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STEREO NOMOGRAMS

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STEREO NOMOGRAMS

When one has occupied oneself intensively with the construction of nomograms, the wish arises to extend the arsenal of nomographic methods and to look for possibilities for the construction of spatial nomograms. In such a nomogram, that must be a representation of the relation $F(abcd) = 0$, the four values of the variables which belong together must lie in a flat plane.

In [3] I discussed this possibility already and, for instance, SOREAU too gives the principle of spatial nomograms in [4], though very summarily and without going into the problem of the realization.

As far as I know LACMANN [1] was the first who occupied himself more fundamentally with the problem, whilst, in a very readable paper [2], SUTOR tried to propagate these nomograms. As it is impossible to give a spatial model to a great number of users, he projected this model from two centres of projection on a flat plane in the complementary colours red and blue. When this image, a stereo nomogram, is viewed through red-blue spectacles, the left eye, armed with the red glass, sees only the right (blue) image. The right eye with the blue glass sees only the left (red) image. The two images together give a spatial representation of the model from which one has started or, dependent on the place from which the drawing on the flat plane is viewed, a collinear transformation of this model. By this transformation a flat plane remains a flat plane. In the spatial model, artificially produced by the applied anaglyphic method, the four variables belonging together will remain in a flat plane, independent of the place in the space of the two eye-centres.

Already about ten years ago I designed some stereo nomograms according to the directives given above and I tried to make reliable readings in these nomograms with a similar real reading plane as SUTOR describes on page 169 of his paper.

Not only for me, but for all the persons co-operating at that time in the tests, these readings led to unsatisfactory results because it must be deemed impossible to bring the real reading plane through the scale points of the three independent variables and *to read at the same time and with the same attitude of the head* [2, page 171] the place where the scale of the dependent variable intersects the reading plane.

I stopped then my experiments and I resumed them not before I found a more satisfactory method for the reading of the dependent variable.

As will appear presently from some examples, this method makes no use of a real reading plane but of two pairs of red and blue reading lines on thin, transparent material. They are laid on the stereo nomogram in such a way that, seen through the red and blue reading spectacles, they give a spatial image of the two intersecting diagonals of the quadrangle formed by the scale points of the four variables which belong together.

If a spatial (stereo) nomogram can be made from the relation

$$F(abcd) = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

between four variables, this relation can be written into the form

$$\begin{vmatrix} X_A & Y_A & Z_A & 1 \\ X_B & Y_B & Z_B & 1 \\ X_C & Y_C & Z_C & 1 \\ X_D & Y_D & Z_D & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

The elements X_A up to and including Z_A on the first row of this determinant are functions of the first variable a, X_B, Y_B and Z_B on the second row of the second variable b, etc. For (2) is the condition which the co-ordinates X_A up to and including Z_D must satisfy if the points A up to and including D lie in a flat plane.

It is obvious that, among others, this manner of writing is always possible when from (1) a flat nomogram can be made with a straight turning axis. This turning axis for the auxiliary variable h couples the nomogram for the relation $F_1(abh) = 0$ to that for $F_2(cdh) = 0$. Together they form a nomogram for $F(abcd) = 0$.

When (see Fig. 1) the nomogram for $F_2(cdh) = 0$ is revolved round the h-scale (Y-axis), then a spatial nomogram is obtained in which the points A, B, C and D are coplanar.

In order to bring (1) into the desired form (2) the elimination-method is used by preference, that is to say, *one tries, after having introduced three auxiliary variables* ξ, η *and* ζ, *to split up the original equation* (1) *with four variables into four equations each with one variable from* (1) *and each linear in* ξ, η *and* ζ.

This method will be applied on the simple formula $\frac{a}{b} = \frac{c}{d}$ for which a nomogram with a straight turning axis can be constructed in a flat plane.

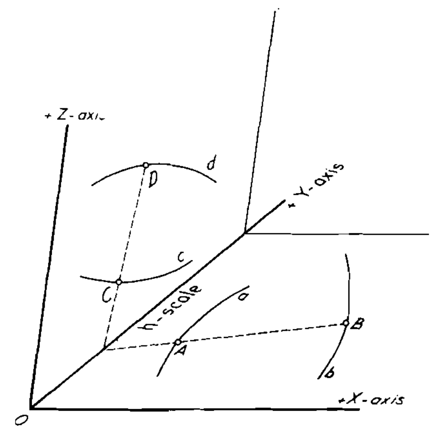

Figure 1.

If: $a = \xi$, $\frac{a}{b} = \eta = \frac{c}{d}$ and $c = \zeta$, then one obtains

$$\begin{aligned} \xi \qquad\quad -a &= 0 \\ \xi - b\eta \qquad\quad &= 0 \\ +\zeta - c &= 0 \\ -d\eta + \zeta \qquad &= 0 \end{aligned}$$

so that

$$\begin{vmatrix} 1 & 0 & 0 & -a \\ 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & -d & 1 & 0 \end{vmatrix} = 0 = \begin{vmatrix} 0 & a & 0 & 1 \\ b & 0 & 0 & 1 \\ 0 & c & 1 & 0 \\ d & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & 0 & 1 \\ b & 0 & 0 & 1 \\ 0 & c & 1 & 1 \\ d & 0 & 1 & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & \mu_1 a & 0 & 1 \\ \mu_2 b & 0 & 0 & 1 \\ 0 & \mu_3 c & \mu_5 & 1 \\ \mu_4 d & 0 & \mu_5 & 1 \end{vmatrix} = 0$$

In this determinant μ_1 up to and including μ_5 are moduli. The relation between μ_1 up to and including μ_4 is $\mu_1 \mu_4 = \mu_2 \mu_3$. One can read from the determinant that, for values which satisfy the equation (1), the points with coordinates

$$\begin{array}{lll} X = 0 & Y = \mu_1 a & Z = 0 \\ X = \mu_2 b & Y = 0 & Z = 0 \\ X = 0 & Y = \mu_3 c & Z = \mu_5 \\ X = \mu_4 d & Y = 0 & Z = \mu_5 \end{array}$$

are coplanar.

The spatial nomogram is marked sketchily in Fig. 2. As AB in this figure is parallel to CD the turning axis (h-scale from Fig. 1) lies in the infinite.

If the Z-axis is turned in such a way that it coincides with the XOY-plane, then the spatial nomogram becomes a flat nomogram with parallel reading lines.

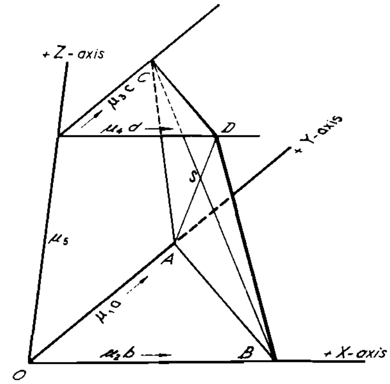

Figure 2.

As was already said, a spatial nomogram for $\frac{a}{b} = \frac{c}{d}$ or for the analogous geodetic formula

$$\frac{e_{\mathrm{m}}}{l_{\mathrm{km}}} = \frac{\delta_{\mathrm{cgr}}}{6.3662 \sin \varphi_{\mathrm{gr}}}$$

is also possible. If the moduli for e-, l-, δ- and φ-scale are chosen $\mu_1 = 3$ mm, $\mu_2 = 30$ mm, $\mu_3 = 2$ mm and $\mu_4 = 20$ mm respectively, then they satisfy the relation $\mu_1 \mu_4 = \mu_2 \mu_3$.

With an arbitrary value for μ_5 this nomogram can be projected in an infinite number of ways from two centres of projection on a flat plane in the colours red and blue. In the stereo nomogram Fig. 3 *) this is done in such a way that the plane of projection coincides with the XOY-plane. The two centres of projection are in the infinite above this plane. The line that joins the two centres (the eye-base) is perpendicular to the bisector of the angle between e- and l-scale ($Z = 0$).

The projection is a parallel one. It has the advantage that it can be computed easily. Another important advantage is that, just as the uniform e- and l-scale, the δ-scale remains a linear one. This facilitates an accurate interpolation between the graduation strokes. In this projection the δ- and φ-scale remain parallel to the e- and l-scale. Scale points with the same calibration on the red and blue δ- (φ-) scale lie on lines parallel to the eye-base (short side of the nomogram). The horizontal distance between the red and the blue δ- (φ-) scale is directly proportional to the modulus μ_5 in Fig. 2. The greater this distance, the greater the distance in the stereo image between the parallel planes of e- and l-scale and δ- and φ-scale respectively. For

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*) All stereo nomograms in this paper were computed and drawn with the utmost care by Mr. M. C. Breemans of the Geodetic Laboratory in Delft. The reproductions were made in an excellent manner by "Topografische Dienst" in Delft. They are at 0.8 of the size for which they are designed. The anaglyphic spectacles were put at my disposal by Zeiss-Aerotopograph, München.

Fig. 3 this distance was designed at 10 mm*). Observations by a number of test persons have proved, that one can see then the nomogram very well as a spatial image without annoying double images.

In order to do readings in the stereo nomogram a red index line and a blue one are laid over the points with given calibration for e and φ, the red one of course over the red scale points and the blue one over the blue scale points of these variables.

As the e-scale lies in the plane of drawing, the red scale points of this variable coincide with the blue ones. With the spectacles one sees these index lines in space as the line AD from Fig. 2. The second pair of index lines is laid over the (red = blue) scale point with the given value of l (point B in Fig. 2) in such a way that, seen through the spectacles, they are interpreted in space as a straight line that intersects AD (in S) as well as the scale for δ (in C, the scale point of the dependent variable).

It can be verified easily without stereoscopic observation that BSC intersects indeed the δ-scale: the red index line and the blue one must intersect the red and the blue δ-scale in points with the same calibration.

With the unarmed eye one can see roughly whether BC intersects or crosses AD: on the plane of drawing the line that joins the red and the blue point S must be parallel to the eye-base (horizontal).

In other cases a pair of opposite sides of the quadrangle instead of the diagonals will be used for readings in the stereo nomogram. This method can be recommended if a good stereoscopic image can be obtained of these sides and their intersection point. The horizontal projection of the intersection point should in such cases, by preference, not fall outside the paper on which the nomogram is drawn. In very special cases – in the text relating to stereo nomogram Fig. 7 I have given an example – one must even use this method, and even in a special manner. All methods, however, have in common, that, during the reading in the nomogram, the observer must pay attention to only one spatial point. The "unveränderte Kopfhaltung" of which Sutor speaks is not necessary. The difficulties of reading with a real reading plane have been overcome by the introduction of an unreal reading plane.

It is obvious that the material on which the index lines are drawn must be very transparent and very thin, for, during the reading of the nomogram, four of these index lines – two red lines and two blue ones – lie over each other. It has appeared that the narrow strips of astralon which have been added to this paper lend themselves admirably to the purpose. In order to avoid parallax as much as possible the index lines "blue" (at the top of the index line), "red" (at the top), "blue" (at the foot of the index line) and "red" (at the foot) must be used in the given sequence in such a way that "blue" and "red" can be read normally (not in mirror writing).

The manner of reading in a stereo nomogram, given above, has been shown in Fig. 3 by means of a stereo key. For $e = 40$ m, $l = 5$ km and $\varphi = 20$ grades $\delta \approx 15.8$ cgr.

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*) On account of the reduction this distance is 8 mm.

The user will notice that the remaining faint images of the blue scales of the variables and the blue index lines are of no influence on an accurate reading of the nomogram.

Of course spatial nomograms (stereo nomograms) with four variables can be extended to nomograms with six and more variables. We assume that, by the introduction of an auxiliary variable d, a relation $F(abcefg) = 0$ can be split into the relations $F_1(abcd) = 0$ and $F_2(efgd) = 0$. If, for each of these relations, a spatial nomogram can be made in such a way that the spatial d-scale in the first nomogram is identical with the d-scale in the second one, then the whole is a representation of the formula with six variables.

An example and an extension of Fig. 2 has been given sketchily in Fig. 4 for the formula $bce - afg = 0$, split into the relations $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{d}$. The a-scale coincides with the e-scale, the b-scale with the f-scale and the c-scale with the g-scale.

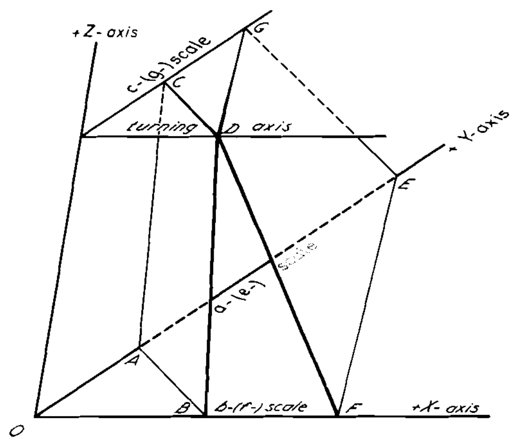

Figure 4.

It is obvious that the possibility of the construction of spatial nomograms (stereo nomograms) is not limited to the formulae which can be represented in a flat plane as a nomogram with a straight turning axis. In general all formulae which can be brought into the form of a determinant of the fourth order with only one variable on each row lend themselves for such a nomogram. This succeeds, among others, with such equations of which a nomogram in a flat plane can be made with point scales for two of the variables and a point field for the other two. This point field is formed by the intersection points of *straight* lines joined to these variables.

This will be illustrated with the goniometric relation

$$\tan a\,(\cot b + \cot c) - \tan d \cot b = 1 \quad \dots\dots\dots\dots \quad (3)$$

It can be brought into the form

$$\cot c \tan a + (\tan a \cot b - 1) - \tan d \cot b = 0$$

When this formula is multiplied by tan b one obtains:

$$\cot c \tan a \tan b + (\tan a - \tan b) - \tan d = 0 \quad \dots\dots\dots \quad (4)$$

(4) can be brought easily into the form of the following determinant of the third order:

$$\begin{vmatrix} 0 & -\cot c & 1 \\ 1 & \tan d & 0 \\ 1 & \tan a - \tan b & \tan a \tan b \end{vmatrix} = 0 = \begin{vmatrix} 0 & -\cot c & 1 \\ 1 & \tan d & 1 \\ \dfrac{1}{1+\tan a \tan b} & \dfrac{\tan a - \tan b}{1+\tan a \tan b} & 1 \end{vmatrix}$$

Apart from moduli the following scale equations for the *ab*-points can be read from this determinant:

$$X = \frac{1}{1+\tan a \tan b} \qquad Y = \frac{\tan a - \tan b}{1+\tan a \tan b}$$

If $\tan b$ ($\tan a$) is eliminated from these relations one finds the equation of the lines joined to a (b). The result is:

$$X(1+\tan^2 a) - Y \tan a - 1 = 0 \qquad (5)$$

$$X(1+\tan^2 b) + Y \tan b - 1 = 0 \qquad (6)$$

The a- (b-) lines are consequently straight lines. The lines joined to the variable a touch a curve. The equation of this curve can be found by eliminating $\tan a$ from (5) and the equation

$$2 \tan a\, X - Y = 0$$

obtained by differentiating (5) with respect to $\tan a$. The result is

$$Y^2 = 4X(X-1)$$

This equation represents a conic section (a hyperbola).

In a similar way one finds that the b-lines from (6) touch the *same* hyperbola $Y^2 = 4X(X-1)$.

Thanks to the fact that the a- and b-lines are straight lines and touch the same conic section, it is, according to the principle of duality, also possible to make a spatial model after (3) *and a stereo nomogram. In this nomogram the common carrier of the point scales for a and b will be a conic section.*

This can be shown in the following way: If in (4): $\tan a \tan b = \xi$, $\tan a - \tan b = \eta$ and $\tan d = \zeta$, then the original equation passes into

$$\xi \cot c + \eta - \zeta = 0 \qquad (7)$$

Besides:

$$\zeta - \tan d = 0 \qquad (8)$$

Elimination of $\tan b$ and $\tan a$ from the relations $\xi = \tan a \tan b$ and $\eta = \tan a - \tan b$ leads to

$$\xi + \eta \tan a - \tan^2 a = 0 \qquad (9)$$

$$\xi - \eta \tan b - \tan^2 b = 0 \qquad (10)$$

(7) up to and including (10) are all linear in ξ, η and ζ and contain only one variable

from (4) each. Apart from a parasitic factor, slipped in by the elimination-process,

$$\begin{vmatrix} 1 & \tan a & 0 & -\tan^2 a \\ 1 & -\tan b & 0 & -\tan^2 b \\ \cot c & 1 & -1 & 0 \\ 0 & 0 & 1 & -\tan d \end{vmatrix} = 0$$

is consequently a manner of writing for (4).

If this determinant is written as

$$\begin{vmatrix} \mu_1 \tan a & \mu_2 \tan^2 a & 0 & 1 \\ -\mu_1 \tan b & \mu_2 \tan^2 b & 0 & 1 \\ \dfrac{\mu_1}{\cot c - 1} & 0 & \dfrac{-\mu_3}{\cot c - 1} & 1 \\ 0 & \mu_2 \tan d & \mu_3 & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (11)$$

then it is reduced to the form (2). In this determinant μ_1, μ_2 and μ_3 are moduli.

If the moduli are chosen in a judicious way, a very good spatial model can be made from (11). The determinant lends itself also very well to the construction of a stereo nomogram, unless the eye-base in the nomogram should be chosen parallel to the X-axis or Y-axis. For, in the first case, the projection of the spatial c-scale on the XOY-plane runs parallel to the eye-base, in the second case and for a perpendicular position of X-axis and Y-axis, the projection of the d-scale. It is then impossible to evoke a spatial image either from the c-scale carrier or from the d-scale carrier.

Though the graduation strokes which are perpendicular to the scale carriers can, nevertheless, help some observers to obtain a stereoscopic image from these scales (this succeeds better the longer the graduation strokes are made), I preferred improvement by remodelling (11) into the form (12). By this operation the projection of the spatial figure on the XOY-plane remains unchanged. However, the projection has turned to the left over an angle φ with respect to the system of axes XY.

$$\begin{vmatrix} \mu_1\cos\varphi \tan a - \mu_2\sin\varphi \tan^2 a & \mu_1\sin\varphi \tan a + \mu_2\cos\varphi \tan^2 a & 0 & 1 \\ -\mu_1\cos\varphi \tan b - \mu_2\sin\varphi \tan^2 b & -\mu_1\sin\varphi \tan b + \mu_2\cos\varphi \tan^2 b & 0 & 1 \\ \dfrac{\mu_1\cos\varphi}{\cot c - 1} & \dfrac{\mu_1\sin\varphi}{\cot c - 1} & \dfrac{-\mu_3}{\cot c - 1} & 1 \\ -\mu_2\sin\varphi \tan d & \mu_2\cos\varphi \tan d & \mu_3 & 1 \end{vmatrix} = 0 \quad . \; . \quad (12)$$

If $\mu_1 = 70$ mm, $\mu_2 = 35$ mm and $\varphi = \text{arc sin } 0.6 = \text{arc cos } 0.8$, then the eye-base can be taken parallel to the X-axis. Moreover, a parallel projection on the XOY-plane or on a plane parallel to the XOY-plane fills the standard paper-size A4 (210×297 mm) in a satisfactory way.

The equations of the red scales in the stereo nomogram run as follows:

$$\left.\begin{array}{ll} X = 56 \tan a - 21 \tan^2 a & Y = 42 \tan a + 28 \tan^2 a \\ X = -(56 \tan b + 21 \tan^2 b) & Y = -42 \tan b + 28 \tan^2 b \\ X = \dfrac{56}{\cot c - 1} & Y = \dfrac{42}{\cot c - 1} \\ X = -21 \tan d & Y = 28 \tan d \end{array}\right\} \quad . \; . \; . \quad (13)$$

For $\mu_3 = 100$ mm – this value is important for the construction of a spatial nomogram but not for the construction of a stereo nomogram – the Z-amounts for the a-, b-, c- and d-points respectively are

$$Z = 0,\; Z = 0,\; Z = \frac{-100}{\cot c - 1},\; Z = 100 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

The advantage of a projection on the XOY-plane would be that the red scales for a and b coincide with the blue ones. A disadvantage, however, is that, for the c-scale ($c = 15° \rightarrow 30°$), all amounts Z are negative. As it is difficult for many observers (I include myself) to evoke a stereoscopic image of a line which lies under the plane of drawing, all Z's have been made positive. This can be attained by adding 140 mm to the amounts (14). They become then:

$$Z = 140,\; Z = 140,\; Z = \frac{20(7 \cot c - 12)}{\cot c - 1} \text{ and } Z = 240$$

For a maximum parallax in the anaglyphic image of 0.06×240 mm $= 14.4$ mm, the amounts X from (13) must therefore be augmented with $\Delta X = 8.40$ mm, $\Delta X = 8.40$ mm, $\Delta X = \dfrac{1.2(7 \cot c - 12)}{\cot c - 1}$ mm and $\Delta X = 14.40$ mm respectively. The amounts Y for the blue scales remain of course unaltered.

The nomogram has been reproduced as Fig. 5. One can read from it that, for $a = 60°$, $b = 45°$ and $c = 30°$, $d = 75°$. The advantage of the stereo nomogram over and above the nomogram with a point field in the flat plane is that, in the stereo nomogram, the scales for a and b are point-scales. Opposite these scales one can make double scales, in the example discussed for instance a graduation for the decimal calibration of the quadrant. This is not possible in the nomogram with a point field in which the a- and b-scales are line-scales.

For the construction of stereo nomograms those formulae with four variables are of course the most important, which can not be represented in a flat plane, neither by a nomogram with a straight turning axis, nor by a nomogram with a point field, consisting of the intersection points of straight lines, joined to two of the variables. In many cases one can see in a simple manner whether such a relation can be brought into the form of a determinant of the fourth order in each of the rows in which only one variable appears. This will be illustrated with an example. I chose for it the formula

$$806.4\, a\{(c+3)\} + \frac{a^2(a-3)}{a-0.1}\{18.9\, b + 378\, c - 300 \tan d + 1039.5\} +$$

$$+ \frac{a^3}{a-0.2}\{37.8\, b + 218.4\, c + 40 \tan d + 466.2\} + 1920 = 0 \; . \; . \; . \; . \quad (15)$$

It is striking that the terms between braces in this relation are linear in the variables

b, c and tan d and that each of these terms is multiplied by a function of the fourth variable a. If:

$$\left.\begin{aligned} \xi &= 806.4\,(c+3) \\ \eta &= 18.9\,b + 378\,c - 300\tan d + 1039.5 \\ \zeta &= 37.8\,b + 218.4\,c + 40\tan d + 466.2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (16)$$

then (15) passes into

$$a\xi + \frac{a^2(a-3)}{a-0.1}\,\eta + \frac{a^3}{a-0.2}\,\zeta + 1920 = 0 \quad \ldots\ldots\ldots\ldots \quad (17)$$

It is linear in ξ, η and ζ and contains only one variable from (15).

As in the equations (16) *b, c and tan d occur linearly, it follows that elimination of c and tan d, b and tan d and b and c from these equations gives expressions for b, c and tan d in which* ξ, η *and* ζ *are also of the first degree.*

The result of this elimination is

$$\left.\begin{aligned} 5\,\xi - 2\eta - 15\zeta - 604.8(5-b) &= 0 \\ \xi \qquad\qquad - 806.4(c+3) &= 0 \\ \xi - 3\eta + 1.5\zeta - 960\tan d \quad &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

By the introduction of the three auxiliary variables ξ, η and ζ (15) has been split up into the four equations (17) and (18) which are all of the first degree in ξ, η and ζ and contain only one variable from (15).

$$\begin{vmatrix} a & \dfrac{a^2(a-3)}{a-0.1} & \dfrac{a^3}{a-0.2} & +1920 \\ 5 & -2 & -15 & -604.8(5-b) \\ 1 & 0 & 0 & -806.4(c+3) \\ 1 & -3 & 1.5 & -960\tan d \end{vmatrix} = 0 =$$

$$= \begin{vmatrix} \dfrac{a^3}{a-0.2} & 20 & \dfrac{a^2(a-3)}{a-0.1} & a \\ -15 & -6.3(5-b) & -2 & 5 \\ 0 & -8.4(c+3) & 0 & 1 \\ 1.5 & -10\tan d & -3 & 1 \end{vmatrix} = 0$$

is therefore a manner of writing for (15).

When the signs of the second and the third column are changed and the first and the second row are divided by a and 5 respectively, this determinant runs:

$$\begin{vmatrix} \dfrac{a^2}{a-0.2} & \dfrac{20}{-a} & \dfrac{a(3-a)}{a-0.1} & 1 \\ -3 & 1.26(5-b) & 0.4 & 1 \\ 0 & 8.4(c+3) & 0 & 1 \\ 1.5 & 10\tan d & 3 & 1 \end{vmatrix} = 0. \quad \ldots\ldots\ldots\ldots \quad (19)$$

The spatial nomogram (stereo nomogram) that can be constructed after (19) has straight scales for b, c and d which are all perpendicular to the XOZ-plane. The scale for a is on a spatial curve. The equation of this curve can be found by eliminating a from the equations

$$X = \frac{a^2}{a-0.2} \qquad Y = \frac{20}{-a} \qquad Z = \frac{a(3-a)}{a-0.1}$$

The result is

$$2X(10+0.1Y)+Z(20+0.1Y)-60 = 0$$

Even for formulae with three variables it is frequently difficult in practice to make a good nomogram with point scales. Theoretically the possibility for such a nomogram is proved if the formula concerned can be brought into the well-known determinant with three rows and three columns. The algebraic operations of this determinant, however, which lead to the constructional determinant, are difficult. The scales of the nomogram must fill the paper-size available as well as possible. The nomogram ruler must intersect the scale of the dependent variable at a favourable angle and the accuracy of the dependent variable must be as great as possible.

When one tries to find a suitable spatial nomogram, the number of difficulties increases in a considerable way as one has 15 degrees of freedom at one's disposal. For, if (19) is multiplied by the transformation modulus

$$\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_5 & \mu_6 & \mu_7 & \mu_8 \\ \mu_9 & \mu_{10} & \mu_{11} & \mu_{12} \\ \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad (20)$$

then another determinant is formed from which the scale equations of another nomogram can be read. It is collinearly related to the original one.

For the d-points for instance these equations run:

$$\left.\begin{aligned} X &= \frac{1.5\mu_1 + 10\mu_2 \tan d + 3\mu_3 + \mu_4}{1.5\mu_{13}+10\mu_{14} \tan d + 3\mu_{15}+\mu_{16}} \\ Y &= \frac{1.5\mu_5 + 10\mu_6 \tan d + 3\mu_7 + \mu_8}{1.5\mu_{13}+10\mu_{14} \tan d + 3\mu_{15}+\mu_{16}} \\ Z &= \frac{1.5\mu_9 + 10\mu_{10} \tan d + 3\mu_{11}+\mu_{12}}{1.5\mu_{13}+10\mu_{14} \tan d + 3\mu_{15}+\mu_{16}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (21)$$

(20) is known when the proportion is known between one of the elements μ_1 up to and including μ_{16} and the 15 other ones. It can be computed from 15 equations of the form (21) with 16 unknowns if, for five arbitrary values of the variables (for instance two of a and one of b, c and d) the co-ordinates X, Y and Z of the scale points are given (not four points in a flat plane).

The computation of this transformation, however, is an extensive job and almost impracticable when no electronic computing apparatus is available. Besides, it does not give any certainty that the spatial nomogram or the stereo nomogram that can be constructed after the transformed determinant satisfies indeed the expectations. As the number of degrees of freedom is so great, one cannot survey how another choice of the X-, Y- and Z-co-ordinates of one or more scale points manifests itself in the moduli μ of the transformation modulus (20). In practice it is therefore recommendable to suffice with a smaller number. By way of experiment and with a limited number of moduli one endeavours to fill the XOY-plane (the plane of drawing) as well as possible with the horizontal projection of the spatial scales. It must be avoided that, even approximately, these scales form together a flat plane. One must take account of that in fixing the moduli which determine the amounts Z. The better this requirement is fulfilled, the better the chance that the scale of the dependent variable intersects the plane through the scale points of the independent variables at a favourable angle. A perpendicular intersection is the best solution of course. In that case a small error in the judgment whether the lines AD and BC from Fig. 2 intersect or cross each other has the smallest influence on the reading of the dependent variable.

In the worked-out example Fig. 6 the transformation modulus (20) has been chosen

$$\begin{vmatrix} 40 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

From the twelve 0's which are found in it, it appears that, thanks to the simplicity of the example, an equal number of degrees of freedom could remain unused.

With $\mu_{11} = 2.5$ mm the Z's in the stereo nomogram are found as the differences in X-direction between the red scale points and the blue ones.

If (19) is multiplied by this transformation modulus then we find the determinant underlying the stereo nomogram. It runs:

$$\begin{vmatrix} \dfrac{40a^2}{a-0.2} & \dfrac{300}{-a} & \dfrac{2.5a(3-a)}{a-0.1} & 1 \\ -120 & 18.9(5-b) & 1 & 1 \\ 0 & 126(c+3) & 0 & 1 \\ 60 & 150 \tan d & 7.5 & 1 \end{vmatrix} = 0$$

One reads from the nomogram that, for $a = -3$, $c = -2.8$ and $d = 38.9$ grades, $b \approx +5.46$.

In Fig. 5 we already discussed an example in which two scales of a spatial (stereo) nomogram were represented on the same curve, in this case a flat curve. This was possible as in (4) the functions of the variables a and b occur in a special manner, namely as a product ($\tan a \tan b$) and as a difference ($\tan a - \tan b$). If there is a relation in which three out of four variables occur in such a regularity, namely as a product, a sum of double products and as a sum, then it appears possible to represent these three variables on the same spatial curve. The fourth scale is, dependent on the structure of the formula, also a spatial curve, a flat curve or, in special cases, a straight line.

In order to show this, one starts from the relation

$$A\cdot f(a)\cdot g(b)\cdot h(c)\cdot k(d)+B\cdot l(d)\{f(a)\cdot g(b)+f(a)\cdot h(c)+g(b)\cdot h(c)\}+ \\ +C\cdot m(d)\{f(a)+g(b)+h(c)\}+D=0 \quad \ldots\ldots \quad (22)$$

in which the variables a, b and c occur in the regularity just mentioned and in which A up to and including D are constants.

If

$$f(a)\cdot g(b)\cdot h(c) = \xi \quad \ldots\ldots \quad (23)$$

$$f(a)\cdot g(b)+f(a)\cdot h(c)+g(b)\cdot h(c) = \eta \quad \ldots\ldots \quad (24)$$

and
$$f(a)+g(b)+h(c) = \zeta \quad \ldots\ldots \quad (25)$$

then (22) passes into

$$A\cdot k(d)\xi+B\cdot l(d)\eta+C\cdot m(d)\zeta+D=0 \quad \ldots\ldots \quad (26)$$

According to (23):

$$g(b)\cdot h(c) = \frac{\xi}{f(a)}$$

and according to (25):

$$g(b)+h(c) = \zeta-f(a)$$

Substituting these values in (24) one obtains:

$$f(a)\{\zeta-f(a)\}+\frac{\xi}{f(a)} = \eta$$

or

$$\xi-f(a)\eta+\{f(a)\}^2\zeta-\{f(a)\}^3=0 \quad \ldots\ldots \quad (27)$$

In a similar way:

$$\xi-g(b)\eta+\{g(b)\}^2\zeta-\{g(b)\}^3=0 \quad \ldots\ldots \quad (28)$$

$$\xi-h(c)\eta+\{h(c)\}^2\zeta-\{h(c)\}^3=0 \quad \ldots\ldots \quad (29)$$

By the introduction of the three auxiliary variables ξ, η and ζ (22) has been split up into the four equations (26) up to and including (29); they are all linear in ξ, η and ζ and all contain only one variable from (22).

$$\begin{vmatrix} A\cdot k(d) & B\cdot l(d) & C\cdot m(d) & D \\ 1 & -f(a) & \{f(a)\}^2 & -\{f(a)\}^3 \\ 1 & -g(b) & \{g(b)\}^2 & -\{g(b)\}^3 \\ 1 & -h(c) & \{h(c)\}^2 & -\{h(c)\}^3 \end{vmatrix} = 0 =$$

$$= \begin{vmatrix} f(a) & \{f(a)\}^2 & \{f(a)\}^3 & 1 \\ g(b) & \{g(b)\}^2 & \{g(b)\}^3 & 1 \\ h(c) & \{h(c)\}^2 & \{h(c)\}^3 & 1 \\ \dfrac{-B\cdot l(d)}{A\cdot k(d)} & \dfrac{C\cdot m(d)}{A\cdot k(d)} & \dfrac{-D}{A\cdot k(d)} & 1 \end{vmatrix} = 0 \quad \ldots\ldots \quad (30)$$

is therefore a notation for (22), apart of course from a parasitic factor which is, however, of no importance in nomographic practice.

If, for instance, the equation

$$abcd-(0.05d+1)(ab+ac+bc)-10=0 \quad \ldots\ldots\ldots\ldots \quad (31)$$

is compared with the type (22) then $A = 1$, $B = -1$, $C = 0$, $D = -10$, $f(a) = a$, $g(b) = b$, $h(c) = c$, $k(d) = d$ and $l(d) = 0.05\,d+1$.

In accordance with (30), (31) can therefore be written as

$$\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ \dfrac{0.05d+1}{d} & 0 & \dfrac{10}{d} & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

From (32) one derives that the scales for a, b and c lie all on the spatial curve $XY-Z=0$. The equation of the d-scale can be found when d is eliminated from the equations

$$X = \frac{0.05d+1}{d} \qquad Z = \frac{10}{d}$$

The result is

$$10X-Z-0.5=0$$

It is a straight line in the XOZ-plane.

When (32) is multiplied by the transformation modulus

$$\begin{vmatrix} 5600 & 0 & 0 & 0 \\ 0 & 2600 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 4 & 9 & 0 & 17 \end{vmatrix}$$

in which ten degrees of freedom have been left unused, then the following determinant is formed:

$$\begin{vmatrix} \dfrac{5600a}{9a^2+4a+17} & \dfrac{2600a^2}{9a^2+4a+17} & \dfrac{50a^3}{9a^2+4a+17} & 1 \\ \dfrac{5600b}{9b^2+4b+17} & \dfrac{2600b^2}{9b^2+4b+17} & \dfrac{50b^3}{9b^2+4b+17} & 1 \\ \dfrac{5600c}{9c^2+4c+17} & \dfrac{2600c^2}{9c^2+4c+17} & \dfrac{50c^3}{9c^2+4c+17} & 1 \\ \dfrac{280(d+20)}{17.2d+4} & 0 & \dfrac{500}{17.2d+4} & 1 \end{vmatrix} = 0 \quad \ldots\ldots \quad (33)$$

It can underlie a suitable spatial nomogram as well as a stereo nomogram. In the stereo nomogram, however, the eye-base can be chosen neither parallel to the X-axis nor parallel to the Y-axis. For, in the first case, it should be difficult to evoke a spatial image of the d-scale because this scale lies in the XOZ-plane. In the second case one should have the utmost difficulty to evoke such an image of the index lines in the nomogram which have a direction differing but little from the direction of the Y-axis.

A satisfying solution that obviates both difficulties can be obtained by applying similar operations upon (33) as given in (12).

When the system of axes is turned over an angle arc sin 0.4 = arc cos 0.916515 to the right and the eye-base is chosen parallel to the new X-axis, then one obtains a stereo nomogram in which a good stereoscopic image can be evoked, from the scales as well as from the index lines. Moreover, it fills the standard paper size A4 for which it was designed in a satisfactory manner. It is shown in Fig. 7.

In the turned system of axes the equations of the red a- (b- and c-) scale points run:

$$X = \frac{5132.48a - 1040a^2}{9a^2 + 4a + 17} \qquad Y = \frac{2240a + 2382.94a^2}{9a^2 + 4a + 17}$$

The scale equations for d (red) are:

$$X = \frac{256.624(d+20)}{17.2d+4} \qquad Y = \frac{112(d+20)}{17.2d+4}$$

The parallaxes in X-direction are $\dfrac{50a^3}{9a^2+4a+17}$ and $\dfrac{500}{17.2d+4}$ respectively. One reads from the nomogram that, for $a = 3.0$, $b = 1.0$ and $c = 2.0$, $d \approx 3.85$.

In case the values of the independent variables a and b, a and c or b and c coincide, the dependent variable cannot be read by using the diagonals of the quadrangle that joins the four scale points belonging together. As was already said on page 6, one must use then a pair of opposite sides. The first side is formed by the stereoscopic image of the red and the blue tangent in the point of the red and the blue curve for a (b, c) that bears the given, same, calibrations for two of these variables. The manner in which the second pair of index lines must be laid needs no further explanation.

One can read that, for $a = b = 2.0$ and $c = 1.26$, $d \approx 4.15$.

On the preceding pages a spatial nomogram (stereo nomogram) was already mentioned in which three scales were lying on the same spatial curve. In some cases it is even possible to represent all four scales on one spatial curve. In order to see that, one has only to extend the theory for nomograms with three variables already given by J. CLARK [5] (see also [6]). For that purpose one starts from the relation

$$\begin{aligned}
&A \cdot f(a) \cdot g(b) \cdot h(c) \cdot k(d) + \\
&B\{f(a) \cdot g(b) \cdot h(c) + f(a) \cdot g(b) \cdot k(d) + f(a) \cdot h(c) \cdot k(d) + g(b) \cdot h(c) \cdot k(d)\} + \\
&+ C\{f(a) \cdot g(b) + f(a) \cdot h(c) + f(a) \cdot k(d) + g(b) \cdot h(c) + g(b) \cdot k(d) + h(c) \cdot k(d)\} + \\
&+ D\{f(a) + g(b) + h(c) + k(d)\} + E = 0 \quad \ldots\ldots\ldots\ldots \quad (34)
\end{aligned}$$

In this relation A up to and including E are constants. The functions $f(a)$ up to and including $k(d)$ of the variables a up to and including d occur symmetrically as a quadruple product, a sum of triple products, a sum of double products and as a sum. It appears that, by the introduction of four auxiliary variables ξ, η, ζ and θ (34) can be split up into five equations, all linear in ξ up to and including θ. In the first one the constants A up to and including E also occur. The four other ones contain each only one variable from the original relation. If

$$f(a)\cdot g(b)\cdot h(c)\cdot k(d)=\xi \quad \ldots \quad (35)$$

$$f(a)\cdot g(b)\cdot h(c)+f(a)\cdot g(b)\cdot k(d)+f(a)\cdot h(c)\cdot k(d)+g(b)\cdot h(c)\cdot k(d)=\eta \quad (36)$$

$$\begin{aligned}&f(a)\cdot g(b)+f(a)\cdot h(c)+f(a)\cdot k(d)+g(b)\cdot h(c)+\\&+g(b)\cdot k(d)+h(c)\cdot k(d)=\zeta\end{aligned} \quad \ldots \quad (37)$$

$$f(a)+g(b)+h(c)+k(d)=\theta \quad \ldots \quad (38)$$

then (34) passes into

$$A\xi+B\eta+C\zeta+D\theta+E=0 \quad \ldots \quad (39)$$

As, according to (35):

$$f(a)=\frac{\xi}{g(b)\cdot h(c)\cdot k(d)}$$

and, according to (36) up to and including (38):

$$\begin{aligned}&g(b)\cdot h(c)\cdot k(d)=\eta-f(a)\,\{g(b)\cdot h(c)+g(b)\cdot k(d)+h(c)\cdot k(d)\}=\\&\eta-f(a)\,[\zeta-f(a)\,\{g(b)+h(c)+k(d)\}]=\\&\eta-f(a)\,[\zeta-f(a)\,\{\theta-f(a)\}]\end{aligned}$$

$$f(a)=\frac{\xi}{\eta-f(a)\,[\zeta-f(a)\,\{\theta-f(a)\}]}$$

whence

$$\xi-f(a)\eta+\{f(a)\}^2\zeta-\{f(a)\}^3\theta+\{f(a)\}^4=0 \quad \ldots \quad (40)$$

In a similar way:

$$\left.\begin{aligned}\xi-g(b)\eta+\{g(b)\}^2\zeta-\{g(b)\}^3\theta+\{g(b)\}^4=0\\\xi-h(c)\eta+\{h(c)\}^2\zeta-\{h(c)\}^3\theta+\{h(c)\}^4=0\\\xi-k(d)\eta+\{k(d)\}^2\zeta-\{k(d)\}^3\theta+\{k(d)\}^4=0\end{aligned}\right\} \quad \ldots \quad (41)$$

From (39), (40) and (41) it follows that the determinant of the fifth order

$$\begin{vmatrix}A & B & C & D & E\\1 & -f(a) & \{f(a)\}^2 & -\{f(a)\}^3 & \{f(a)\}^4\\1 & -g(b) & \{g(b)\}^2 & -\{g(b)\}^3 & \{g(b)\}^4\\1 & -h(c) & \{h(c)\}^2 & -\{h(c)\}^3 & \{h(c)\}^4\\1 & -k(d) & \{k(d)\}^2 & -\{k(d)\}^3 & \{k(d)\}^4\end{vmatrix}=0=$$

$$= \begin{vmatrix} A & -B & C & -D & E \\ 1 & f(a) & \{f(a)\}^2 & \{f(a)\}^3 & \{f(a)\}^4 \\ 1 & g(b) & \{g(b)\}^2 & \{g(b)\}^3 & \{g(b)\}^4 \\ 1 & h(c) & \{h(c)\}^2 & \{h(c)\}^3 & \{h(c)\}^4 \\ 1 & k(d) & \{k(d)\}^2 & \{k(d)\}^3 & \{k(d)\}^4 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (42)$$

is a manner of writing for (34).

By an algebraic operation four of five elements from the first row can always be made 0. The minor of the fifth element, a determinant with four rows and four columns, underlies the spatial nomograms (stereo nomograms) mentioned above.

If one compares

$$abcd+3(a+b+c+d)-40 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

with (34), it appears that

$A = 1$, $B = C = 0$, $D = 3$, $E = -40$, $f(a) = a$, $g(b) = b$, $h(c) = c$ and $k(d)=d$

Brought into the form (42) (43) therefore runs:

$$\begin{vmatrix} 1 & 0 & 0 & -3 & -40 \\ 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \\ 1 & c & c^2 & c^3 & c^4 \\ 1 & d & d^2 & d^3 & d^4 \end{vmatrix} = 0 =$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & a & a^2 & 3+a^3 & 40+a^4 \\ 1 & b & b^2 & 3+b^3 & 40+b^4 \\ 1 & c & c^2 & 3+c^3 & 40+c^4 \\ 1 & d & d^2 & 3+d^3 & 40+d^4 \end{vmatrix} = \begin{vmatrix} a & a^2 & 3+a^3 & 40+a^4 \\ b & b^2 & 3+b^3 & 40+b^4 \\ c & c^2 & 3+c^3 & 40+c^4 \\ d & d^2 & 3+d^3 & 40+d^4 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{a}{40+a^4} & \dfrac{a^2}{40+a^4} & \dfrac{3+a^3}{40+a^4} & 1 \\ \dfrac{b}{40+b^4} & \dfrac{b^2}{40+b^4} & \dfrac{3+b^3}{40+b^4} & 1 \\ \dfrac{c}{40+c^4} & \dfrac{c^2}{40+c^4} & \dfrac{3+c^3}{40+c^4} & 1 \\ \dfrac{d}{40+d^4} & \dfrac{d^2}{40+d^4} & \dfrac{3+d^3}{40+d^4} & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (44)$$

All scales of the spatial nomogram that can be constructed from this determinant lie on the same spatial curve.

The equation of this curve runs:

$$3X^4+XY^3-Z(40X^4+Y^4) = 0$$

In order to come to the stereo nomogram represented in Fig. 8, (44) is multiplied by the transformation modulus

$$\begin{vmatrix} \frac{10000}{3} & 0 & \frac{1000}{3} & 0 \\ 12400 & -18400 & 6800 & 0 \\ 0 & 12.5 & 75 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

From the constructional determinant which is then obtained one can read as scale equations for the a- (b-, c-, d-) points:

$$\left.\begin{aligned} X &= \frac{\frac{1000}{3}(a^3+10a+3)}{40+a^4} \\ Y &= \frac{12400a-18400a^2+6800\,(3+a^3)}{40+a^4} \\ Z &= \frac{12.5a^2+75\,(3+a^3)}{40+a^4} \end{aligned}\right\} \qquad (45)$$

In order that the co-ordinate numbers should not be too great all amounts X can be reduced by 25 mm and all amounts Y by 450 mm. The amounts Z have been reduced by 5.625 mm. For a (b, c, d) $= 0$ $Z = 0$ in that case. For that calibration the red and the blue scale point coincide. The eye-base in the nomogram was taken parallel to the X-axis. One reads from the nomogram that, for $a = 8.0$, $b = 1.0$ and $c = 3$, $d \approx 0.148$. In case two values of the independent variables are alike the readings on the nomogram are done according to the directives given on page 16.

In behalf of teaching purposes a spatial model, a wire figure, was made after (43) in the workshop of the Geodetic Institute in Delft. Its sizes in X-direction are three times as great as indicated by the equations (45) and in Y-direction two times as great. The amounts Z are the fortyfold of the parallaxes which can be computed from (45). A stereo photo of the spatial model has been reproduced as Fig. 9. The example just mentioned has been marked in it. In addition to the diagonals the sides of the quadrangle have also been denoted by wires. These sides, however, are of no importance for the readings in the nomogram.

In the above I have tried to give an extension to the nomographic possibilities by means of stereo nomograms. I think that there will be few objections to making greater use of these computing aids than up to this time. The simple manner of reading in these nomograms, introduced in this paper, can promote this use. Especially stereo nomograms can be used in technical sciences for those formulae with four or more variables which must be computed frequently and which can not be represented in a flat plane.

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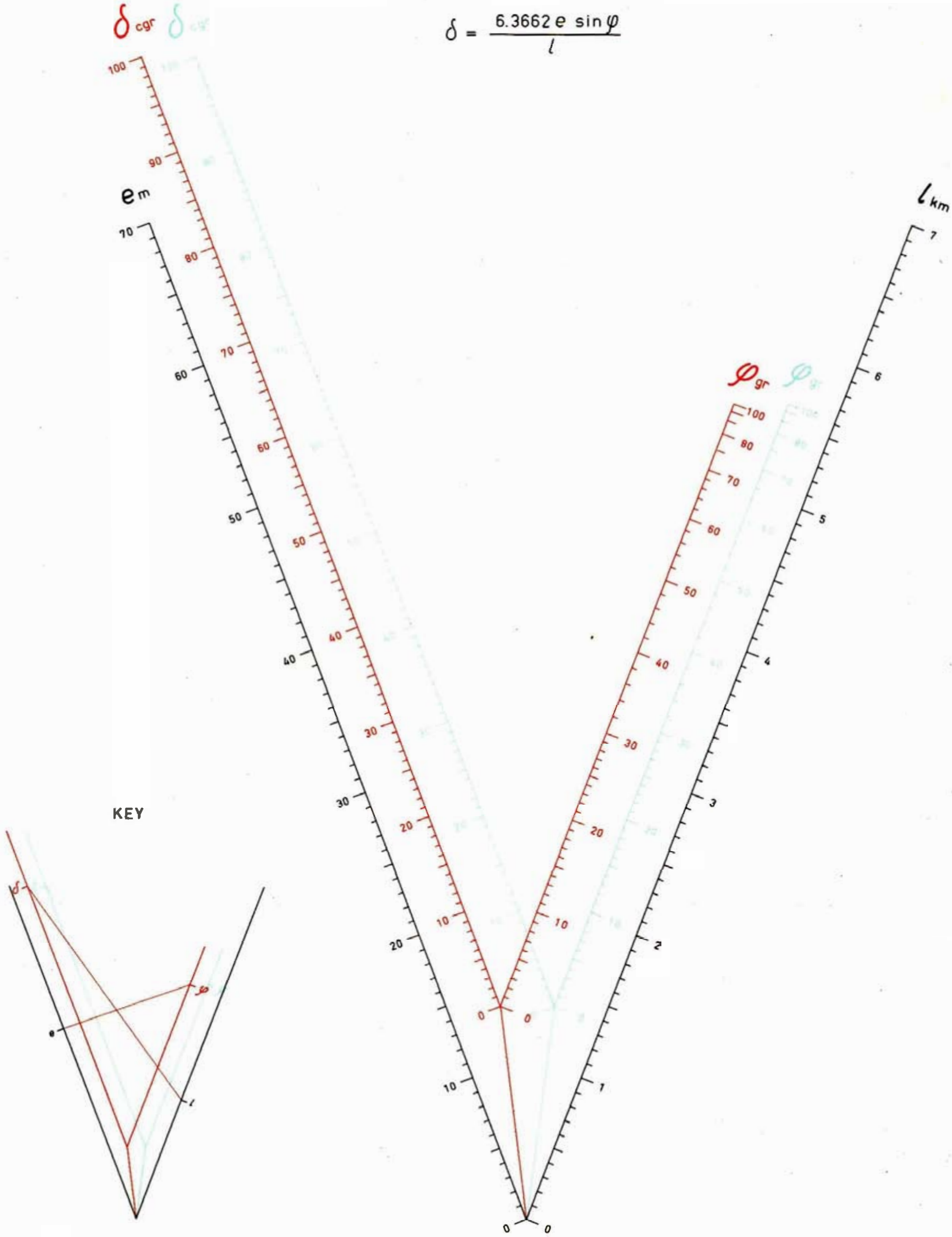
STEREONOMOGRAM
for

$$\delta = \frac{6.3662\, e \, \sin\varphi}{l}$$

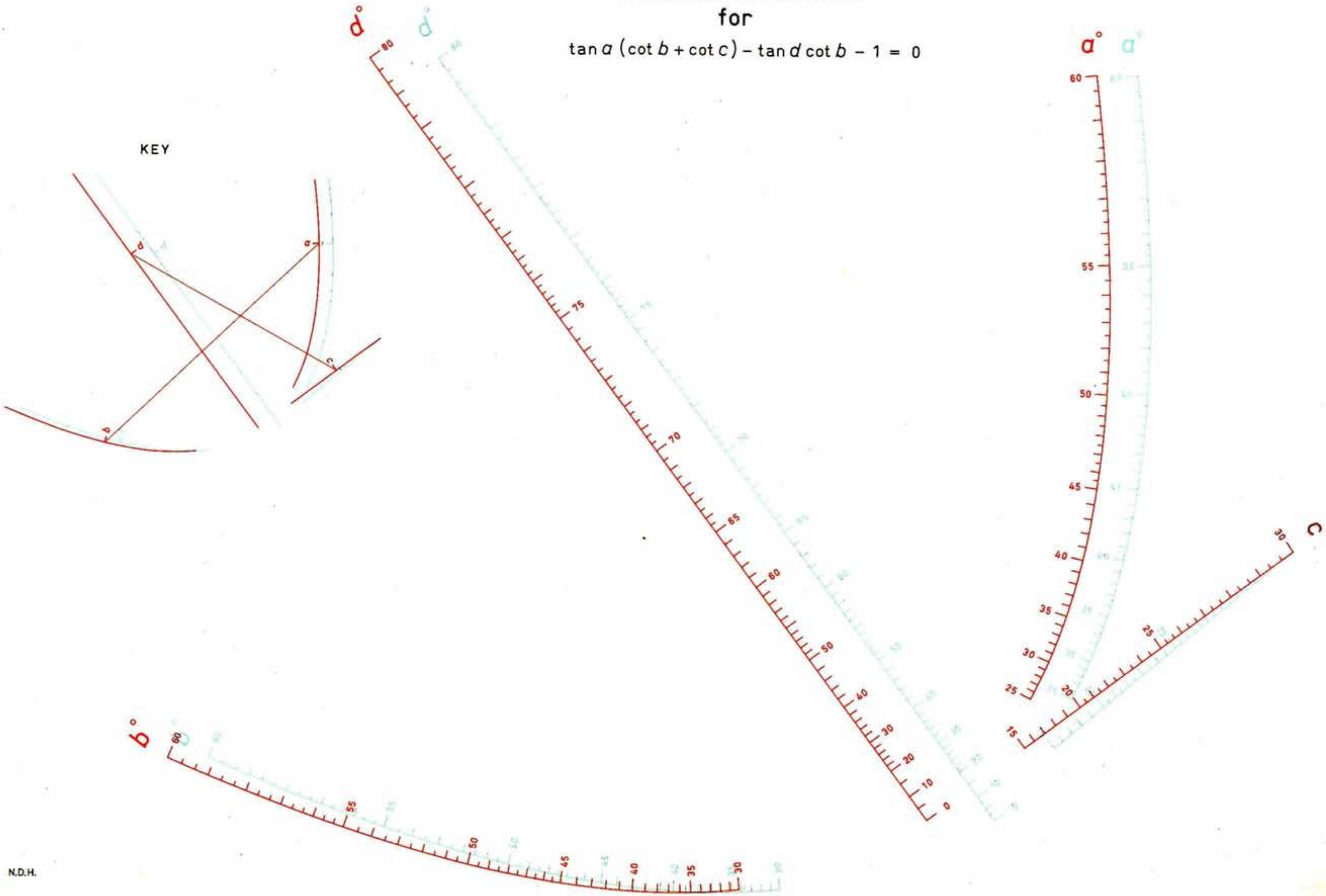


KEY

N.D.H.

Figure 3

STEREONOMOGRAM
for
$$\tan a\,(\cot b + \cot c) - \tan d \cot b - 1 = 0$$



N.D.H.

Figure 5

STEREONOMOGRAM
for

$$806.4\,a\,(c+3) + \frac{a^2(a-3)}{a-0.1}\,(18.9\,b + 378\,c - 300\tan d + 1039.5) +$$
$$+ \frac{a^3}{a-0.2}\,(37.8\,b + 218.4\,c + 40\tan d + 466.2) + 1920 = 0$$

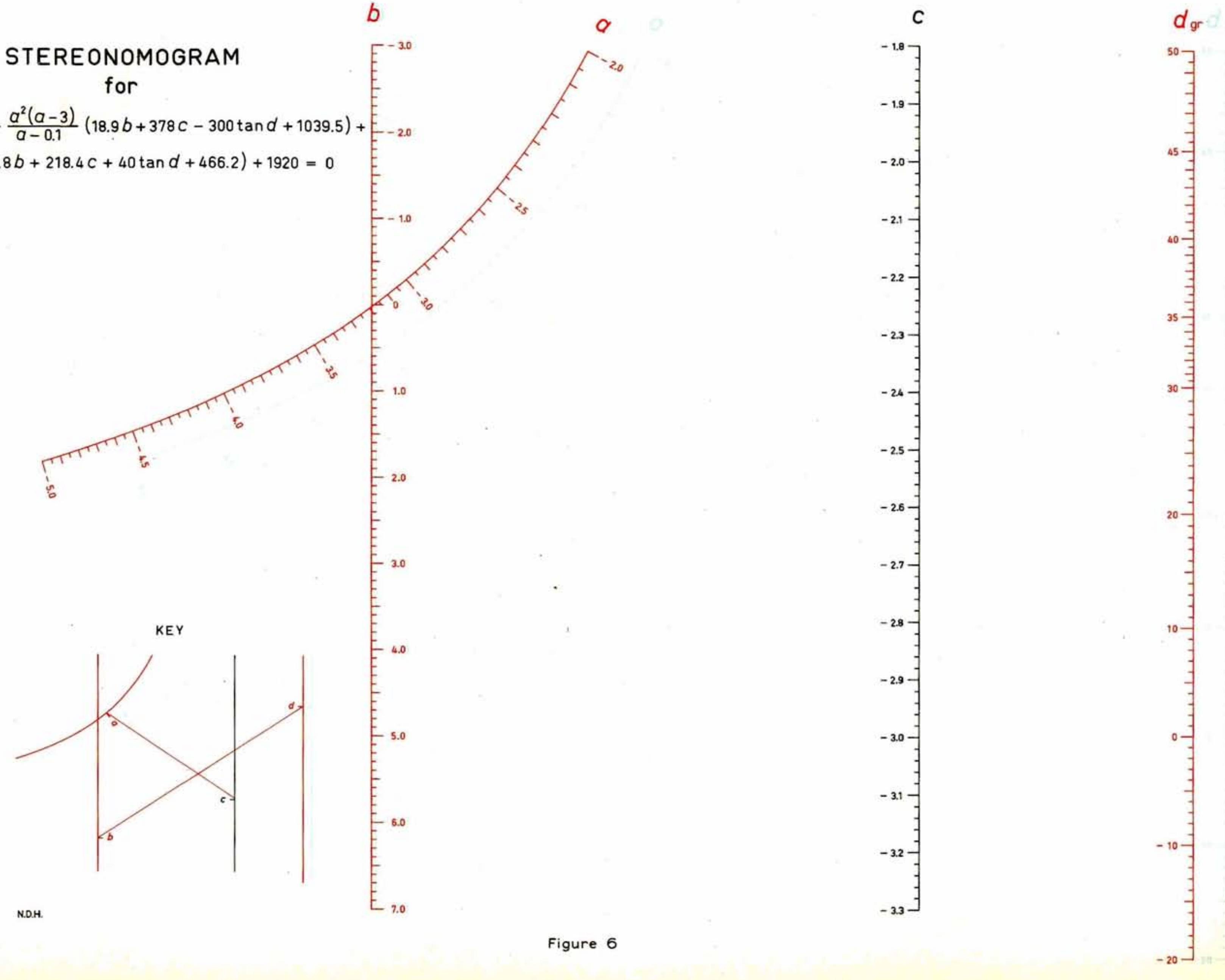
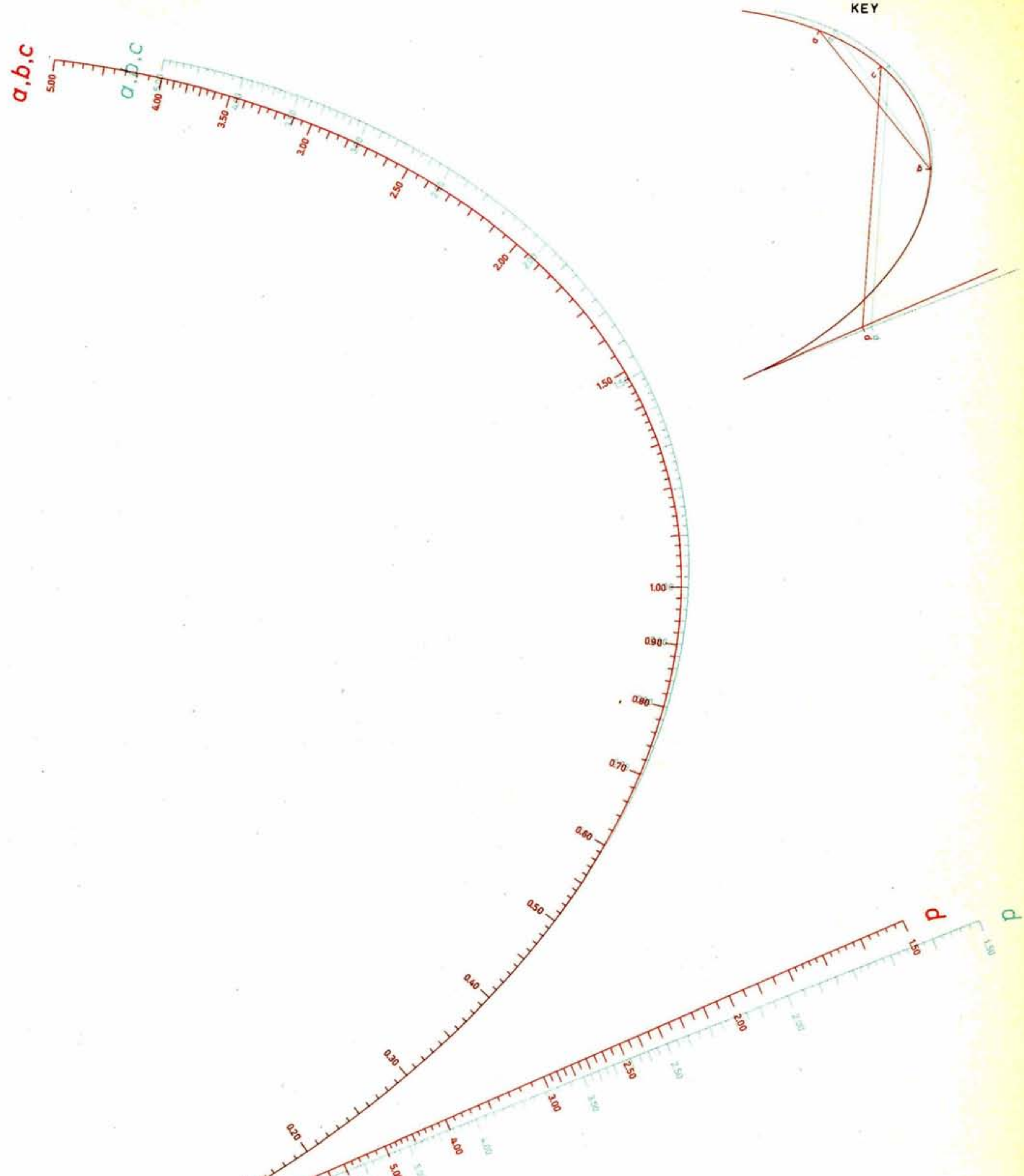


Figure 6



STEREONOMOGRAM
for
$$abcd - (0.05d + 1)(ab + ac + bc) - 10 = 0$$

N.D.H.

Figure 7

STEREONOMOGRAM
for
$abcd + 3(a+b+c+d) - 40 = 0$

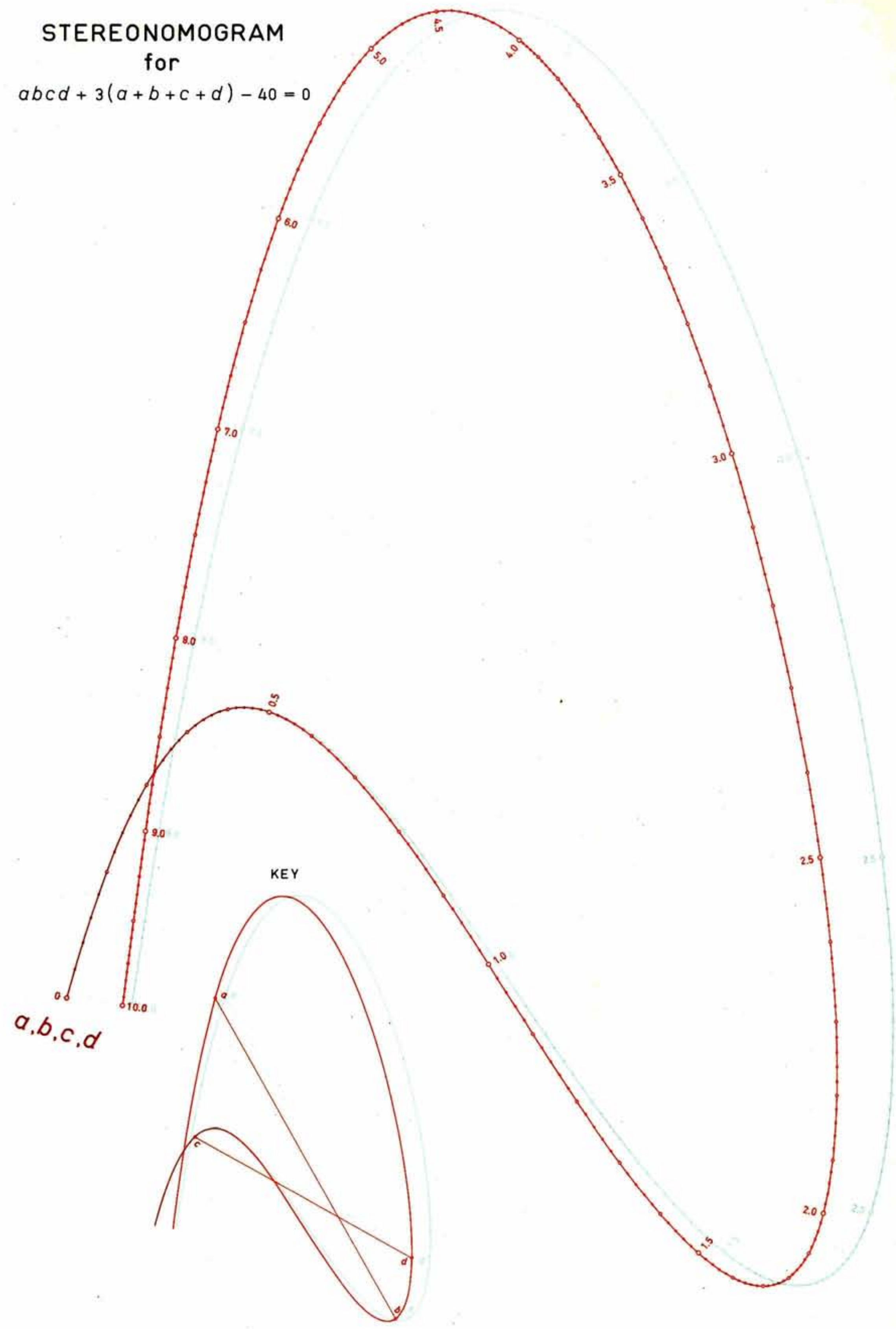


Figure 8

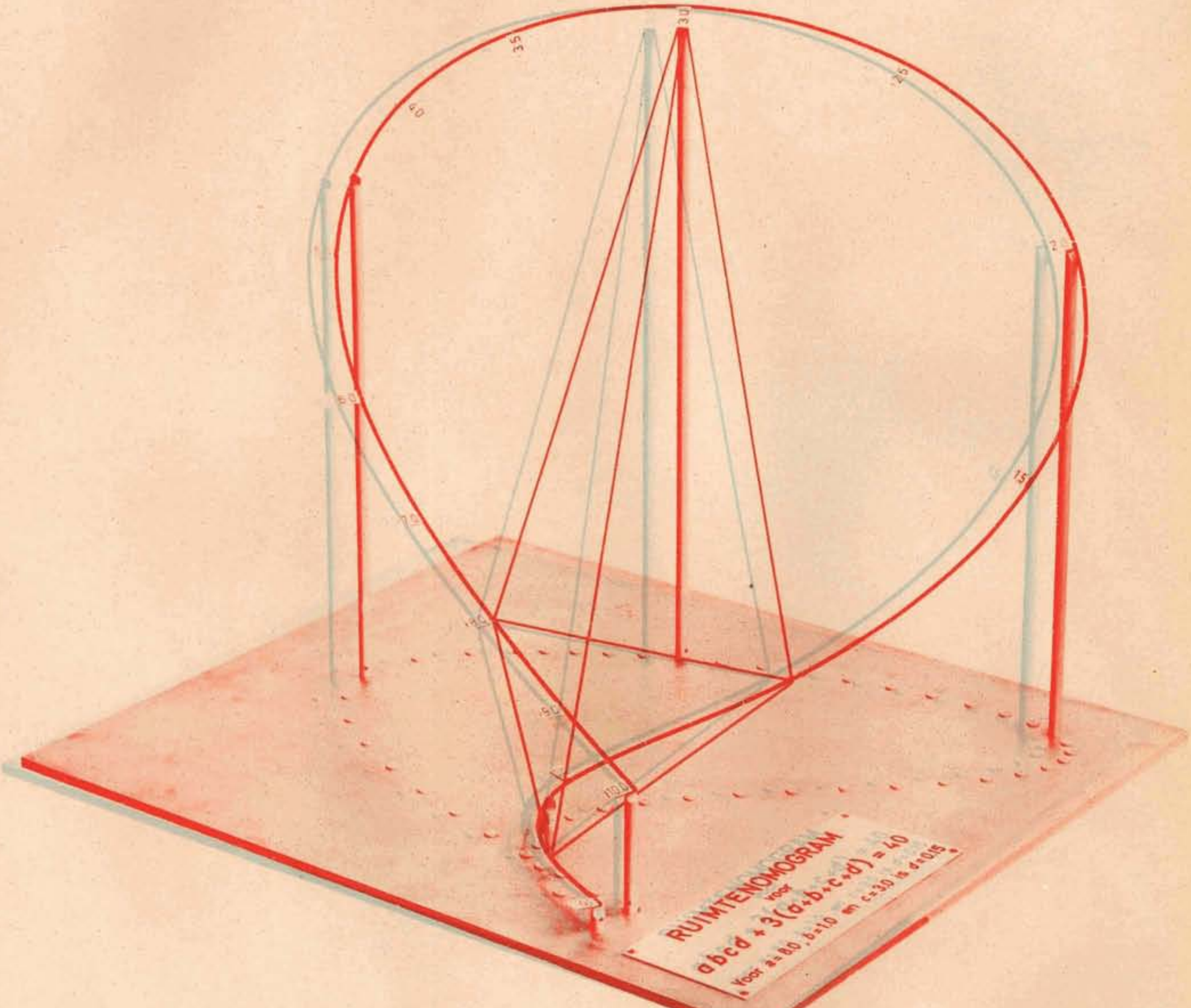

RUIMTENOMOGRAM
voor
abcd + 3(a+b+c+d) = 40
Voor a=8,0 , b=1,0 en c=3,0 is d=0,15
30
35
40
25

Figure 9