and [BAARDA et al. 1956]. If this aspect is taken account of, one could speak of “accuracy of figure” in the sense of EISENHART [EISENHART 1952].

From a theoretical as well as from a practical point of view it would be best if both “precision of figure” and “accuracy of figure” could be treated together; both theories are intimately connected.

The combination might then result in rules concerning a strongly generalized concept “strength of figure”.

Personally, I am convinced that only then the “specifications for fundamental geodetic networks” will have actual significance for geodetic practice. Some traces are indeed found in the provisional specifications of Special Studygroup No. 1:14 adopted at Helsinki 1960.

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