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QUATERNION ALGEBRA APPLIED TO POLYGON THEORY IN THREE DIMENSIONAL SPACE

by

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PREFACE

The present publication is the result of a study which started in 1969, and is based on a preliminary study made by Professor W. Baarda in the early sixties. It describes a functional model for the adjustment of spatial geodetic networks in which horizontal and vertical angles, distances and astronomical quantities (longitude, latitude and azimut) are measured.
I am not only very much indebted to Professor Baarda for the very stimulating and encouraging discussions we had already had during my student period, but especially in the later period when I had become a "practitoner", suffering from a growing distance between my daily environment and the field of science and research.
Furthermore, thanks are due to Mr. Brouwer and Dr. van Dalen, who gave many suggestions to improve the accessibility of the text to readers who are not familiar with quaternion equations, to Professor Alberda who was so kind to check and improve the translation, and of course to my wife who typed the manuscript, leaving exactly enough space for each of the about 1000 formulae.

H. Quee,
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CONTENTS

Chapter 0.

INTRODUCTION.

In this study a functional model for the adjustment of spatial geodetic networks is described. The model can be used both for terrestrial first order networks including astronomical observation variates and for engineering surveys, e.g. for the measurement of deformations, or for setting out high structures.

0.1 Starting points.

In this study, the use of quaternion algebra for the formulation of spatial functional relations and difference equations is of vital importance [14] . This choice is based on the historical background of the theory, and, in particular, on the way in which it could be linked with the methodological starting points formulated by W. Baarda within the scope of his two-dimensional polygon theory in the complex plane [2].
These starting points are used in the preceding theory and may be shortly summarized as follows :

- An exclusive use is made of operationally defined coordinate systems, or, in W. Baarda's terminology : S-coordinate systems [3].
- Observation variates are put together in so-called "form quantities", because the definition of a coordinate system has to be based on variables that are invariant in similarity-transformations. This means the use of distance-ratios, which requires an algebraic system in which division is defined.
- The model may not contain assumptions or artificial structures (e.g. an ellipsoid) necessitating "model-corrections" of observation variates.
- The adjustment model is set up primarily according to the method of condition equations, i.e. the "standard problem I" in J.M. Tienstra's terminology. It is true that the method of observation equations (standard problem II) is much more usual in international literature, but it is considered to be less satisfactory for the present theoretical investigation, because of the sometimes vague definition of unknown variates, the unsystematic way of introducing approximate values, and the greater risk of singularities slipping in [4].

In view of these starting points, quaternion algebra proves to be a magnificent and efficacious and indispensable tool. For vectors whose dimension is more than two, it is the only associative algebra in which division is defined and in which there are no zero-divisors (contrary to vector calculus).

It is true that the use of quaternion algebra for our purpose gives rise to some problems, but these are limited to the practical elaboration of the system of formulae :

- Because of the absence of the commutative property of multiplication, the formulae generally contain one extra factor (the relations themselves) or one extra term (the difference equations) in comparison with analogue formulae in the two-dimensional theory; for example, consider the transformation of system i to system r :

$$q^{(r)} = \lambda_{ri}\, p_{ri}\, q^{(i)} p_{ri}^{-1}$$

$$\Delta q^{(r)} = q^{(r)} \Delta \ln \lambda_{ri} + (\Delta p . p^{-1})_{ri}\, q^{(r)} - q^{(r)} (\Delta p . p^{-1})_{ri} + \lambda_{ri}\, p_{ri}\, \Delta q^{(i)} p_{ri}^{-1}$$

- The vectors described are three-dimensional, the spatial rotation has three independent parameters, however, quaternions have four components (with basis-units 1, i, j, k). This means that the first component of a quaternion is equal to zero, if the quaternion represents a vector :

  $$q_{ik} = 0 + i\, x_{ik} + j\, y_{ik} + k\, z_{ik} \qquad \text{(vector)}$$

  Chapter 3 shows that in dimensionless difference quantities (in which all four components $\neq 0$!) there is, both for observation variates and coordinate quantities a linear dependence between the components of the relative difference-variate :

  $$(q^{-1} \Delta q)_{ik}$$

  For the rotation-quaternion :

  $$p_{ri} = d_{ri} + i\, a_{ri} + j\, b_{ri} + k\, c_{ri}$$

  in which all four components $\neq 0$, the situation is even slightly more complicated. This is discussed in Chapter 1.
- In quaternion algebra, there is no complete "function theory", as is the case with complex numbers. Quaternion functions cannot be integrated, though they can be differentiated. This is sufficient for the purpose of this study.

In addition to the theoretical considerations concerning the choice of quaternion algebra, it is of great importance that here we have an elegant methodical structure for geodetic methods in all three dimensions : one-dimensional levelling, two-dimensional "plane surveying" and three-dimensional first order networks and networks for the construction of high buildings.

In [3] it has already been shown that a one-dimensional network is a "special case" of a two-dimensional network, as far as the structure of the adjustment model, and especially the description of the precision, are concerned. In the present study (see Chapter 4) it is shown that the two-dimensional structure is, in turn, a special case of the three-dimensional one. This uniformity of structure means that the overall system developed by the "Delft school" for the description of stochastic aspects is universally applicable to three-dimensional problems (internal and external reliability ; S-transformations and criterion matrices; the $\bar{\lambda}$-theory [5] , [6] , [3] , [7].

0.2 Historical background of the theory.

As early as 1960, hence a considerable time before the finalisation of the theory pertaining to the "polygon theory in the complex plane", Baarda concluded that quaternion algebra would be the most appropriate tool for the function model of three-dimensional polygon networks. Furthermore, he made an initial exploration of the practical elaboration [8] ; in these manuscripts some cardinal points of the system of formulae are solved, such as the use of isomorphy between quaternions and matrices, the definition of a spatial analogue for the two-dimensional Π-quantity, and, closely connected with it, the three-dimensional coordinate condition and its difference equation. Rotations are also briefly described: this aspect was worked out by E.Vermaat some years later [23] . In his graduation paper the present author worked out these studies to a provisional termination [18] . The model described there displays a number of "grey" spots: for example, the linear dependences within the condition model have not been obtained from algebraic analysis, but from computer-aided determination of the rank of matrices. Furthermore, the interpretation of a number of concepts and auxiliary quantities is "geometric" rather than "algebraic" in nature. The gravest shortcoming was the total absence of the transfer to S-coordinates. In the period elapsed since then, the theory has been completed and perfected. The main points studied were :

- the transfer to S-coordinates after adjustment by the method of condition equations, and the links between the transformation designed for this purpose and the general three-dimensional S-transformation, developed in the same period by W. Baarda and later by M. Molenaar [17] .
- the analysis of the linear dependences in the condition model.
- the analysis of the special position accorded to the first azimuth (see section 4.2).

As time went by, the progress of the investigation was slowed down more and more by the exigencies of the author's daily work, where he was, at first, mainly occupied with the implementation of the two-dimensional polygon theory, as developed by Baarda, in cartographic measurements and in engineering survey networks. Nevertheless, this practical environment and the study in the three-dimensional theory have had positive effects on each other. For example, there turned out to be a strong similarity between, on the one hand, the way in which horizontal orientation unknowns θ_i in networks with non-parallel first axes are transferred via observation variates from one side of the network to the next :

$$\theta_1 = A_{12} - r_{12}$$

$$\theta_2 = \epsilon\left(\theta_1, r_{12}, \zeta_{12}, \varphi_1, \lambda_{12}, \varphi_2\right) - r_{21}$$

$$\theta_3 = \theta\left(\theta_2, r_{23}, \zeta_{23}, \varphi_2, \lambda_{23}, \varphi_3\right) - r_{32} .$$

etc. (see 3.30 and 3.40).

On the other hand, the way in which the initial arguments φ_i of interlinked alignment elements (i) of a track depends upon the initial argument φ_A and the angles ϕ of alignment elements :

$$\varphi_1 = \varphi_A$$

$$\varphi_2 = \varphi_1 + \phi\left(\cdots p_1^j \cdots\right) .$$

$$\varphi_3 = \varphi_2 + \phi\left(\cdots p_2^j \cdots\right) .$$

etc.

(p_i^j : parameters of the alignment element i, etc;
$\phi = 0$ for straight element)

The recognition of this agreement led, in 1974, to the development of an original practical algorithm for the automated solution of alignment equations from conditions, such as the constraint condition [19] .
In addition, a profound study in the years 1977-1978, leading to a geodetic system for the control of automatic track maintenance machines, clearly showed that particularly the theory of the S-transformations is indispensable in the formulation of purpose dependent standards for practical geodetic activities [20] .
This example refers to the complex plane; as soon as a three-dimensional measuring process is used in the case of setting out, or deformation-measurements of high civil engineering structures the same applies there, and a good functional model is indispensable.

0.3 Practical applications.

The model described here is the missing link in an operational theory for terrestrial-geodetic networks. Here, we must make a distinction between two fields of application, each with its own theoretical and practical problems: the first order geodetic networks (slightly inclined "plane" networks of which the points are more than 10 km spaced apart, and with astronomical orientation of local systems) and the networks for the determination of deformation and for setting out of high buildings and bridges (small networks with great differences in height; the direction of the local gravity and thus the first axis of the theodolite are considered to be parallel).
The problems involved in the conventional procedures in the first order networks are clearly outlined in some papers by W. Baarda [9] : the necessary corrections of observation variates; the regional adaptation of ellipsoids; the units of length, which cannot be equal to the instrumental units of length; the problems encountered during the connection of these networks; the vague determination of the third dimension and the inaccurate definition of the so-called Laplace equation.
It would seem possible to solve part of these problems by the addition of zenith angles and the determination of longitude and latitude in all (or most of the) stations, and also by the measurement of distances; in accordance with this procedure some test networks have been measured since 1965, particularly in mountainous areas in Germany and Switzerland [22], [21], [12].
However, with regard to these test networks, it becomes apparent from publications that the procedure chosen does not comply with the starting points formulated in this introduction: the adjustment is not done in a "S-system",only the method of observation equations is used, and the use of distance ratios is left out of consideration altogether.

Regarding the second field of application, that of small networks for civil engineering problems, only a small number of publications is available. This may well be caused by the fact that in practical geodesy, confronted with "spatial objects", no three-dimensional measurement procedure is chosen, (one might choose spatial radius vectors, possibly supplemented by measurement of some height-differences per floor or storey), but the problem is split up into a two-dimensional procedure for the planimetry and a one-dimensional procedure for the differences in height. This may lead to very complicated problems in the implementation of the measurements, the horizontal position of the higher storeys being defined very poorly. An example illustrating this is described in [11] .

0.4 Suggestions for further research.

This study only covers the description of a functional model for three-dimensional terrestrial networks, in which optional astronomical observation variates are admitted.
Further studies, focused on practical applications are required on, inter alia, the following problems :
0.4.1.
After the model has been programmed, it will be possible, with the aid of other computer programs of the Department of Geodesy of the Delft University of Technology, designated by the collective name SCAN, to study the optimal construction of networks for the two fields of application.
0.4.2.
The interaction between zenith angles and the astronomical observation variates and also the effects of all these observation variates on precision and on the internal and external reliability.
0.4.3.
What is the relation between vertical refraction and the so-called Z-conditions in the sides of the network, arising from the direct and reverse measurement of zenith angles ?
(see also [13], [10], [1] for the problems encountered in measuring zenith angles).
0.4.4.
In chapter 2.3 is suggested, to choose a measuring procedure in which astronomical longitudes (and possibly latitudes) are measured simultaneously in each pair of stations, in order to eliminate the influence of star coordinates and polar motion, and to reduce the influence of time. This has to be elaborated further, both practically and theoretically.
0.4.5.
In "engineering survey networks", the direction of the vertical (the first axis of the theodolite) is not determined by astronomical observations.
In which cases it is to be preferred to introduce two unknowns for the direction of the vertical in every networkpoint; in which cases is it possible to start from the assumption that these are all parallel to each other ? How should the network be designed in these various situations ?

In all these problems, the purpose of the network, and especially the question whether the "vertical component" (perpendicular to the earth surface) is by itself significant or only serves to improve the "horizontal component", play an important role.

0.5 Guide lines for the reader.

In Chapter 1 the algebraic apparatus is described : arithmetic procedures; the geometrical interpretation; rotations, difference equations and isomorphic matrices.
In Chapter 2 the introduction of terrestrial and geodetic-astronomical

observation variates is described.
In Chapter 3 the fundamental quantities described in Chapter 2, are linked to more complex structures: successive rotations, vector rotations. Subsequently, the first linear dependency is derived and inverse functions are established (differences of observation variates, expressed in differences of coordinate quantities). Finally the transfer of orientation unknowns θ_i is discussed, and, simultaneously, that of the length factors.
Chapter 4 deals with three important differences between the three-dimensional and the two-dimensional model; these differences are caused by the fact that the quaternion quotient :

$$\varphi_{jik} = q_{ik}\, q_{ij}^{-1}$$

is not fully invariant in similarity transformations, contrary to the analogue quantity :

$$\underline{\Delta\Pi}_{jik} = \frac{\ln z_{ik}}{\ln z_{ij}}$$

in the two-dimensional model. The differences referred to concern :

- The role of the orientation unknowns and the first azimuth.
- The fact that the relations must be established in one of the local systems, and the effects thereof on estimators and weight coefficients of observation variates.
- The introduction of S-coordinates by the inclusion of the stochastic "basis transformation" $\underline{p}_{Rr}$; $\underline{\bar{\lambda}}_{Rr}$

  When using the adjustment method of condition equations, this transformation is entered in the formula by which, after adjustment, coordinate quantities $\underline{q}^{(R)}$ are computed from the estimators $\underline{X}^i$ of observation variates :

$$\underline{q}^{(R)} = \underline{\bar{\lambda}}_{Rr}\, \underline{p}_{Rr}\, \underline{q}^{(r)}\, \underline{p}_{Rr}^{-1}$$

with :

$\underline{q}^{(R)}$: "S-coordinates"

$\underline{q}^{(r)} = q(\ldots,\underline{X}_i,\ldots)$

$\underline{\bar{\lambda}}_{Rr} = \bar{\lambda}_{Rr}(\ldots,\underline{X}_i,\ldots)$

$\underline{p}_{Rr} = p_{Rr}(\ldots,\underline{X}_i,\ldots)$

When using the method of observation equations, the basis transformation, in the form of four unknowns, is entered in the correction equations.

Regarding this chapter, the method of observation equations seems to be less sensible for these complications, so it may be once more concluded that this method is theoretically weaker than the method of condition equations, because it may be applied on the basis of a much more superficial analysis, thus involving the risk that the model is incomplete or incorrect.
This chapter also considers the numbers of quantities and condition equations in the adjustment model for a closed polygon.
Finally in Chapter 5 the condition model is given, starting from W. Baarda's theory for the complex plane and building on the conclusions in the Chapters 3 and 4. It becomes apparent that the structure remains strongly affiliated

with that in the complex plane, be it that there are more types of observation variates and more types of conditions in it and that the relations between the conditions are considerably more complicated. Finally, the correction equations for the adjustment model of observation equations are established.

Chapter 1

QUATERNION ALGEBRA.

1.1 Units and definitions.

Quaternion algebra was formulated about 1843 by W.R.Hamilton [14]. It is a hypercomplex algebra with four base elements

$1\ ,\ i\ ,\ j\ ,\ k\ .$

As in algebra with complex numbers, the following applies :

$$\begin{aligned} 1\,1 &= 1 \\ i\,i &= -1 \\ j\,j &= -1 \\ k\,k &= -1 \end{aligned} \qquad (1.1)$$

The scalar unit 1 is an inactive operand in multiplications by the three others :

$$\begin{aligned} 1\,i &= i\,1 = i \\ 1\,j &= j\,1 = j \\ 1\,k &= k\,1 = k \end{aligned} \qquad (1.2)$$

The three "imaginary" units generate each other in accordance with cyclic multiplication rules :

$$\begin{aligned} i\,j &= k \\ j\,k &= i \\ k\,i &= j \end{aligned} \qquad (1.3^a)$$

They are non-commutative :

$$\begin{aligned} j\,i &= -k \\ k\,j &= -i \\ i\,k &= -j \end{aligned} \qquad (1.3^b)$$

A quaternion $\mathcal{Q}$ has four base components, e.g. w, x, y and z :

$$\mathcal{Q} = w + i\,x + j\,y + k\,z \qquad (1.4)$$

We introduce the following terms and notations :

$$\left.\begin{array}{ll}
\text{-the scalar part of Q :} & Sc\{Q\} = w \\
\text{-the vector part of Q :} & Vc\{Q\} = ix+jy+kz \\
\text{-the i-component of Q :} & Vi\{Q\} = x \\
\text{-the j-component of Q :} & Vj\{Q\} = y \\
\text{-the k-component of Q :} & Vk\{Q\} = z
\end{array}\right\} \qquad (1.5)$$

$$\text{-the norm of Q :} \quad N\{Q\} = w^2+x^2+y^2+z^2 \longrightarrow \qquad (1.6)$$

Hence :

$$\mathcal{Q} = Sc\{\mathcal{Q}\} + Vc\{\mathcal{Q}\} =$$

$$= Sc\{\mathcal{Q}\} + i\,Vi\{\mathcal{Q}\} + j\,Vj\{\mathcal{Q}\} + k\,Vk\{\mathcal{Q}\}. \qquad (1.7)$$

1.2 Addition, subtraction, multiplication and division.

We consider two quaternions :

$$\mathcal{Q}_1 = w_1 + i\,x_1 + j\,y_1 + k\,z_1$$

$$\mathcal{Q}_2 = w_2 + i\,x_2 + j\,y_2 + k\,z_2$$

The sum and the difference of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are then defined as :

$$\mathcal{Q}_1 + \mathcal{Q}_2 = w_1 + w_2 + i\,[x_1+x_2] + j\,[y_1+y_2] + k\,[z_1+z_2].$$

$$\mathcal{Q}_1 - \mathcal{Q}_2 = w_1 - w_2 + i\,[x_1-x_2] + j\,[y_1-y_2] + k\,[z_1-z_2].$$

Multiplication by a scalar a gives :

$$a\mathcal{Q}_1 = a w_1 + i\,ax_1 + j\,ay_1 + k\,az_1.$$

Applying the rules of multiplication (1.1), (1.2) and (1.3), the product of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ becomes :

$$\begin{aligned}
\mathcal{Q}_1\mathcal{Q}_2 = {} & w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 + \\
& + i\,[w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2] + \\
& + j\,[w_1y_2 + y_1w_2 + z_1x_2 - x_1z_2] + \\
& + k\,[w_1z_2 + z_1w_2 + x_1y_2 - y_1x_2]
\end{aligned} \qquad (1.8)$$

Now it becomes directly apparent that the product is non-commutative :

$$\mathcal{Q}_1\mathcal{Q}_2 \neq \mathcal{Q}_2\mathcal{Q}_1.$$

However, the following still applies :

$$\mathrm{Sc}\{\mathcal{Q}_1\mathcal{Q}_2\} = \mathrm{Sc}\{\mathcal{Q}_2\mathcal{Q}_1\}. \tag{1.9}$$

The product is commutative, if the vector components of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are "parallel", or if :
(with a and b being scalars)

$$a\,\mathrm{Vc}\{\mathcal{Q}_1\} + b\,\mathrm{Vc}\{\mathcal{Q}_2\} = 0 \quad \text{than :} \quad \mathcal{Q}_1\mathcal{Q}_2 = \mathcal{Q}_2\mathcal{Q}_1 \tag{1.10}$$

We define the "conjugate" of $\mathcal{Q}$ as :

$$\mathcal{Q}^T = \mathrm{Sc}\{\mathcal{Q}\} - \mathrm{Vc}\{\mathcal{Q}\} \tag{1.11}$$

therefore :

$$(\mathcal{Q}^T)^T = \mathcal{Q}. \tag{1.12}$$

It also follows from (1.8) :

$$(\mathcal{Q}_1\mathcal{Q}_2)^T = \mathcal{Q}_2^T\mathcal{Q}_1^T \tag{1.13}$$

Further, see (1.5) and (1.6) :

$$\begin{aligned}\mathcal{Q}\mathcal{Q}^T = \mathcal{Q}^T\mathcal{Q} &= \left[\mathrm{Sc}\{\mathcal{Q}\} + \mathrm{Vc}\{\mathcal{Q}\}\right]\left[\mathrm{Sc}\{\mathcal{Q}\} - \mathrm{Vc}\{\mathcal{Q}\}\right] = \\ &= \mathrm{Sc}^2\{\mathcal{Q}\} - \mathrm{Vc}^2\{\mathcal{Q}\} = \\ &= w^2 + x^2 + y^2 + z^2 = \\ &= N\{\mathcal{Q}\}.\end{aligned} \tag{1.14}$$

This means :

$$\frac{\mathcal{Q}\mathcal{Q}^T}{N\{\mathcal{Q}\}} = \frac{\mathcal{Q}^T\mathcal{Q}}{N\{\mathcal{Q}\}} = 1.$$

so, by definition, the inverse of $\mathcal{Q}$ reads :

$$\boxed{\mathcal{Q}^{-1} = \frac{\mathcal{Q}^T}{N\{\mathcal{Q}\}} \quad \rightarrow \quad \mathcal{Q}\mathcal{Q}^{-1} = \mathcal{Q}^{-1}\mathcal{Q} = 1.} \tag{1.15}$$

or :

$$\mathcal{Q}^{-1} = \frac{1}{w^2+x^2+y^2+z^2}\left[w - i\,x - j\,y - k\,z\right]$$

Remark :
Because of (1.6), it follows from N $\{\mathcal{Q}\} = 0$ that $\mathcal{Q} = 0$; therefore the inverse of $\mathcal{Q}$ is always defined, except when $\mathcal{Q} = 0$.
Consequently, in quaternion algebra there occur no zero divisors.

Because furthermore, with (1.15) :

$$\mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_2^{-1} \mathcal{Q}_1^{-1} = 1$$

the following applies by definition :

$$\mathcal{Q}_2^{-1} \mathcal{Q}_1^{-1} = [\mathcal{Q}_1 \mathcal{Q}_2]^{-1} \qquad (1.16)$$

This can be extended to products with more than two factors ; suppose :

$$\mathcal{R} = \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3$$

then, with (1.16) :

$$\begin{aligned} \mathcal{R}^{-1} &= \{[\mathcal{Q}_1 \mathcal{Q}_2]\mathcal{Q}_3\}^{-1} = \\ &= \mathcal{Q}_3^{-1} [\mathcal{Q}_1 \mathcal{Q}_2]^{-1} = \\ &= \mathcal{Q}_3^{-1} \mathcal{Q}_2^{-1} \mathcal{Q}_1^{-1} . \end{aligned} \qquad (1.17)$$

1.3 A geometrical interpretation of quaternions.

The imaginary units i, j and k may be regarded as unit vectors in R_3, composing together a right-handed trirectangular trihedral (see fig.1)

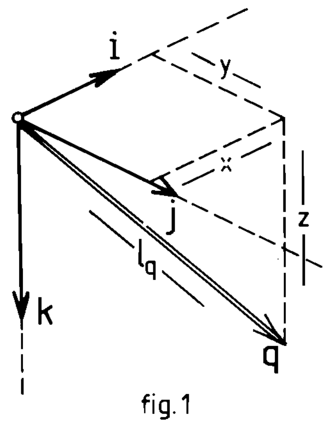


fig.1

Then a quaternion q with scalar part :

$$Sc\{q\} = 0$$

becomes a vector in R_3 :

$$q = 0 + ix + jy + kz .$$

From (1.6) it follows that :

$$\begin{aligned} \sqrt{N\{q\}} &= \sqrt{x^2 + y^2 + z^2} = \\ &= \ell_q \ : \ \text{"length" of } q \end{aligned} \qquad (1.18)$$

1.3.1
The geometrical significance of the quaternion quotient.

We consider two quaternions q_1 and q_2 whose scalar parts vanish :

$$q_1 = 0 + ix_1 + jy_1 + kz_1$$

$$q_2 = 0 + ix_2 + jy_2 + kz_2$$

Then, according to (1.8) the product of q_1 and q_2 is :

$$\begin{aligned} q_1 q_2 = &- x_1 x_2 - y_1 y_2 - z_1 z_2 + i[y_1 z_2 - z_1 y_2] + \\ &+ j[z_1 x_2 - x_1 z_2] + [x_1 y_2 - y_1 x_2] = \end{aligned}$$

$$= -x_1 x_2 - y_1 y_2 - z_1 z_2 + \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \qquad (1.19)$$

Because we consider x_1, y_1, and z_1, respectively x_2, y_2 and z_2 as components of two vectors in a rectangular cartesian coordinate system in R_3, the laws of "vector analysis" can be applied to (1.19), so :

"scalar product" :

$$x_1 x_2 + y_1 y_2 + z_1 z_2 = \underline{q}_1 \cdot \underline{q}_2 = \ell_1 \ell_2 \cos \bar{\alpha}$$

"vector product" :

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = q_1 \times q_2 = \underline{e}\, \ell_1 \ell_2 \sin \bar{\alpha} \qquad (1.20)$$

Here $\bar{\alpha}$ is the angle between two vectors and e is the unit normal vector on the plane through the two vectors, which, because the x-, y-, z-system is a right-handed trihedral system, fits in with the sense of rotation of q_1 to q_2.

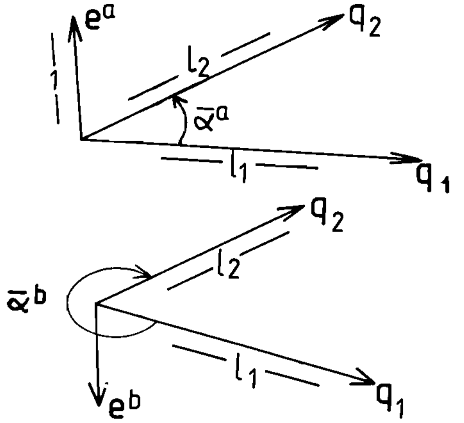


fig. 2

There are two possibilities : (see fig.2)

a : reversed sense of rotation $q_1 \rightarrow q_2$:

$e = e^a$ "upwards"

$\bar{\alpha} = \bar{\alpha}^a$

b : clock wise sense of rotation $q_1 \rightarrow q_2$:

$e = e^b$ "downwards" $(e^b = -e^a)$

$\bar{\alpha} = \bar{\alpha}^b = 2\pi - \bar{\alpha}^a$

In view of (1.19) and (1.20), the product of q_1 and q_2 is :

$$\begin{aligned} q_1 q_2 &= -\ell_1 \ell_2 \cos \bar{\alpha}^a + e^a \ell_1 \ell_2 \sin \bar{\alpha}^a = \\ &= -\ell_1 \ell_2 \cos \bar{\alpha}^b + e^b \ell_1 \ell_2 \sin \bar{\alpha}^b \end{aligned} \qquad (1.21)$$

$$N\{e^a\} = N\{e^b\} = 1.$$

According to (1.15) and (1.11) :

$$q_2^{-1} = \frac{q_2^T}{N\{q_2\}} = \frac{-q_2}{\ell_2 \ell_2}$$

If, in (1.21) q_2 is replaced by q_2^{-1}, (1.21) passes consequently into :

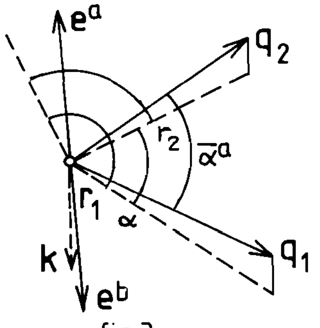


fig.3

$$q_1 q_2^{-1} = \frac{\ell_1}{\ell_2} \cos \bar{\alpha}^a - e^a \frac{\ell_1}{\ell_2} \sin \bar{\alpha}^a \qquad (1.22)$$

Here the length-ratio

$$v = \frac{\ell_1}{\ell_2}$$

comes into play, consequently :

$$\ln v = \ln \ell_1 - \ln \ell_2 .$$

We now follow the pattern of Baarda's "Polygon theory in the complex plane" [2] and define the angle α in the horizontal plane as the difference of two directions r :

$$\alpha = r_1 - r_2$$

The angle $\bar{\alpha}$ (in the plane of q_1 and q_2) that fits with this choice is :

$$\bar{\alpha} = \bar{\alpha}^a$$

Since the graduations of the horizontal circle of a theodolite are numbered clockwise (seen from above), a positive rotation on this circle is right-handed and fits in with e^b. Replacing in (1.22)e^a by e^b one obtains :

$$\boxed{\begin{array}{l} q_1 q_2^{-1} = v \cos \bar{\alpha} + e^b v \sin \bar{\alpha} \\ \text{with: } \; v = \dfrac{\ell_1}{\ell_2} \quad ; \; \bar{\alpha} \approx r_1 - r_2 . \end{array}} \qquad (1.23)$$

The approximate equality for $\bar{\alpha}$ in (1.23) is only valid when q_1 and q_2 are near-horizontal.

From (1.23) it becomes apparent that the four components of the quaternion quotient $q_1 q_2^{-1}$ determine the shape of a triangle and also describe the spatial position of the plane of that triangle;

Suppose :

$$q_1 q_2^{-1} = D + iA + jB + kC$$

$$e = 0 + i n_1 + j n_2 + k n_3 \quad ; \quad n_1^2 + n_2^2 + n_3^2 = 1$$

consequently :

$$\left.\begin{array}{l}A\\B\\C\\D\end{array}\right\}\text{ govern }\left\{\begin{array}{l}\left.\begin{array}{l}v\\\bar{\alpha}\end{array}\right\}\text{ the "shape" of the triangle}\\ \left.\begin{array}{l}n_2\\n_3\end{array}\right\}\text{ two out of the three components of the unitnormalvector}\end{array}\right.$$

The computation rules for v, $\bar{\alpha}$, n_2 en n_3 are :

$$V = \sqrt{D^2 + A^2 + B^2 + C^2}$$

$$\cos\bar{\alpha} = \frac{D}{V}$$

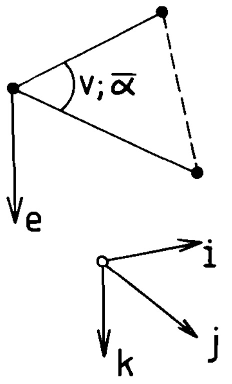


fig. 4

If q_1 and q_2 lie approximately in the i-, j- plane of the system of coordinates and e "points downwards" (i.e. the theodolite is not upside down) the following applies : (see fig. 4)

$$n_3 \approx +1$$

then $\bar{\alpha}$ must be chosen such that :

$$\text{sign}\{\sin\bar{\alpha}\} = \text{sign}\{C\}$$

$$n_1 = \frac{A}{V\sin\bar{\alpha}} \quad ; \quad n_2 = \frac{B}{V\sin\bar{\alpha}} \quad ; \quad n_3 = \frac{C}{V\sin\bar{\alpha}}$$

1.3.2

Decomposition into orthogonal components.

We consider the quaternions :

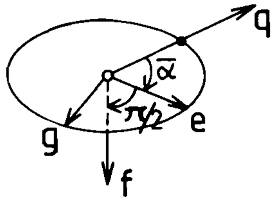


fig. 5

$$q \; ; \quad Sc\{q\} = 0 \; ; \quad N\{q\} = \ell_q^2$$

$$e \; ; \quad Sc\{e\} = 0 \; ; \quad N\{e\} = 1$$

According to (1.23) we obtain : (see fig. 5)

$$e\,q^{-1} = \frac{1}{\ell_q}\left[\cos\bar{\alpha} + f\sin\bar{\alpha}\right] .$$

Here, f is the unit normal vector on the plane through e and q, so :

$$N\{f\} = 1 \; ; \quad Sc\{f\} = 0 \; ; \quad \rightarrow \; f^{-1} = -f$$

therefore :

$$[e\,q^{-1}]^{-1} = \ell_q\left[\cos\bar{\alpha} - f\sin\bar{\alpha}\right] \; \rightarrow$$

$$e[e\,q^{-1}]^{-1} = \ell_q\left[e\cos\bar{\alpha} - ef\sin\bar{\alpha}\right] =$$

$$= \ell_q\left[e\cos\bar{\alpha} + ef^{-1}\sin\bar{\alpha}\right] .$$

According to (1.23) :

$$ef^{-1} = \cos\frac{\pi}{2} + g\sin\frac{\pi}{2} =$$

$$= g \cdot$$

Here g is the unit normal vector on the plane through e and f, so g lies in the plane of e and q (see fig. 6);

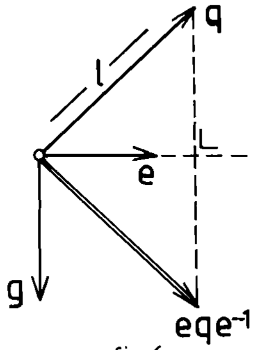


fig.6

Consequently :

$$e[eq^{-1}]^{-1} = \ell[e\cos\bar{\alpha} + g\sin\bar{\alpha}]$$

or :

$$\boxed{\begin{array}{l} eqe^{-1} = \ell_q[e\cos\bar{\alpha} + g\sin\bar{\alpha}] \\ \quad\quad \text{is the "mirror image" of } q \\ \quad\quad \text{in relation to } e \end{array}} \qquad (1.24)$$

Remark : $eqe^{-1} = e^{-1}qe$

This means : (see fig. 7)

$$\left.\begin{array}{ll} \bar{q} = \frac{1}{2}[q - e^{-1}qe] : & \text{is the component of } q \\ & \text{perpendicular to } e \\ \bar{\bar{q}} = \frac{1}{2}[q + e^{-1}qe] : & \text{is the component of } q \\ & \text{parallel to } e \end{array}\right\} \quad (1.25)$$

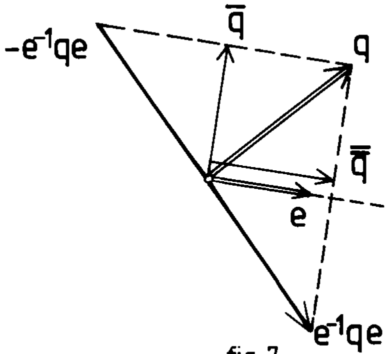


fig.7

Remark :
Instead of the unit normal vector e used here, a vector d with N {d} ≠ 1 can also be used in (1.24) and (1.25).

1.3.3
Rotations.

We wish to rotate a quaternion (vector) q with Sq {q} = 0 over an angle e about an axis (vector) e ; θ is a right-handed rotation with respect to e.

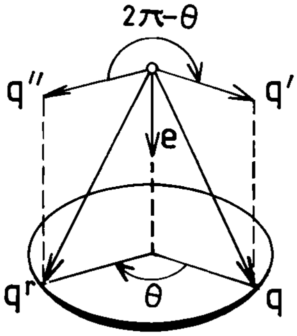


fig.8

Assume : (see fig.8)

q^r is the vector after rotation
q' is the component of q ⊥ e
q" is the component of q^r⊥ e

From (1.25) it follows that :

$$q' = \frac{1}{2}[q - e^{-1}qe] \qquad (1.26)$$

$$q'' = \frac{1}{2}[q^r - e^{-1}q^r e] \qquad (1.27)$$

According to (1.23) and because N {q'} = N {q"}, so v=1, it applies that :

$$q'' q'^{-1} = \cos\theta + e \sin\theta$$
$$\downarrow$$
$$q'' = [\cos\theta + e \sin\theta]\, q'$$

We substitute (1.26) in the right-hand member and (1.27) in the left-hand member of this equation :

$$\tfrac{1}{2}\left[q^r - e^{-1} q^r e\right] = \left[\cos\theta + e \sin\theta\right] \tfrac{1}{2}\left[q - e^{-1} q\, e\right] \qquad \text{I}$$

The components of q and q^r parallel to e are equal to each other; therefore, see(1.25) :

$$\tfrac{1}{2}\left[q^r + e^{-1} q^r e\right] = \tfrac{1}{2}\left[q + e^{-1} q\, e\right] \qquad \text{II}$$

The addition of the equations I and II now leads to :

$$q^r = \left[\cos\tfrac{1}{2}\theta + e \sin\tfrac{1}{2}\theta\right] q \left[\cos\tfrac{1}{2}\theta - e \sin\tfrac{1}{2}\theta\right] \qquad \text{III}$$

Now assume that the "rotation quaternion" p is defined as :

$$\boxed{p = \cos\tfrac{1}{2}\theta + e \sin\tfrac{1}{2}\theta} \qquad (1.28)$$

then III becomes the general rotation formula of quaternion algebra :

$$\boxed{q^r = p\, q\, p^{-1}} \qquad (1.29)$$

Two important properties apply here :

$$N\{q^r\} = N\{q\}.$$
$$Sc\{q^r\} = Sc\{q\}. \qquad (1.30)$$

In (1.29) the norm of a rotation quaternion need not equal unity. To show this, let h be a scalar, let N {p} = 1 and define :

$$\bar{p} = h\, p$$
$$\downarrow$$
$$\bar{p}^{-1} = \frac{1}{h} p^{-1}$$
$$\downarrow$$
$$q^r = \bar{p}\, q\, \bar{p}^{-1} =$$
$$= p\, q\, p^{-1}. \qquad (1.31)$$

From the derivation of the rotation quaternion it follows that the four components comply with the following "form-rule" :

$$\bar{p} = h\left[d + i\,s a + j\,s b + k\,s c\right]$$

$$\text{with :} \quad a^2 + b^2 + c^2 = 1. \qquad \text{I} \qquad (1.32)$$

$$d^2 + s^2 = 1 \qquad \text{II}$$

This will prove important for the differentiation of rotation quaternions, because also

$$p + \Delta p$$

must of course comply with (1.32).

The expression (1.29) can also be used for the description of a rotation of the coordinate system over an angle θ about an axis e :

$$\left[i^1, j^1, k^1\right] \xrightarrow[e]{\theta} \left[i^2, j^2, k^2\right]$$

A rotation of the coordinate system over an angle θ about an axis e is, in fact, equivalent to the rotation of the vectors over $-\theta$ about e, so, with the following rotation quaternion :

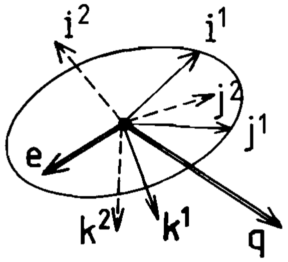


fig.9

$$p = \cos \tfrac{1}{2}\theta - e \sin \tfrac{1}{2}\theta \qquad (1.32^1)$$

Let the vector q be described on two different systems (1) and (2) : (see fig. 9)

$$q^{(1)} = w^1 + i^1 x^1 + j^1 y^1 + k^1 z^1$$

$$q^{(2)} = w^2 + i^2 x^2 + j^2 y^2 + k^2 z^2$$

Then, introducing the notation $p = p_{21}$ for the rotation quaternion transforming $q^{(1)}$ into $q^{(2)}$, the rotation formula is :

$$\boxed{\begin{array}{l} q^{(2)} = p_{21}\, q^{(1)}\, p_{21}^{-1} \\ \text{with : } p_{21} = \cos \tfrac{1}{2}\theta - e \sin \tfrac{1}{2}\theta \end{array}} \qquad (1.33)$$

The coordinate system rotates about e; therefore :

$$e^{(1)} = e^{(2)}$$

which means that the rotation quaternion p_{21} itself is invariant relative to the rotation of system (1) to system (2). This also becomes apparent from :

$$p_{21}\, p_{21}^{(1)}\, p_{21}^{-1} = p_{21}^{(2)} =$$

$$= p_{21} \,. \qquad (1.34)$$

And, since from the definition of the rotation quaternion it directly follows that

$$p_{12} = p_{21}^{-1} \quad ; \quad p_{21} = p_{12}^{-1} \qquad (1.35)$$

it also applies that :

$$p_{12}\, p_{21}^{(2)}\, p_{12}^{-1} = p_{21}^{-1}\, p_{21}^{(2)}\, p_{12}^{-1} =$$

$$= p_{12}^{-1} =$$

$$= p_{21}^{(1)} . \qquad (1.36)$$

We can combine (1.34) and (1.36) to :

$$p_{21}^{(1)} = p_{21}^{(2)}$$

$$p_{12}^{(1)} = p_{12}^{(2)} \qquad (1.37)$$

This implies that the components of a rotation quaternion apply to both systems, whose rotation relative to each other is described by that quaternion.

1.3.4
Successive rotations.

On what system should the rotation quaternion be described in the case of successive rotations of the system ?
We first consider two successive vector rotations of the vector q : (see fig. 10)

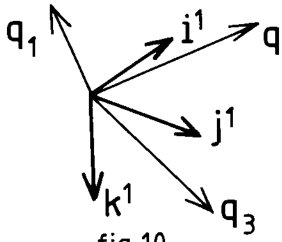


fig.10

$$q_1 \xrightarrow{\bar{p}} q_2 \xrightarrow{\bar{\bar{p}}} q_3 .$$

The vectors q_1, q_2 and q_3 and the rotations $\bar{p}$ and $\bar{\bar{p}}$ are all described relative to the system:

$$[i^1, j^1, k^1] .$$

According to (1.28) and (1.29) the rotation formulae are :

First step : $q_2^{(1)} = \bar{p}\; q_1^{(1)}\; \bar{p}^{-1}$

with : $\bar{p} = \cos \frac{1}{2}\bar{\theta} + \bar{e} \sin \frac{1}{2}\bar{\theta}$. (1.38)

Second step : $q_3^{(1)} = \bar{\bar{p}}\; q_2^{(1)}\; \bar{\bar{p}}^{-1}$

with : $\bar{\bar{p}} = \cos \frac{1}{2}\bar{\bar{\theta}} + \bar{\bar{e}} \sin \frac{1}{2}\bar{\bar{\theta}}$ (1.39)

Subsequently, we consider the two opposite rotations of the system :

$$[i^1, j^1, k^1] \xrightarrow{p_{21}} [i^2, j^2, k^2] \xrightarrow{p_{32}} [i^3, j^3, k^3] .$$

Now the following must apply :

$$q_3^{(3)} = q_2^{(2)} = q_1^{(1)} \qquad (1.40)$$

According to (1.33) the rotation formula for the first step reads :

$$q_1^{(2)} = p_{21}\, q_1^{(1)}\, p_{21}^{-1}$$

$$\text{with: } p_{21} = \cos \tfrac{1}{2}\bar{\theta} - \bar{e}^{(1) \text{ or } (2)} \sin \tfrac{1}{2}\bar{\theta} = \qquad (1.41)$$

$$= \bar{p}^{-1}.$$

Hence :

$$\boxed{q_1^{(2)} = \bar{p}^{-1}\, q_1^{(1)}\, \bar{p}} \qquad (1.42)$$

Subsequently, for the second step :

$$q_1^{(3)} = p_{32}\, q_1^{(2)}\, p_{32}^{-1}$$

In this formula, p_{32} is, however, according to (1.37), described on system (2) or system (3):

$$p_{32} = p_{32}^{(2) \text{ or } (3)} =$$

$$= p_{21}\, p_{32}^{(1)}\, p_{21}^{-1}$$

In this expression $p_{32}^{(1)}$ represents the opposite rotation of $\bar{\bar{p}}$, so :

$$p_{32}^{(1)} = \cos \tfrac{1}{2}\bar{\bar{\theta}} - \bar{\bar{e}} \sin \tfrac{1}{2}\bar{\bar{\theta}} =$$

$$= \bar{\bar{p}}^{-1}. \qquad (1.43)$$

Therefore :

$$q_1^{(3)} = \bar{p}^{-1}\, \bar{\bar{p}}^{-1}\, \underbrace{\bar{p}\, q_1^{(2)}\, \bar{p}^{-1}}\, \bar{\bar{p}}\, \bar{p}$$

From (1.42) it follows :

$$\bar{p}\, q_1^{(2)}\, \bar{p}^{-1} = q_1^{(1)}$$

Hence :

$$\boxed{q_1^{(3)} = \bar{p}^{-1}\, \bar{\bar{p}}^{-1}\, q_1^{(1)}\, \bar{\bar{p}}\, \bar{p}} \qquad (1.44)$$

To verify this result, we apply (1.42) to q_2 instead of q_1 :

$$q_2^{(2)} = \bar{p}^{-1}\, q_2^{(1)}\, \bar{p} \quad ;$$

by(1.38) : $= \bar{p}^{-1}\, \bar{p}\, q_1^{(1)}\, \bar{p}^{-1}\, \bar{p} =$

$$= q_1^{(1)} \qquad (1.45^a)$$

and, subsequently (1.44) to q_3 instead of q_1 :

$$q_3^{(3)} = \bar{p}^{-1} \bar{\bar{p}}^{-1} q_3^{(1)} \bar{\bar{p}} \bar{p} \;;$$

by (1.39) : $= \bar{p}^{-1} q_2^{(1)} \bar{p} =$

by (1.38) : $= q_1^{(1)} .$ (1.45^b)

by (1.45^a) and (1.45^b), (1.40) has been verified.

Finally, we convert (1.44) to the form with system rotations instead of vector rotations :

$$(1.41) : \quad \bar{p}^{-1} = p_{21}^{(1)\text{ or }(2)}$$

$$(1.42) : \quad \bar{\bar{p}}^{-1} = p_{32}^{(2)\text{ or }(3)} =$$

$$= p_{21}^{-1} \, p_{32}^{(2)} \, p_{21}$$

Substitution in (1.44) leads to :

$$q_i^{(3)} = p_{21} \, p_{21}^{-1} \, p_{21}^{-1} \, p_{32}^{(2)} \, p_{21} \, q_i^{(1)} \, p_{21}^{-1} \, p_{32}^{(2)-1} \, p_{21} \, p_{21}^{-1} =$$

$$= p_{32}^{(2)} \, p_{21} \, q_i^{(1)} \, p_{21}^{-1} \, p_{32}^{(2)-1} .$$

In view of (1.37) this may be read as :

$$q_i^{(3)} = p_{32}^{(3)\text{ or }(2)} \, p_{21}^{(2)\text{ or }(1)} \, q_i^{(1)} \, p_{21}^{(2)\text{ or }(1)^{-1}} \, p_{32}^{(3)\text{ or }(2)^{-1}} .$$

It being agreed that rotation quaternions are always described on one of their own systems, the top-indices may be omitted; thus the general rotation formula for system rotations becomes :

$$\boxed{q_i^{(3)} = p_{32} \, p_{21} \, q_i^{(1)} \, p_{21}^{-1} \, p_{32}^{-1}} \tag{1.46}$$

or, in view of (1.35) :

$$q_i^{(3)} = p_{32} \, p_{21} \, q_i^{(1)} \, p_{12} \, p_{23} \tag{1.47}$$

1.4 Differentiation of quaternions.

1.4.1
The difference quantities of quaternion functions.

We consider the quaternion $\mathcal{Q}$:

$$\mathcal{Q} = w + i x + j y + k z \tag{1.48}$$

Suppose the components w, x, y and z are functions of (scalar) quantities a_i :

$$w = w(\dots, a_i, \dots)\,.$$

$$x = x(\dots, a_i, \dots)\,.$$

$$y = y(\dots, a_i, \dots)\,. \qquad (1.49)$$

$$z = z(\dots, a_i, \dots)\,.$$

so that :

$$\varphi = w(\dots, a_i, \dots) + i\, x(\dots, a_i, \dots) + j\, y(\dots, a_i, \dots) + k\, z(\dots, a_i, \dots)\,. \qquad (1.50)$$

The introduction of the difference quantities Δa_i then leads to the difference quantity $\Delta\varphi$ of the quaternion, according to :

$$\varphi + \Delta\varphi = w(\dots, a_i + \Delta a_i, \dots) + i\, x(\dots, a_i + \Delta a_i, \dots) + \text{etc}\,.$$

Expanding the four functions w, x, y and z in Taylor's series, and neglecting terms of the second and higher orders, we obtain :

$$\begin{aligned}
\varphi + \Delta\varphi = {} & w(\dots, a_i, \dots) + \sum_i \frac{\partial w}{\partial a_i} \Delta a_i + \\
& + i\left[x(\dots, a_i, \dots) + \sum_i \frac{\partial x}{\partial a_i} \Delta a_i\right] + \\
& + j\left[y(\dots, a_i, \dots) + \sum_i \frac{\partial y}{\partial a_i} \Delta a_i\right] + \\
& + k\left[z(\dots, a_i, \dots) + \sum_i \frac{\partial z}{\partial a_i} \Delta a_i\right]
\end{aligned}$$

In view of (1.50) we thus obtain :

$$\boxed{\Delta\varphi = \sum_i \frac{\partial w}{\partial a_i} \Delta a_i + i \sum_i \frac{\partial x}{\partial a_i} \Delta a_i + j \sum_i \frac{\partial y}{\partial a_i} \Delta a_i + \sum_i \frac{\partial z}{\partial a_i} \Delta a_i} \qquad (1.51)$$

Subsequently, we consider the quaternion function R of several quaternions φ_i :

$$R = R(\dots, \varphi_i, \dots)\,.$$

The introduction of quaternion differences $\Delta\varphi_i$, see (1.51), then leads to the difference ΔR of a quaternion function :

$$R + \Delta R = R(\dots, \varphi_i + \Delta\varphi_i, \dots)\,. \qquad (1.52)$$

In this formula, too, the right-hand member can be expanded in Taylor's series. But, because of the non-commutativity of multiplication, it is essential to take account of the sequence of the factors.

$$R = \varphi_1 \varphi_2$$

$$\begin{aligned}
R + \Delta R &= [\varphi_1 + \Delta\varphi_1][\varphi_2 + \Delta\varphi_2] = \\
&= \varphi_1 \varphi_2 + \varphi\, \Delta\varphi_2 + \Delta\varphi_1\, \varphi_2 + \cdots
\end{aligned}$$

The difference quantity of the inverse quaternion :

Let :

$$R = \mathcal{Q}^{-1}$$

and :

$$R + \Delta R = [\mathcal{Q} + \Delta\mathcal{Q}]^{-1}$$

so, multiplied by $\mathcal{Q} + \Delta\mathcal{Q}$:

$$[R + \Delta R][\mathcal{Q} + \Delta\mathcal{Q}] = 1 .$$

hence : $R\mathcal{Q} + R\,\Delta\mathcal{Q} + \Delta R\,\mathcal{Q} = 1 .$

and, since $R\mathcal{Q} = 1$:

$$\Delta R\,\mathcal{Q} = -R\,\Delta\mathcal{Q} = -\mathcal{Q}^{-1}\Delta\mathcal{Q} .$$

hence :

$$\Delta R = -\mathcal{Q}^{-1}\Delta\mathcal{Q}\,\mathcal{Q}^{-1}$$

or :

$$\boxed{\Delta[\mathcal{Q}]^{-1} = -\mathcal{Q}^{-1}\Delta\mathcal{Q}\,\mathcal{Q}^{-1} .} \qquad (1.53)$$

1.4.2.

The difference quantity of rotation quaternions

We consider the rotation quaternion p for the system rotation; see (1.33) :

$$p = \cos \tfrac{1}{2}\theta - e \sin \tfrac{1}{2}\theta$$

According to (1.52) :

$$p + \Delta p = \cos \tfrac{1}{2}[\theta + \Delta\theta] - [e + \Delta e] \sin \tfrac{1}{2}[\theta + \Delta\theta] \qquad (1.54)$$

If $p = p_{12}$, the following applies in this expression :

$$e = e^{(1)\ \text{or}\ (2)}$$

Of course this does not apply to the difference quantity Δe; suppose :

$$\Delta e = \Delta e^{(1)} \qquad (1.55)$$

is defined only on the (1)-system; then :

$$\Delta e^{(2)} = p_{21}\,\Delta e^{(1)}\,p_{21}^{-1} \neq \Delta e^{(1)} .$$

We will now consider the meaning of this for p + Δ p ; p +Δp must comply with the "form-rule" (1.32). This means :

$$\text{I} : N\{e + \Delta e\} = 1 .$$

$$\text{II} : \cos^2 \tfrac{1}{2}[\theta + \Delta\theta] + \sin^2 \tfrac{1}{2}[\theta + \Delta\theta] = 1$$

II has been complied with.

I means, since $N\{e\} = 1$:

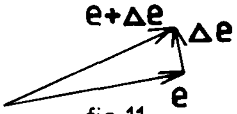


fig. 11

$$\text{either}: \Delta e = 0 \qquad (1.56^a)$$

$$\text{or}: \Delta e \perp e \quad (N\{\Delta e\} \ll N\{e\}) \qquad (1.56^b)$$

(1.56^a) is complied with, if the axis of rotation is, for example , defined as one of the three unit vectors of the system :

$$e = i \quad or \quad e = j \quad or \quad e = k$$

From section 2.3 it will become apparent that this situation applies to the five steps into which an "astronomical" rotation is split up.

(1.56^b) is complied with, if $Sc\{e\ \Delta e\} = 0$.

When : $e=0+ia+jb+kc$,

this is the case if :

$$a\,\Delta a + b\,\Delta b + c\,\Delta c = 0$$

This is the case if c is defined as a function of a and b :

$$c = \sqrt{1-a^2-b^2}$$
$$\downarrow$$
$$\Delta c = \frac{-a}{c}\Delta a - \frac{b}{c}\Delta b\,.$$

This means that in a rotation quaternion, a maximum number of three independent variables can occur :
the angle of rotation θ and two out of three components of the axis of rotation e. This is in agreement with the function of the rotation quaternion.
Further elaboration of (1.54), using :

$$\cos \tfrac{1}{2}\Delta\theta = 1\,.$$

$$\sin \tfrac{1}{2}\Delta\theta = \tfrac{1}{2}\Delta\theta$$

results in :

$$p + \Delta p = \cos\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\tfrac{1}{2}\theta\,\Delta\theta - [e+\Delta e][\sin\tfrac{1}{2}\theta + \tfrac{1}{2}\cos\tfrac{1}{2}\theta\,\Delta\theta]$$

Premultiplication by :

$$p^{-1} = \cos\tfrac{1}{2}\theta + e\sin\tfrac{1}{2}\theta$$

then results in :

$$\boxed{p^{-1}\Delta p = -\tfrac{1}{2}e\,\Delta\theta - \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,\Delta e - e\,\Delta e\,\sin^2\tfrac{1}{2}\theta\,.} \qquad (1.57)$$

From this it becomes apparent that, if (1.56) has been fulfilled :

$$Sc\{p^{-1}\Delta p\} = -\sin^2\tfrac{1}{2}\theta \cdot Sc\{e\,\Delta e\} = 0 \qquad (1.58)$$

This also applies to a rotation composed of several steps, for example :

$$p_{rc} = p_{ra}\, p_{a\ell}\, p_{\ell c}$$

Then we obtain, according to (1.52), etc. :

$$\Delta p_{rc} = \Delta p_{ra}\, p_{a\ell}\, p_{\ell c} + p_{ra}\, \Delta p_{a\ell}\, p_{\ell c} + p_{ra}\, p_{a\ell}\, \Delta p_{\ell c}$$

$$p^{-1}_{rc} = p^{-1}_{\ell c}\, p^{-1}_{a\ell}\, p^{-1}_{ra}$$

Hence :

$$(p^{-1}\Delta p)_{rc} = p^{-1}_{\ell c}\, p^{-1}_{a\ell}\, (p^{-1}\Delta p)_{ra}\, p_{a\ell}\, p_{\ell c} + p^{-1}_{\ell c}\, (p^{-1}\Delta p)_{a\ell}\, p_{\ell c} + (p^{-1}\Delta p)_{\ell c}$$

In this expression we find, according to (1.58) :

$$Sc\{(p^{-1}\Delta p)_{ra}\} = Sc\{(p^{-1}\Delta p)_{a\ell}\} = Sc\{(p^{-1}\Delta p)_{\ell c}\} = 0 .$$

therefore, in view of (1.30^b), it also applies that :

$$Sc\{(p^{-1}\Delta p)_{rc}\} = 0 . \qquad (1.59)$$

The structure of the quantity $(p^{-1}\Delta\, p)$ is discussed in greater detail in section 2.3 .

The coordinate system in which $(p^{-1}\Delta\, p)$ is defined.

In (1.55) it was already found that $\Delta\, e$ and therefore also Δp, in contrast with p, are defined on one of the two systems; therefore this also applies to $(p^{-1}\Delta\, p)$:

$$(p^{-1}\Delta p)_{ra} = \cos \tfrac{1}{2}[\theta + \Delta\theta]_{ra} + [e^{(r)\text{ or }(a)} + \Delta e^{(a)}]_{ra} \sin[\theta + \Delta\theta]_{ra} =$$
$$= (p^{-1}\Delta p)^{(a)}_{ra} \qquad (1.60)$$

Now, by definition :

$$(p^{-1}\Delta p)^{(r)}_{ra} = p_{ra}\, (p^{-1}\Delta p)^{(a)}_{ra}\, p^{-1}_{ra}$$

hence :

$$\boxed{(p^{-1}\Delta p)^{(r)}_{ra} = (\Delta p\;\; p^{-1})^{(a)}_{ra}} \qquad (1.61)$$

If p is of the type (1.56^a), i.e. $\Delta e = 0$:

$$Vc\{p\} = -\, e \sin \tfrac{1}{2}\theta .$$

$$Vc\{\Delta p\} = -\, e\, \tfrac{1}{2}\cos \tfrac{1}{2}\theta\;\; \Delta\theta .$$

so : (a is a scalar)

$$Vc\{p\} = a\; Vc\{\Delta p\} \quad \text{if} \quad \Delta e = 0$$

In view of (1.10) Δp and p are now indeed commutative : this means :

$$\Delta p^{(r)}_{ra} = p_{ra}\, \Delta p^{(a)}_{ra}\, p^{-1}_{ra} =$$

$$= \Delta p^{(a)}_{ra}\, p_{ra}\, p^{-1}_{ra} .$$

hence :

$$\Delta p^{(r)}_{ra} = \Delta p^{(a)}_{ra} \qquad \text{if} \quad \Delta e = o .$$

therefore also :

$$\boxed{(p^{-1}\Delta p)^{(r)}_{ra} = (p^{-1}\Delta p)^{(a)}_{ra} \qquad \text{if} \quad \Delta e = o} \tag{1.62}$$

This formula too, will turn out to be important in section 2.3, where astronomical rotations, for which $\Delta e=0$, are discussed.

1.5 Isomorphism with matrices.

In the manuscript [8] W. Baarda has already developed the basic thoughts used in this section.

The partial isomorphism between quaternions and matrices of the order 4 discussed here is very important in two respects :

-It constitutes the basis for the application of the present theory by means of computer programmes,

-The notation of quaternion equations, especially difference equations, in isomorphic matrices is easier interpreted than the notation in quaternions; this may be an advantage in the study of theory and in the analysis of linear dependencies.

A matrix which is isomorphic with quaternions with respect to the quaternion-product, may be derived in a "natural way", i.e. directly from the rules of multiplication (1.8).

We consider the quaternion product :

$$\mathcal{Q} = q_1\, q_2 \tag{1.63}$$

$$q_1 = w_1 + i\,x_1 + j\,y_1 + k\,z_1 .$$

$$q_2 = w_2 + i\,x_2 + j\,y_2 + k\,z_2$$

$$\mathcal{Q} = W + i X + j Y + k Z$$

The components W,X,Y,Z of the quaternion product $\mathcal{Q}$ are composed according to (1.8). On the basis of Baarda's idea, we now arrange these terms according to the components of q_2 and q_1; so :

$$W = w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 = w_2 w_1 - x_2 x_1 - y_2 y_1 - z_2 z_1 .$$

$$X = x_1 w_2 + w_1 x_2 - z_1 y_2 + y_1 z_2 = x_2 w_1 + w_2 x_1 + z_2 y_1 - y_2 z_1 .$$

$$Y = y_1 w_2 + z_1 x_2 + w_1 y_2 - x_1 z_2 = y_2 w_1 - z_2 x_1 + w_2 y_1 + x_2 z_1$$

$$Z = z_1 w_2 - y_1 x_2 + x_1 y_2 + w_1 z_2 = z_2 w_1 + y_2 x_1 - x_2 y_1 + w_2 z_1 .$$

This can be represented by matrices in the following two ways :

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} w_1 & -x_1 & -y_1 & -z_1 \\ x_1 & w_1 & -z_1 & y_1 \\ y_1 & z_1 & w_1 & -x_1 \\ z_1 & -y_1 & x_1 & w_1 \end{pmatrix} \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_2 & -x_2 & -y_2 & -z_2 \\ x_2 & w_2 & z_2 & -y_2 \\ y_2 & -z_2 & w_2 & x_2 \\ z_2 & y_2 & -x_2 & w_2 \end{pmatrix} \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad (1.64)$$

To denote these matrices we choose :

$\overrightarrow{(\ )}$ for the column matrices

$(\ )$ for the square matrix, showing the original sequence (q_1, q_2)

$(\bar{\ })$ for the square matrix, showing the inverted sequence (q_2, q_1)

As basic letter we take, of course, the letter used as symbol for the quaternions in (1.63); thus (1.64) will become :

$$\boxed{(\vec{\mathcal{Q}}) = (q_1)(\vec{q_2}) = (\bar{q_2})(\vec{q_1})} \qquad (1.65)$$

This can be extended to products of square matrices ; from (1.64) it follows directly that :

$$(\mathcal{Q}) = (q_1)(q_2) . \qquad a) .$$
$$(\bar{\mathcal{Q}}) = (\bar{q_2})(\bar{q_1}) . \qquad b) . \qquad (1.66)$$

Furthermore, it becomes directly apparent that both the matrix in (1.66^a) (normal sequence) and that in (1.66^b) (inverted sequence) can be applied for the sum and the difference of quaternions.

Because of (1.15) the inverse of q_1 is :

$$q_1^{-1} = \frac{1}{N\{q_1\}} [w_1 - i x_1 - j y_1 - k z_1]$$

The isomorphic matrix for q^{-1} can thus be obtained by inverting in (1.64) the signs of the components x, y and z, and by leaving those of w unchanged. In doing so, the transposed matrix is obtained !

Hence : (* transpose of matrix)

$$(q^{-1}) = \frac{1}{N\{q\}} (q)^* \qquad a) .$$
$$(\bar{q}^{-1}) = \frac{1}{N\{q\}} (\bar{q})^* \qquad b) . \qquad (1.67)$$

Furthermore, it now becomes apparent that :

$$(q^{-1})(q) = \frac{1}{N\{q\}} (q)^*(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1.68)$$

Through post-multiplication of (1.68) by the inverse matrix $(q)^{-1}$, we obtain:

$$(q^{-1}) = (q)^{-1} \tag{1.69}$$

There are consequently two types of isomorphic matrices for quaternions, viz. one for the normal sequence and one for the inverted sequence of the factors of a quaternion product. This will prove to be very important, because owing to this, the awkward non-commutativity can be avoided ; in the column version (see 1.65) of the matrix product the two types can, in fact, be used in mixed form.

Let :

$$\mathcal{Q} = q_1 q_2 q_3 \, .$$

then, according to (1.65) :

$$(\overrightarrow{\mathcal{Q}}) = (q_1)(q_2)(\vec{q}_3) \qquad a) \, .$$

but also : (notation : $(q_1 q_2) = (q_1)\ (q_2)$, etc.)

$$(\overrightarrow{\mathcal{Q}}) = (\bar{q}_3)(q_1)(\vec{q}_2) \qquad b) \, . \qquad (1.70)$$

and also, see also (1.66^b) :

$$(\overrightarrow{\mathcal{Q}}) = (\overline{q_2 q_3})(\vec{q}_1) = (\bar{q}_3)(\bar{q}_2)(\vec{q}_1) \, . \qquad c)$$

Each of the factors of a quaternion product can therefore be entered as last factor in the isomorphic matrix product. With reference to difference equations, this affords the possibility of placing the difference quaternion at the end.

Rotations.

The general rotation formula (1.33) :

$$q^{(2)} = p_{21}\, q^{(1)}\, p_{21}^{-1}$$

therefore reads in isomorphic matrices : (see 1.66^a)

$$(q^{(2)}) = (p_{21})\,(q^{(1)})\,(p_{21})^{*}$$

$$N\{p_{21}\} = 1 \quad !$$

Here, too, the sequence may be changed, by passing to the "column" version (1.65); we place the factor $q^{(1)}$ at the end of the right hand member :

$$\boxed{(\overrightarrow{q^{(2)}}) = (\bar{p}_{21})^{*}(p_{21})(\overrightarrow{q^{(1)}}) \, .} \tag{1.71}$$

If here :

$$p = d + i\,a + j\,b + k\,c$$

$$q^{(1)} = w + i\,x + j\,y + k\,z$$

$$q^{(2)} = W + i\,X + j\,Y + k\,Z$$

this becomes, in accordance with (1.64) :

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d^2+a^2-b^2-c^2 & 2[-dc+ab] & 2[db+ae] \\ 0 & 2[dc+ab] & d^2-a^2+b^2-c^2 & 2[-da+bc] \\ 0 & 2[-db+ac] & 2[da+bc] & d^2-a^2-b^2+c^2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \qquad (1.72)$$

Chapter 2

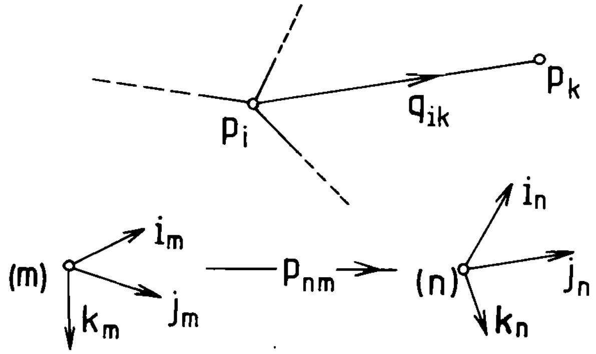
THE INTRODUCTION OF GEODETIC AND ASTRONOMICAL OBSERVATION VARIATES.

2.1 Introduction

In this section we shall consider how the geodetic and geodetic-astronomical observation variates can be linked with the set of concepts developed in Chapter 1.
The basic considerations are given in the present section ; it deals, in particular, with the introduction of length units.

Starting from the formulation by Baarda [8], this is worked out, in greater detail, in section 2.2 for the three types of "terrestrial" observation variates, which are combined for each side of a network P_iP_k in the quaternion :

$$q_{ik}$$



and in section 2.3 for the astronomical observation variates, with which, for each couple of local systems (m) and (n) a rotation quaternion :

$$p_{mn}$$

is established.

The relation between the astronomical quantities "longitude" λ and "latitude" φ on the one hand and the zenith angle on the other is established by the k-vectors of the local systems, to be defined as the direction of local gravity, i.e. as "zero direction" for longitude, latitude and zenith angle. The astronomical orientation unknowns (azimuth) and the terrestrial directions r are then defined in the plane of the i- and j-vectors.

Rotations.

Let an orthogonal trihedral of unit vectors be defined in each station P_m :

$$i_m,\ j_m,\ k_m \ : \text{(m) -system} \tag{2.1}$$

These "local systems" can pass into each other through a rotation (similarity transformation without translation and scale). We consider the rotation of the (m)-system to the (n)-system, in the first place as vector rotation of the unit vectors according to (1.28) and (1.29) :

$$\begin{aligned} i_n^{(m)} &= p\ i_m^{(m)}\ p^{-1} \\ j_n^{(m)} &= p\ j_m^{(m)}\ p^{-1} \\ k_n^{(m)} &= p\ k_m^{(m)}\ p^{-1} \end{aligned} \tag{2.2}$$

with the rotation quaternion :

$$p = \cos \tfrac{1}{2}\,\theta_{nm} + e_{nm} \sin \tfrac{1}{2}\,\theta_{nm} \qquad (2.2')$$

Furtheron we shall use the system rotation according to (1.33); so :

$$\boxed{\begin{array}{l} q^{(n)} = p_{nm}\, q^{(m)}\, p_{nm}^{-1} \\ \text{with: } p_{nm} = \cos \tfrac{1}{2}\,\theta_{nm} - e_{nm} \sin \tfrac{1}{2}\,\theta_{nm} \end{array}} \qquad (2.3)$$

Units of length

From (2.2) it follows that the "lengths" of the unit vectors of all systems are equal to each other :

$$N\{i_n^{(m)}\} = N\{i_m^{(m)}\} = 1 \quad ; \text{ etc.}$$

"algebraic unit of length"

The way of introducing geometry thus means that the "algebraic" unit of length (= length of the unit vectors) acts as unit of length of the computation system. The question of its "magnitude" is, in principle, not important; on this subject, one could imagine the metre, or any other artificial unit. From section 4.2 it will appear that, for reasons connected with computation, it will be wise to choose this unit such that the lengths of the sides of the network in the computation system are of the order of magnitude 1. (For example, it is possible to choose a multiple of the metre, viz. 500 or 1.000 or 10.000 m.)

In addition, we must define an instrumental unit of length for each local system. Though it will appear from section 2.2 that the computation model may be established, using quotients of these instrumental units of length, we shall explicitely define the relation between the instrumental units and the algebraic unit by the quantities λ_m :

λ_m is the length (magnitude) of the instrumental unit of length of the (m)-system, expressed in the algebraic unit (=length of the unit vectors)

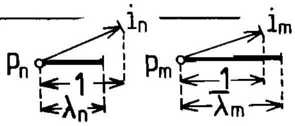


fig. 12

(2.4)

From section 3.4 it will appear that the instrumental units of length must be considered as derived variates. In section 4.4 they are linked, as a group, to the length of one of the sides of the network P_rP_s, which, for this purpose, is considered as non-stochastic and thus acts, in fact, as the "absolute" unit of length for the network; this is an aspect of the theory of the "S-transformations" (Dutch : Schrankingstransformaties).

Summarizing, we may thus recognize three "types" of length units : (see fig. 13)

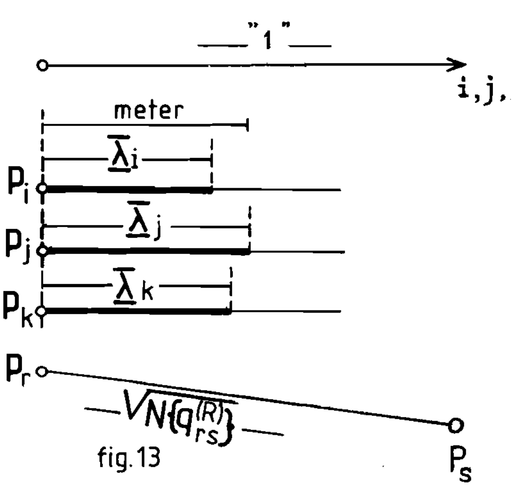


fig.13

1. the "algebraic" unit of length ; an aid for computation, its magnitude being approximately equal to the average lengths of the sides : f metre.
2. The group of instrumental length units; stochastic realisations of the metre, therefore :

$$\underline{\bar{\lambda}}_i \approx 1/f$$

3. The "unit of the S-system"; the non-stochastic length of the side P_rP_s

All notations of vectors, units of length and length factors used in the following chapters are summarized in next scheme : (stochastic variates are underlined)

Vectors q $\overset{}{P_i \circ\!\!-\!\!-\!\!\circ P_k}$	$N\{q\}$	Units of length $\circ\xrightarrow{-1-}$ all i,j,k	Length factors
$\underline{q}_{ik}^{(i)}$	$\underline{s}_{ik}$	$\underline{\bar{\lambda}}_i \approx \frac{1}{f}$ (≈ 1 m.)	$\underline{\bar{\lambda}}_{ri} = \frac{\underline{\bar{\lambda}}_i}{\underline{\bar{\lambda}}_r} = \frac{\underline{s}_{rj}}{\underline{s}_{jr}}\frac{\underline{s}_{ji}}{\underline{s}_{ij}}$
$\underline{q}_{ik}^{(r)}$	$\underline{\bar{\lambda}}_{ri}\,\underline{s}_{ik}$	$\underline{\bar{\lambda}}_r \approx \frac{1}{f}$	
$\bar{\underline{q}}_{ik}^{(r)}$	$\bar{\lambda}^{\circ}\,\underline{\bar{\lambda}}_{ri}\,\underline{s}_{ik}$	$\bar{\lambda}^{\circ} = \frac{1}{f}$	
$\bar{\bar{\underline{q}}}_{ik}^{(r)}$	$\bar{\lambda}^{\circ}[\underline{\bar{\lambda}}_{ri}+\underline{\varepsilon}][\underline{s}_{ik}+\underline{\varepsilon}]$		
$\underline{q}_{ik}^{(R)}$	$\underline{\bar{\lambda}}_{Rr}\,\bar{\lambda}^{\circ}[\underline{\bar{\lambda}}_{ri}+\underline{\varepsilon}][\underline{s}_{ik}+\underline{\varepsilon}]$		$\underline{\bar{\lambda}}_{Rr} = \frac{s_{rs}}{\underline{s}_{rs}+\underline{\varepsilon}}$.
$\underline{q}_{rs}^{(R)}$	$\frac{s_{rs}}{\underline{s}_{rs}+\underline{\varepsilon}}\,\frac{1}{f}\,\underbrace{\underline{\bar{\lambda}}_{rr}}_{\hookrightarrow =1}[\underline{s}_{rs}+\underline{\varepsilon}] = \frac{s_{rs}}{f}$	← " S-transformation"	

$\underline{\varepsilon}$: adjustment corrections .

2.2 The terrestrial-geodetic observation variates.

Remark : In the following sections, the observation variables will always be considered as stochastic variates and will therefore be underlined.

Definitions.

From station P_i, the distances $\underline{s}_{ik}$ to stations P_k are measured :

$\underline{s}_{ik}$ is the length of side P_iP_k expressed in the instrumental length unit $\bar{\lambda}_i$($\bar{\lambda}_i$ is approximately 1 metre) "distance measure".	(2.5)

By means of a theodolite established in P_i, the two "polar coordinates" of the spatial vector $P_i P_k$ can be measured :
the zenith angle $\underline{\zeta}_{ik}$ and the direction $\underline{r}_{ik}$:

$\underline{\zeta}_{ik}$ is the angle between the vector $P_i P_k$ and the upward direction of the first axis of the theodolite set up in P_i ; (2.6)

$\underline{r}_{ik}$ is the angle between the zero direction of the horizontal circle and the projection of the vector $P_i P_k$ on the plane of that circle. (2.7)

Since the first axis of the theodolite is perpendicular to the horizontal circle, the local system (orthogonal trihedral i, j, k) can be defined by linking it up as follows with the theodolite : (see fig. 14)

- the i-vector lies in the plane of the horizontal circle, in the "zero direction"
- the k-vector lies in the part of the first axis "pointing downwards"
- the j-vector completes a right-handed system i, j, k.

(2.8)

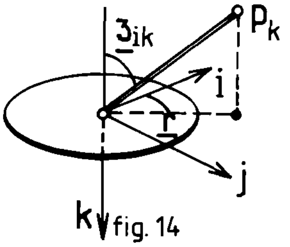


fig. 14

By (1.18), the spatial vector $P_i P_k$ can now be expressed as quaternion q_{ik} with a scalar part = 0, by conversion of the rectangular coordinate differences x, y, z to polar coordinates s, r, ζ :

$$\boxed{\begin{array}{l}\underset{\downarrow}{\underline{q}^{(i)}_{ik}} = 0 + i\,\underline{s}_{ik} \cos \underline{r}_{ik} \sin \underline{\zeta}_{ik} + j\,\underline{s}_{ik} \sin \underline{r}_{ik} \sin \underline{\zeta}_{ik} - k\,\underline{s}_{ik} \cos \underline{\zeta}_{ik} \\ \qquad \sqrt{N\{\underline{q}^{(i)}_{ik}\}} = \underline{s}_{ik}\end{array}} \tag{2.9}$$

We can pass to the "algebraic" unit of length through multiplication by $\bar{\lambda}_i$: (=approx. 1/f)

$$\bar{\underline{q}}^{(i)}_{ik} = \bar{\lambda}_i\, \underline{q}^{(i)}_{ik} \quad ; \quad \sqrt{N\{\bar{\underline{q}}^{(i)}_{ik}\}} = \bar{\lambda}_i\, \underline{s}_{ik} \tag{2.10}$$

The rotation to another local system (r) is made, using the rotation quaternion p_{ri} ; see (2.3):

$$\bar{\underline{q}}^{(r)}_{ik} = \underline{p}_{ri}\, \bar{\underline{q}}^{(i)}_{ik}\, \underline{p}^{-1}_{ri} \tag{2.11}$$

On the analogy of (2.10), we now pass to :

$$\bar{\underline{q}}^{(r)}_{ik} = \bar{\lambda}_r\, \underline{q}^{(r)}_{ik} \tag{2.12}$$

The substitution of (2.10) and (2.12) in (2.11) then gives the transformation between the two local systems (i) and (r) :

$$\bar{\lambda}_r \underline{q}_{ik}^{(r)} = \bar{\lambda}_i \underline{p}_{ri} \underline{q}_{ik}^{(i)} \underline{p}_{ri}^{-1} \tag{2.12'}$$

In order to obtain a more logical arrangement of the observation variates on the one hand and the transformation variates on the other, we elaborate this as follows :

$$\underline{q}_{ik}^{(r)} = \frac{\bar{\lambda}_i}{\bar{\lambda}_r} \underline{p}_{ri} \underline{q}_{ik}^{(i)} \underline{p}_{ri}^{-1}$$

Finally, the introduction of the "length factor"

$$\bar{\lambda}_{ri} = \frac{\bar{\lambda}_i}{\bar{\lambda}_r} \tag{2.13}$$

leads to the general transformation formula for local instrumental systems :

$$\boxed{\underline{q}_{ik}^{(r)} = \bar{\lambda}_{ri} \underline{p}_{ri} \underline{q}_{ik}^{(i)} \underline{p}_{ri}^{-1}} \tag{2.14}$$

is the vector $P_i P_k$, described relative to the unit vectors and the unit of length of the (r)-system

Remark : The algebraic unit of length is no longer involved here.

Subsequently, we consider, similar to Baarda, [8] the left-division of two vectors measured from P_i, viz. $P_i P_k$ and $P_i P_j$; see fig. 15 :

$$\boxed{\underline{\mathcal{Q}}_{jik}^{(i)} = \underline{q}_{ik}^{(i)} \underline{q}_{ij}^{(i)-1}} \tag{2.15}$$

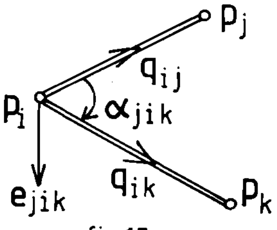


fig.15

In view of (1.23) this is :

$$\underline{\mathcal{Q}}_{jik}^{(i)} = \underline{v}_{jik} \left[\cos \underline{\bar{\alpha}}_{jik} + \underline{e}_{jik} \sin \underline{\bar{\alpha}}_{jik} \right]$$

with : $\underline{v}_{jik} = \frac{\underline{s}_{ik}}{\underline{s}_{ij}}$ (2.16)

Therefore, also the unit of length of the local (i)-system does not occur in (2.15).

The variate $\mathcal{Q}_{jik}$ is therefore dimensionless, covering all the terrestrial observation variates of station P_i ; it describes the form of a triangle $P_j P_i P_k$ and also the position of the spatial plane of that triangle relative to the (i)-system.
The properties of $\mathcal{Q}_{jik}$ thus agree, to a high extent, with those of the Π-quantity in Baarda's polygon theory in the complex plane.

Difference quantities.

By means of (1.50 etc.), (2.9) can be differentiated to the three observation variates :

$$\begin{aligned} \Delta \underline{q}_{ik}^{(i)} &= 0 + i\left[\cos r \sin z \, \underline{\Delta s} - s \sin r \sin z \, \underline{\Delta r} + s \cos r \cos z \, \underline{\Delta z}\right]_{ik} + \\ &+ j\left[\sin r \sin z \, \underline{\Delta s} + s \cos r \sin z \, \underline{\Delta r} + s \sin r \cos z \, \underline{\Delta z}\right]_{ik} + \\ &+ k\left[-\cos z \, \underline{\Delta s} + s \sin z \, \underline{\Delta z}\right]_{ik} \end{aligned} \tag{2.17}$$

Premultiplication by :

$$q^{-1\,(i)}_{ik} = 0 + \frac{1}{s}\left[-i\cos r\sin\zeta - j\sin r\sin\zeta + k\cos\zeta\right]_{ik}$$

then gives the dimensionless variate :

$$\begin{aligned}(q^{-1}\underline{\Delta q})^{(i)}_{ik} &= \frac{1}{s}\,\underline{\Delta s} + i\left[-\cos r\sin\zeta\cos\zeta\,\underline{\Delta r} - \sin r\,\underline{\Delta\zeta}\right]_{ik} + \\ &+ j\left[-\sin r\sin\zeta\cos\zeta\,\underline{\Delta r} + \cos r\,\underline{\Delta\zeta}\right]_{ik} + \\ &+ k\left[-\sin^2\zeta\,\underline{\Delta r}\right]_{ik}\end{aligned} \qquad (2.18)$$

In view of (1.64), the quantity $(q^{-1}\underline{\Delta}q)$ can be represented as an isomorphic column matrix, by splitting up the right-hand member into a coefficient matrix and a column matrix of differences :

$$\left(\overrightarrow{(q^{-1}\underline{\Delta q})^{(i)}_{ik}}\right) = \begin{pmatrix} Sc\{(q^{-1}\underline{\Delta q})^{(i)}_{ik}\} \\ Vi\{\quad ..\quad\} \\ Vj\{\quad ..\quad\} \\ Vk\{\quad ..\quad\} \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & 0 & 0 \\ 0 & -\cos r\sin\zeta\cos\zeta & -\sin r \\ 0 & -\sin r\sin\zeta\cos\zeta & \cos r \\ 0 & -\sin^2\zeta & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{\Delta s}_{ik} \\ \underline{\Delta r}_{ik} \\ \underline{\Delta\zeta}_{ik} \end{pmatrix} \qquad (2.19)$$

or, introducing $\Delta\underline{\ln s}_{ik}$ and $\sin\zeta\,\underline{\Delta r}_{ik}$ as difference quantities :

$$\begin{pmatrix} Sc\{(q^{-1}\underline{\Delta q})^{(i)}_{ik}\} \\ Vi\{\quad ..\quad\} \\ Vj\{\quad ..\quad\} \\ Vk\{\quad ..\quad\} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos r\cos\zeta & -\sin r \\ 0 & -\sin r\cos\zeta & \cos r \\ 0 & -\sin\zeta & 0 \end{pmatrix} \begin{pmatrix} \Delta\underline{\ln s}_{ik} \\ \sin\zeta\,\underline{\Delta r}_{ik} \\ \underline{\Delta\zeta}_{ik} \end{pmatrix} \qquad (2.20)$$

Subsequently, (2.15) is differentiated; applying the rules established in (1.50) etc.:

$$\underline{\Delta\varphi}^{(i)}_{jik} = \underline{\Delta q}_{ik}\,q^{-1}_{ij} - q_{ik}\,q^{-1}_{ij}\,\underline{\Delta q}_{ij}\,q^{-1}_{ij}$$

Through premultiplication by q^{-1}_{ik} and post multiplication by q_{ij}, this passes into :

$$\boxed{q^{-1}_{ik}\,\underline{\Delta\varphi}_{jik}\,q_{ij} = (q^{-1}\underline{\Delta q})_{ik} - (q^{-1}\underline{\Delta q})_{ij}} \qquad (2.21)$$

This formula bears a strong resemblance to a formula known from Baarda's polygon theory in the complex plane, (2.2.13) in [2] :

$$\underline{\Delta\Pi}_{jik} = \underline{\Delta\Lambda}_{ik} - \underline{\Delta\Lambda}_{ij} \qquad (2.22)$$

In the latter formula, moreover :

$$\underline{\Delta\Lambda}_{ik} = \Delta \underline{\ln s}_{ik} + i \, \underline{\Delta r}_{ik}$$

The symbol i is now the imaginary unit of the complex numbers.

Furthermore, applying the properties of a network in the complex plane :

a) : $\zeta = \pi/2$: all the points lie in one horizontal plane

$$(q^{-1}\underline{\Delta q})^{(i)}_{ik} = \Delta \underline{\ln s}_{ik} - k \, \underline{\Delta r}_{ik} + [-i \sin r + j \cos r] \, \underline{\Delta\zeta}_{ik}$$

b) : $\underline{\Delta\zeta} = 0$: the zenith angles are not observation variates.

$$(q^{-1}\underline{\Delta q})^{(i)}_{ik} = \Delta \underline{\ln s}_{ik} - k \, \underline{\Delta r}_{ik} \tag{2.23}$$

Compare (2.22)

2.3 The geodetic-astronomical observation variates.

2.3.1
Definitions

In section 2.2, the local systems of the stations P_m are defined as follows : (2.8)

- the unit vector k_m points in the direction of the "first axis" of the theodolite established in P_m ; this is an instrumental realisation of the vertical ;

- the unit vector i_m is situated in the plane perpendicular to k_m, parallel to the zero direction of the horizontal circle.

- the unit vector j_m completes a right handed system of orthogonal unit vectors i_m, j_m, k_m.

The unit vectors k of various stations can be interrelated by describing the spatial direction of each k-vector by means of two polar coordinates ; for this, we choose the system of the astronomical longitude $\underline{\lambda}$ and latitude $\underline{\varphi}$, measurable through astronomical observations ; see fig. 16 :

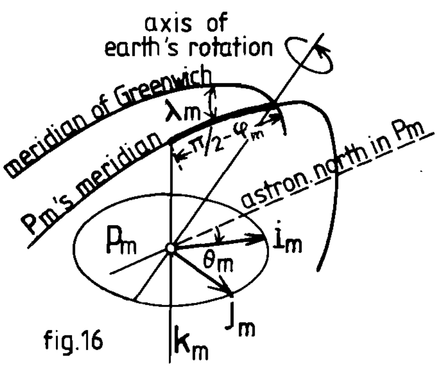


fig.16

The angle in the plane of the horizon between i_m and the astronomical north of P_m :

$$\underline{\theta}_m \tag{2.24}$$

is the "astronomical orientation" of the (m)-system.

We shall now describe the rotation of the (m)-system to another local system, the (n)-system, as a rotation in steps, according to (1.46).

For this, we define two ancillary systems by (m) :

-the (m')-system :
the k-vector of the (m')-system is parallel to k_m :
the i-vector of (m') lies in the direction of the astronomical north in P . (2.25)
the j-vector completes a right-handed system $i_{m'}$, $j_{m'}$, $k_{m'}$

-the (m")-system :
the j-vector of the (m")-system is parallel to $j_{m'}$ (2.26)
the k-vector of (m") lies in the polar direction of the φ, λ -system
the i-vector completes a right handed system $i_{m''}$, $j_{m''}$, $k_{m''}$.

The rotation from (m) to (n) can now be described in five successive steps with, consecutively, $\underline{\theta}_m$, $\underline{\varphi}_m$, $\underline{\lambda}_{mn}$, $\underline{\varphi}_n$ and $\underline{\theta}_n$ as angle of rotation :

$$\left.\begin{array}{ll}
1: & \underline{p}_{m'm} = \cos\tfrac{1}{2}\underline{\theta}_m + k \sin\tfrac{1}{2}\underline{\theta}_m \\
2: & \underline{p}_{m'm''} = \cos\tfrac{1}{2}[\pi/2 - \underline{\varphi}_m] + j \sin\tfrac{1}{2}[\pi/2 - \underline{\varphi}_m] \cdot \\
3: & \underline{p}_{m''n''} = \cos\tfrac{1}{2}[\lambda_m - \underline{\lambda}_n] - k \sin\tfrac{1}{2}[\underline{\lambda}_m - \underline{\lambda}_n] \cdot \\
4: & \underline{p}_{n''n'} = \cos\tfrac{1}{2}[\pi/2 - \underline{\varphi}_n] - j \sin\tfrac{1}{2}[\pi/2 - \underline{\varphi}_n] \cdot \\
5: & \underline{p}_{n'n} = \cos\tfrac{1}{2}\underline{\theta}_n - k \sin\tfrac{1}{2}\underline{\theta}_n \cdot
\end{array}\right\} \qquad (2.27)$$

Remarks :
-the choice of the sign in $p_{m'm}$ and $p_{nn'}$ means that $\underline{\theta}$ turns clockwise (as seen from above) ;
-the sign in $p_{m''m'}$ and $p_{n'n''}$ means that $\underline{\varphi}$ is positive on the northern hemisphere ;
-the sign in $p_{n''m''}$ means that $\underline{\lambda}$ is counted positive towards the east.

The five steps of (2.27) are all described relative to their "own" systems, and, according to (1.46), they can all be directly multiplied to :

$$\underline{p}_{nm} = \underline{p}_{nn'}\,\underline{p}_{n'n''}\,\underline{p}_{n''m''}\,\underline{p}_{m''m'}\,\underline{p}_{m'm} \underset{(2.27).}{=} \underline{p}^{-1}_{n'n}\,\underline{p}^{-1}_{n''n'}\,\underline{p}^{-1}_{m''n''}\,\underline{p}^{-1}_{m'm''}\,\underline{p}^{-1}_{m'm} \qquad (2.28)$$

This leads, introducing $\underline{\lambda}_{nm} = \underline{\lambda}_m - \underline{\lambda}_n$, to :

$$\begin{aligned}
\underline{p}_{nm} = {} & \cos\tfrac{1}{2}\underline{\lambda}_{nm} \cos\tfrac{1}{2}[\underline{\varphi}_m - \underline{\varphi}_n] \cos\tfrac{1}{2}[\underline{\theta}_m - \underline{\theta}_n] + \sin\tfrac{1}{2}\underline{\lambda}_{nm} \sin\tfrac{1}{2}[\underline{\varphi}_m + \underline{\varphi}_n] \sin\tfrac{1}{2}[\underline{\theta}_m - \underline{\theta}_n] + \\
& + i\left[-\cos\tfrac{1}{2}\underline{\lambda}_{nm} \sin\tfrac{1}{2}[\underline{\varphi}_m - \underline{\varphi}_n] \sin\tfrac{1}{2}[\underline{\theta}_m + \underline{\theta}_n] + \sin\tfrac{1}{2}\underline{\lambda}_{nm} \cos\tfrac{1}{2}[\underline{\varphi}_m + \underline{\varphi}_n] \cos\tfrac{1}{2}[\underline{\theta}_m + \underline{\theta}_n]\right] + \\
& + j\left[-\cos\tfrac{1}{2}\underline{\lambda}_{nm} \sin\tfrac{1}{2}[\underline{\varphi}_m - \underline{\varphi}_n] \cos\tfrac{1}{2}[\underline{\theta}_m + \underline{\theta}_n] - \sin\tfrac{1}{2}\underline{\lambda}_{nm} \cos\tfrac{1}{2}[\underline{\varphi}_m + \underline{\varphi}_n] \sin\tfrac{1}{2}[\underline{\theta}_m + \underline{\theta}_n]\right] + \\
& + k\left[\cos\tfrac{1}{2}\underline{\lambda}_{nm} \cos\tfrac{1}{2}[\underline{\varphi}_m - \underline{\varphi}_n] \sin\tfrac{1}{2}[\underline{\theta}_m - \underline{\theta}_n] - \sin\tfrac{1}{2}\underline{\lambda}_{nm} \sin\tfrac{1}{2}[\underline{\varphi}_m + \underline{\varphi}_n] \cos\tfrac{1}{2}[\underline{\theta}_m - \underline{\theta}_n]\right].
\end{aligned} \qquad (2.29)$$

In (2.29), the astronomical longitudes $\underline{\lambda}$ only occur as difference quantities, contrary to the latitudes $\underline{\varphi}$ and the orientation variates $\underline{\theta}$. It would be

possible to choose a corresponding measuring procedure, i.e. (approximately) simultaneous measurement of the longitudes at both ends of each network side ; beside the fact that time would play a less decisive role, also the influence of polar motion and the definition of "star-coordinates" would be deminished.
In addition, it will appear from the difference quantities (2.34) that in networks covering a limited part of the surface of the earth (which means that the k-vectors are all approximately parallel and the differences in longitude and latitude are small), also with respect to the astronomical latitudes φ chiefly the differences are important.

2.3.2.
Difference quantities

In section 1.4 rules have been deduced for the differentiation of rotation quaternions; within the scope of these "form rules", see (1.56), they can be differentiated in two ways :

a) according to (1.56^a) : $\Delta e = 0$
The five steps of (2.27) comply with this, because here the unit vectors act as axis of rotation e :

(2.27/1 and 3 and 5) : $e = k$
(2.27/2 and 4) : $e = j$

b) according to (1.56^b) : $Sc \left\{ e \Delta e \right\} = 0$

Method b :
If (2.29) is differentiated directly to all astronomical variates, and subsequently multiplied on the left by p_{nm}^{-1} , it becomes apparent that :

$$Sc \left\{ (\underline{p^{-1}\Delta} p) \right\} = o$$

In view of (1.57), it follows from this that :

$$Sc \left\{ (\underline{e^{-1}\Delta} e) \right\} = o$$

We now use method a) by differentiating the five steps of (2.27) :

$$\left.\begin{array}{ll}
1: & (\underline{p^{-1}\Delta} p)_{m'm}^{(m')\text{ or }(m)} = o + i\,o + j\,o + k\ \tfrac{1}{2}\underline{\Delta\theta}_m \\
2: & (\underline{p^{-1}\Delta} p)_{m''m'}^{(m'')\text{ or }(m')} = o + i\,o - j\ \tfrac{1}{2}\underline{\Delta\varphi}_m + k\,o \\
3: & (\underline{p^{-1}\Delta} p)_{n''m''}^{(n'')\text{ or }(n'')} = o + i\,o + j\,o - k\ \tfrac{1}{2}\underline{\Delta\lambda}_{nm} \\
4: & (\underline{p^{-1}\Delta} p)_{n'n''}^{(n')\text{ or }(n'')} = o + i\,o + j\ \tfrac{1}{2}\underline{\Delta\varphi}_n + k\,o \\
5: & (\underline{p^{-1}\Delta} p)_{nn'}^{(n)\text{ or }(n')} = o + i\,o + j\,o - k\ \tfrac{1}{2}\underline{\Delta\theta}_n
\end{array}\right\} \qquad (2.30)$$

The difference equation of (2.28) is :

$$(p^{-1}\underline{\Delta} p)^{(m)}_{nm} = p_{mn'}\,(p^{-1}\underline{\Delta} p)^{(n')}_{nn'}\,p^{-1}_{mn'} + p_{mn''}\,(p^{-1}\underline{\Delta} p)^{(n'')}_{n'n''}\,p^{-1}_{mn''} + \\ + p_{mm''}\,(p^{-1}\underline{\Delta} p)^{(m'')}_{n''m''}\,p^{-1}_{mm''} + p_{mm'}\,(p^{-1}\underline{\Delta} p)^{(m')}_{m''m'}\,p^{-1}_{mm'} + (p^{-1}\underline{\Delta} p)_{m'm} \qquad (2.31)$$

The substitution of (2.30) in (2.31) then gives :

$$\begin{aligned}(p^{-1}\underline{\Delta} p)^{(m)}_{nm} = 0 + \Big[& i\left[\cos\theta_m(\sin\varphi_n\cos\varphi_m - \cos\lambda_{nm}\cos\varphi_n\sin\varphi_m) - \sin\theta_m\sin\lambda_{nm}\cos\varphi_n\right] + \\ & + j\left[-\sin\theta_m(\sin\varphi_n\cos\varphi_m - \cos\lambda_{nm}\cos\varphi_n\sin\varphi_m) - \cos\theta_m\sin\lambda_{nm}\cos\varphi_n\right] + \\ & + k\left[-\cos\lambda_{nm}\cos\varphi_n\cos\varphi_m - \sin\varphi_n\sin\varphi_m\right]\Big]\ \tfrac{1}{2}\underline{\Delta}\theta_n + \\ + \Big[& i\left[\cos\lambda_{nm}\sin\theta_m - \sin\lambda_{nm}\sin\varphi_m\cos\theta_m\right] + \\ & + j\left[\cos\lambda_{nm}\cos\theta_m + \sin\lambda_{nm}\sin\varphi_m\sin\theta_m\right] + \\ & - k\ \sin\lambda_{nm}\cos\varphi_m\Big]\ \tfrac{1}{2}\Delta\varphi_n \\ + \Big[& i\cos\varphi_m\cos\theta_m - j\cos\varphi_m\sin\theta_m - k\sin\varphi_m\Big]\ \tfrac{1}{2}\underline{\Delta\lambda}_{nm} \\ + \Big[& -i\sin\theta_m - j\cos\theta_m\Big]\ \tfrac{1}{2}\underline{\Delta\varphi}_m + k\ \tfrac{1}{2}\underline{\Delta\theta}_m \end{aligned} \qquad (2.32)$$

or, expressed in an isomorphic column matrix, with symbolic indication of the coefficients in the first and second columns of the coefficient matrix :

$$\begin{pmatrix} Sc\{(p^{-1}\underline{\Delta} p)^{(m)}_{nm}\} \\ Vi\{\quad '' \quad\} \\ Vj\{\quad '' \quad\} \\ Vk\{\quad '' \quad\} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{\partial Vi}{\partial\theta_n} & \dfrac{\partial Vi}{\partial\varphi_n} & \cos\varphi_m\cos\theta_m & -\sin\theta_m & 0 \\ \dfrac{\partial Vj}{\partial\theta_n} & \dfrac{\partial Vj}{\partial\varphi_n} & -\cos\varphi_m\sin\theta_m & -\cos\theta_m & 0 \\ \dfrac{\partial Vk}{\partial\theta_n} & \dfrac{\partial Vk}{\partial\varphi_n} & -\sin\varphi_m & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tfrac{1}{2}\underline{\Delta\theta}_n \\ \tfrac{1}{2}\underline{\Delta\varphi}_n \\ \tfrac{1}{2}\underline{\Delta\lambda}_{nm} \\ \tfrac{1}{2}\underline{\Delta\varphi}_m \\ \tfrac{1}{2}\underline{\Delta\theta}_m \end{pmatrix} \qquad (2.33)$$

2.3.3

Networks with parallel k-unit-vectors

In a network with parallel k-vectors, we obtain :

$$\varphi_m = \varphi_n \quad \text{and} \quad \lambda_m = \lambda_n \quad (\rightarrow \lambda_{nm} = 0)$$

Then, the coefficients in (2.33) become :

$$\begin{pmatrix} Sc\{(p^{-1}\underline{\Delta}p)_{nm}^{(m)}\} \\ Vi\{\quad '' \quad\} \\ Vj\{\quad '' \quad\} \\ Vk\{\quad '' \quad\} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \sin\theta_m & \cos\varphi_m\cos\theta_m & -\sin\theta_m & 0 \\ 0 & \cos\theta_m & -\cos\varphi_m\sin\theta_m & -\cos\theta_m & 0 \\ -1 & 0 & -\sin\varphi_m & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\underline{\Delta}\theta_n \\ \frac{1}{2}\underline{\Delta}\varphi_n \\ \frac{1}{2}\underline{\Delta}\lambda_{nm} \\ \frac{1}{2}\underline{\Delta}\varphi_m \\ \frac{1}{2}\underline{\Delta}\theta_m \end{pmatrix} \qquad (2.34)$$

From the coefficients in the first matrix of the right-hand member it appears that in a network with parallel k-vectors also the astronomical latitudes φ and the orientation variates θ only act as difference quantities (for the longitudes λ this is always the case).

Chapter 3.

IMPORTANT DIFFERENCE EQUATIONS.

3.1 Chains of astronomical rotations.

In section 2.3, the internal structure of the astronomical rotation quaternion has been analysed as a rotation in five consecutive steps (2.27). What happens, if we connect several astronomical rotations ?
We consider four local systems (r), (a), (b), and (i), of which three mutual rotations are known :

$$p_{ra} \,,\; p_{ab} \,,\; p_{bi}$$

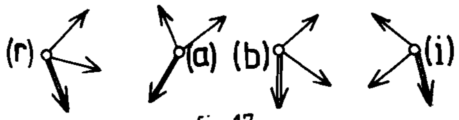


fig.17

The rotation from (i) to (r) is then : see (1.46)

$$\overset{(r)\ or\ (i)}{p_{ri}} = p_{ra}\, p_{ab}\, p_{bi} \qquad (3.1)$$

3.1.1
The relative difference quantity

The difference equation of (3.1) reads :

$$(p^{-1}\Delta p)^{(i)}_{ri} = p_{ia}(p^{-1}\Delta p)^{(a)}_{ra}\, p^{-1}_{ia} + p_{ib}(p^{-1}\Delta p)^{(b)}_{ab}\, p^{-1}_{ib} + (p^{-1}\Delta p)^{(i)}_{bi} \,. \qquad (3.2)$$

By means of (2.27), each of the three rotations in (3.1) can be split up into five component factors :

$$p_{ri} = \underbrace{p_{rr'}\, p_{r'r''}\, p_{r''a''}\, p_{a''a'}\, \overbrace{p_{a'a}\, p_{aa'}}}_{=p_{ra}}\, p_{a'a''}\, p_{a''b''}\, p_{b''b'}\, p_{b'b}\, p_{bb'}\, p_{b'b''}\, p_{b''i''}\, p_{i''i'}\, p_{i'i} \,. \qquad (3.3)$$

In view of (1.46), all factors are described in this formula relative to their "own" systems.
Now, the following applies :

$$(2.27/1) : \quad p_{a'a} = \cos \tfrac{1}{2}\theta_a + k \sin \tfrac{1}{2}\theta_a \,.$$

$$(2.27/5) : \quad p_{aa'} = \cos \tfrac{1}{2}\theta_a - k \sin \tfrac{1}{2}\theta_a \,.$$

hence :

$$p_{a'a}\, p_{aa'} = 1 \;; \qquad p_{b'b}\, p_{bb'} = 1 \,.$$

Likewise, with (2.27/2) and (2.27/4) :

$$p_{a''a'}\, p_{a'a''} = 1 \;; \qquad p_{b''b'}\, p_{b'b''} = 1 \,.$$

With this (3.3) passes into the simple form :

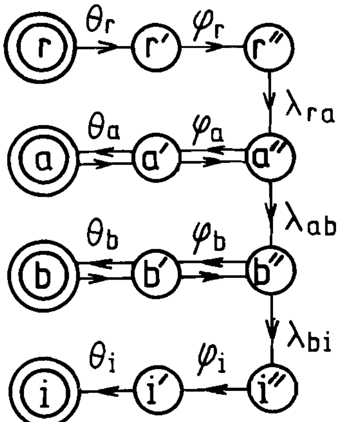


fig.18

$$\boxed{p_{ri} = p_{rr'}\, p_{r'r''}\, p_{r''a''}\, p_{a''b''}\, p_{b''i''}\, p_{i''i'}\, p_{i'i}} \qquad (3.4)$$

or, as a function of astronomical quantities :

$$p_{ri} = p_{r\ldots i} = p\left(\theta_r , \varphi_r , \lambda_{r\ldots i} , \varphi_i , \theta_i\right) \cdot \tag{3.4'}$$

The terms with φ_a, θ_a, φ_ℓ, and θ_ℓ will therefore also disappear from the difference formula (3.2) ; we substitute (2.31) in (3.2), with successively ($n \rightarrow r$; $m \rightarrow a$) ; ($n \rightarrow a$; $m \rightarrow b$) and ($n \rightarrow b$; $m \rightarrow i$) :

$$\begin{aligned}
(\overset{-1}{p}\Delta p)^{(i)}_{ri} = {} & p_{ia}\Big[p_{ar'}(\overset{-1}{p}\Delta p)_{rr'}\overset{-1}{p}_{ar'} + p_{ar''}(\overset{-1}{p}\Delta p)_{r'r''}\overset{-1}{p}_{ar''} + p_{aa''}(\overset{-1}{p}\Delta p)_{r''a''}\overset{-1}{p}_{aa''} + \\
& + p_{aa'}(\overset{-1}{p}\Delta p)_{a''a'}\overset{-1}{p}_{aa'} + (\overset{-1}{p}\Delta p)_{a'a}\Big]\overset{-1}{p}_{ia} + \\
& + p_{ib}\Big[p_{ba'}(\overset{-1}{p}\Delta p)_{aa'}\overset{-1}{p}_{ba'} + p_{ba''}(\overset{-1}{p}\Delta p)_{a'a''}\overset{-1}{p}_{ba''} + p_{bb''}(\overset{-1}{p}\Delta p)_{a''b''}\overset{-1}{p}_{bb''} + \\
& + p_{bb'}(\overset{-1}{p}\Delta p)_{b''b'}\overset{-1}{p}_{bb'} + (\overset{-1}{p}\Delta p)_{b'b}\Big]\overset{-1}{p}_{ib} + \\
& p_{ib'}(\overset{-1}{p}\Delta p)_{bb'}\overset{-1}{p}_{ib'} + p_{ib''}(\overset{-1}{p}\Delta p)_{b'b''}\overset{-1}{p}_{ib''} + p_{ii}(\overset{-1}{p}\Delta p)_{b''i''}\overset{-1}{p}_{ii''} + \\
& + p_{ii'}(\overset{-1}{p}\Delta p)_{i''i'}\overset{-1}{p}_{ii'} + (\overset{-1}{p}\Delta p)_{i'i} \cdot
\end{aligned} \tag{3.5}$$

In this formula, the terms containing $\Delta\theta_a$ are :

$$\begin{aligned}
& p_{ia}(\overset{-1}{p}\Delta p)_{a'a}\overset{-1}{p}_{ia} + p_{ib}\,p_{ba'}(\overset{-1}{p}\Delta p)_{aa'}\overset{-1}{p}_{ba'}\overset{-1}{p}_{ib} = \\
& = p_{ia}(\overset{-1}{p}\Delta p)_{a'a}\overset{-1}{p}_{ia} + p_{ia}\,p_{aa'}(\overset{-1}{p}\Delta p)^{(a')}_{aa'}\overset{-1}{p}_{aa'}\overset{-1}{p}_{ia} = \\
& = p_{ia}\Big[(\overset{-1}{p}\Delta p)_{a'a} + (\overset{-1}{p}\Delta p)^{(a')}_{aa'}\Big]\overset{-1}{p}_{ia} \quad .
\end{aligned}$$

Here, according to (2.30/1) the following applies :

$$(\overset{-1}{p}\Delta p)_{a'a} = 0 + k\, {}^1\!/_2\, \Delta\theta_a \;\cdot$$

The rotation $p_{aa'}$ is of the type $\Delta e = 0$ (1.56^a), therefore (1.62) applies to

$$\begin{aligned}
(\overset{-1}{p}\Delta p)^{(a)}_{aa'} & = (\overset{-1}{p}\Delta p)^{(a')}_{aa'} = \\
& = 0 - k\, {}^1\!/_2\, \Delta\theta_a
\end{aligned}$$

therefore, in (3.5), the two terms with $\Delta\theta_a$ eliminate each other.
It can be demonstrated in a similar way that the terms with $\Delta\varphi_a$, $\Delta\varphi_\ell$ and $\Delta\varphi_\ell$ disappear ; thus (3.5) passes into :

$$\begin{aligned}
(\overset{-1}{p}\Delta p)^{(i)}_{ri} = {} & p_{ia}\Big[p_{ar'}(\overset{-1}{p}\Delta p)_{rr'}\overset{-1}{p}_{ar'} + p_{ar''}(\overset{-1}{p}\Delta p)_{r'r''}\overset{-1}{p}_{ar''} + \\
& + p_{aa''}(\overset{-1}{p}\Delta p)_{r''a''}\overset{-1}{p}_{aa''}\Big]\overset{-1}{p}_{ia} + p_{ib}\,p_{bb''}(\overset{-1}{p}\Delta p)_{a''b''}\overset{-1}{p}_{bb''}\overset{-1}{p}_{ib} + \\
& + p_{ii''}(\overset{-1}{p}\Delta p)_{b''i''}\overset{-1}{p}_{ii''} + p_{ii'}(\overset{-1}{p}\Delta p)_{i''i'}\overset{-1}{p}_{ii'} + (\overset{-1}{p}\Delta p)_{i'i} \cdot
\end{aligned} \tag{3.6}$$

By placing the factors $p_{aa''} \cdots \overset{-1}{p}_{aa''}$ of the third, fourth and fifth terms outside brackets, they become :

$$p_{aa''}\Big[(\overset{-1}{p}\Delta p)_{r''a''} + p_{a''b''}(\overset{-1}{p}\Delta p)_{a''b''}\overset{-1}{p}_{a''b''} + p_{a''i''}(\overset{-1}{p}\Delta p)_{b''i''}\overset{-1}{p}_{a''i''}\Big]\overset{-1}{p}_{aa''} \tag{3.6'}$$

these are the terms containing $\Delta\lambda_{ra}$, $\Delta\lambda_{ab}$, $\Delta\lambda_{bi}$.

According to 2.30/3, the following applies in this formula :

$$(\overset{-1}{p}\Delta p)_{r''a''} = 0 - k\ \tfrac{1}{2}\Delta\lambda_{ra}$$

$$(\overset{-1}{p}\Delta p)_{a''b''} = 0 - k\ \tfrac{1}{2}\Delta\lambda_{ab}$$

$$(\overset{-1}{p}\Delta p)_{b''i''} = 0 - k\ \tfrac{1}{2}\Delta\lambda_{bi}$$

and, according to (2.27/3) :

$$p_{a''b''} = \cos\tfrac{1}{2}\lambda_{ab} - k\sin\tfrac{1}{2}\lambda_{ab}$$

$$p_{a''i''} = p_{a''b''}\, p_{b''i''} = p_{a''b''}\left[\cos\tfrac{1}{2}\lambda_{bi} - k\sin\tfrac{1}{2}\lambda_{bi}\right]$$

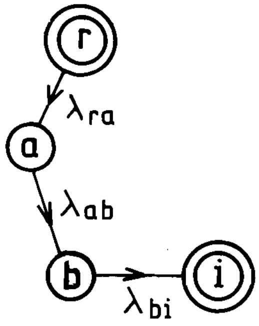


fig.19

therefore, the term (3.6') becomes :

$$-p_{aa''}\,k\,\overset{-1}{p}_{aa''}\left[\tfrac{1}{2}\Delta\lambda_{ra} + \tfrac{1}{2}\Delta\lambda_{ab} + \tfrac{1}{2}\Delta\lambda_{bi}\right]$$

In (3.6) this is pre- and post multiplied by $p_{ia} \cdots\cdot \overset{-1}{p}_{ia}$; therefore, the coefficients of the differences of longitudes become :

$$-p_{ia}\,p_{aa''}\,k\,\overset{-1}{p}_{aa''}\,\overset{-1}{p}_{ia} = -p_{ii'}\,p_{i'i''}\,p_{i''b''}\,p_{b''a''}\,k\,\overset{-1}{p}_{b''a''}\,\overset{-1}{p}_{i''b''}\,\overset{-1}{p}_{i'i''}\,\overset{-1}{p}_{ii'} =$$

$$\left.\begin{array}{l} p_{i''b''} \,//\, k \\ p_{b''a''} \,//\, k \end{array}\right\} \rightarrow \qquad = -p_{ii'}\,p_{i'i''}\,k\,\overset{-1}{p}_{i'i''}\,\overset{-1}{p}_{ii'} =$$

$$= 0 + i\cos\theta_i\cos\varphi_i - j\sin\theta_i\cos\varphi_i - k\sin\varphi_i\,.$$

Thus (3.5) finally becomes :

$$\boxed{\begin{aligned}(\overset{-1}{p}\Delta p)^{(i)}_{ri} = {} & p_{ir'}(\overset{-1}{p}\Delta p)_{rr'}\overset{-1}{p}_{ir'} + p_{ir''}(\overset{-1}{p}\Delta p)_{r'r''}\overset{-1}{p}_{ir''} + \\ & -p_{ia''}\,k\,\overset{-1}{p}_{ia''}\ \tfrac{1}{2}\left[\Delta\lambda_{ra} + \Delta\lambda_{ab} + \Delta\lambda_{bi}\right] + \\ & + p_{ii'}(\overset{-1}{p}\Delta p)_{i''i'}\overset{-1}{p}_{ii'} + (\overset{-1}{p}\Delta p)_{i'i}\end{aligned}} \qquad (3.7)$$

with : $(\overset{-1}{p}\Delta p)_{rr'} = -k\ \tfrac{1}{2}\Delta\theta_r$.

$$(\overset{-1}{p}\Delta p)_{r'r''} = j\ \tfrac{1}{2}\Delta\varphi_r$$

$$(\overset{-1}{p}\Delta p)_{i''i'} = -j\ \tfrac{1}{2}\Delta\varphi_i$$

$$(\overset{-1}{p}\Delta p)_{i'i} = k\ \tfrac{1}{2}\Delta\theta_i$$

3.1.2

The coefficients of the astronomical variates

From (3.7) some conclusions may be directly drawn concerning the coefficients of the differences of astronomical variates :

1. In the coefficients of $\Delta\theta_r$, $\Delta\varphi_r$, $\Delta\varphi_i$ and $\Delta\theta_i$ the longitudes and latitudes of the "intermediate" systems (a) and (b) do not occur. This means that these coefficients are independent of the "route" chosen, from the (i)- to the (r)-system.
2. The coefficients of all longitude differences $\Delta\lambda$ are equal to each other. Here, only the quantities θ_i and φ_i occur.
3. If the systems (r) and (i) lie on a small part of the earth's surface (i.e. k_i is approximately parallel to k_r), the coefficients of $\Delta\varphi_i$ and $\Delta\varphi_r$ are approximately equal to each other (with opposite signs). The same applies to $\Delta\theta_i$ and $\Delta\theta_r$.

3.2 The vector transformation

3.2.1

The difference equation for the vector transformation

We consider (2.14) :

$$q_{ik}^{(r)} = \bar{\lambda}_{ri}\, p_{ri}\, q_{ik}^{(i)}\, p_{ri}^{-1} \tag{3.8}$$

The difference equation of this formula is :

$$\Delta q_{ik}^{(r)} = q_{ik}^{(r)} \frac{\Delta\bar{\lambda}_{ri}}{\bar{\lambda}_{ri}} + p_{ri}\,(p^{-1}\Delta p)_{ri}\, p_{ri}^{-1}\, q_{ik}^{(r)} - q_{ik}^{(r)}\, p_{ri}\,(p^{-1}\Delta p)_{ri}\, p_{ri}^{-1} + \bar{\lambda}_{ri}\, p_{ri}\, \Delta q_{ik}^{(i)}\, p_{ri}^{-1} \tag{3.9}$$

Premultiplication by :

$$q_{ik}^{-1\,(r)} = \frac{1}{\bar{\lambda}_{ri}}\, p_{ri}^{-1}\, q_{ik}^{-1\,(i)}\, p_{ri} \tag{3.10}$$

then results in :

$$\boxed{(q^{-1}\Delta q)_{ik}^{(r)} = \Delta \ln \bar{\lambda}_{ri} + q_{ik}^{-1\,(r)}\,(p^{-1}\Delta p)_{ri}^{(r)}\, q_{ik}^{(r)} - (p^{-1}\Delta p)_{ri}^{(r)} + p_{ri}\,(q^{-1}\Delta q)_{ik}^{(i)}\, p_{ri}^{-1}} \tag{3.11}$$

If the transformation quantities $\bar{\lambda}_{ri}$ and p_{ri} consist of a number of factors, as described for p_{ri} in section 3.1, the quantities in the right-hand member of (3.11) are functions of the following differences :

(2.13) : $\Delta \ln \bar{\lambda}_{ri}$ is the sum of $\Delta \ln \bar{\lambda}_{r..} \ldots\ldots \Delta \ln \bar{\lambda}_{..i}$ (length factors)

(3.7) : $(p^{-1}\Delta p)_{ri}$: $\Delta\theta_r$, $\Delta\varphi_r$, $\Delta\lambda_{r\ldots i}$, $\Delta\varphi_i$, $\Delta\theta_i$

(astronomical observation variates).

(2.20) : $(q^{-1}\Delta q)_{ik}^{(i)}$: $\Delta \ln s_{ik}$, Δr_{ik}, $\Delta \mathfrak{z}_{ik}$ (geodetic observation variates).

Subsequently, we consider a variant of (3.11), by placing the factors $p_{ri} \ldots\ldots p_{ri}^{-1}$ outside brackets :

$$(q^{-1}\Delta q)_{ik}^{(r)} = p_{ri}\left[\Delta \ln \bar{\lambda}_{ri} + q_{ik}^{(i)-1}\,(p^{-1}\Delta p)_{ri}\, q_{ik}^{(i)} - (p^{-1}\Delta p)_{ri} + (q^{-1}\Delta q)_{ik}^{(i)}\right] p_{ri}^{-1} \tag{3.12}$$

By means of (1.65) - (1.71), this equation can be put in isomorphic matrices :

$$\left(\overrightarrow{(q^{-1}\Delta q)_{ik}^{(r)}}\right) = \left(\overline{p_{ri}}\right)^{*}\left(p_{ri}\right)\left[\left(\overrightarrow{\Delta \ln \bar{\lambda}}_{ri}\right) + \left[\frac{1}{N\{q_{ik}\}}\left(\overline{q_{ik}}\right)\left(q_{ik}\right)^{*} - (\delta)\right]\left(\overrightarrow{(p^{-1}\Delta p)_{ri}^{(i)}}\right) + \left(\overrightarrow{(q^{-1}\Delta q)_{ik}^{(i)}}\right)\right] p_{ri}^{-1} \qquad (3.13)$$

Suppose :

$$q_{ik}^{(i)} = 0 + i\,x + j\,y + k\,z \quad ; \quad N\{q_{ik}^{(i)}\} = S_{ik}^{2} \quad ;$$

Now (3.13) becomes :

$$\begin{pmatrix} Sc\{(q^{-1}\Delta q)_{ik}^{(i)}\} \\ Vi\{ \quad " \quad \} \\ Vj\{ \quad " \quad \} \\ Vk\{ \quad " \quad \} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\begin{matrix} \text{see} \\ (1.72) \end{matrix}} & \\ 0 & & & \end{pmatrix} \left[\begin{pmatrix} \Delta \ln \bar{\lambda}_{ri} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-2(y^2+z^2)}{s^2} & \frac{2xy}{s^2} & \frac{2xz}{s^2} \\ 0 & \frac{2xy}{s^2} & \frac{-2(x^2+z^2)}{s^2} & \frac{2yz}{s^2} \\ 0 & \frac{2xz}{s^2} & \frac{2yz}{s^2} & \frac{-2(x^2+y^2)}{s^2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ Vi\{(p^{-1}\Delta p)_{ri}^{(i)}\} \\ Vj\{ \quad " \quad \} \\ Vk\{ \quad " \quad \} \end{pmatrix} + \begin{pmatrix} Sc\{(q^{-1}\Delta q)_{ik}^{(i)}\} \\ Vi\{ \quad " \quad \} \\ Vj\{ \quad " \quad \} \\ Vk\{ \quad " \quad \} \end{pmatrix} \right] \qquad (3.14)$$

In view of (2.33) here is :

$$\begin{pmatrix} 0 \\ Vi\{(p^{-1}\Delta p)_{ri}^{(i)}\} \\ Vj\{ \quad " \quad \} \\ Vk\{ \quad " \quad \} \end{pmatrix} = \text{terms with } \Delta\theta_r, \Delta\varphi_r, \Delta\lambda_{r\ldots i}, \Delta\varphi_i \cdots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \Delta\theta_i$$

so the coefficients of $\Delta\theta_i$ in (3.14) are :

$$\cdots + \begin{pmatrix} 0 \\ \frac{-2xz}{s^2} \\ \frac{-2yz}{s^2} \\ \frac{-2(x^2+y^2)}{s^2} \end{pmatrix} \tfrac{1}{2}\Delta\theta_i + \cdots$$

In view of (2.19), the following applies in (3.14), conversing the polar coordinates r_{ik} and $\Im_{ik}$ in rectangular coordinate differences x, y and z :

$$\begin{pmatrix} Sc\{(q^{-1}\Delta q)^{(i)}_{ik}\} \\ Vi\{ \quad '' \quad \} \\ Vj\{ \quad '' \quad \} \\ Vk\{ \quad '' \quad \} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta \ln s_{ik} + \begin{pmatrix} 0 \\ \frac{xz}{s^2} \\ \frac{yz}{s^2} \\ \frac{-(x^2+y^2)}{s^2} \end{pmatrix} \Delta r_{ik} + \begin{pmatrix} 0 \\ \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix} \Delta \Im_{ik}$$

therefore, the coefficients of Δr_{ik} in the part of (3.14) between braces are : .... $+ \begin{pmatrix} 0 \\ \frac{xz}{s^2} \\ \frac{yz}{s^2} \\ \frac{-(x^2+y^2)}{s^2} \end{pmatrix} \Delta r_{ik} + \cdots$

This means that the coefficients of $\Delta\theta_i$ and Δr_{ik} are equal to each other in all four components of $(q^{-1}\Delta q)_{ik}$ (3.15)

Furthermore, it appears directly from (3.14) that :

$$Sc\{(q^{-1}\Delta q)\} = \Delta \ln \bar{\lambda}_{ri} + \Delta \ln s_{ik} \,. \tag{3.16}$$

3.2.2

The difference equation of the vector transformation

We apply (3.11) to two vectors measured from P_i :

$$(q^{-1}\Delta q)^{(r)}_{ik} = \Delta \ln \bar{\lambda}_{ri} + q^{-1(r)}_{ik} (p^{-1}\Delta p)^{(r)}_{ri}\, q^{(r)}_{ik} - (p^{-1}\Delta p)^{(r)}_{ri} + p_{ri}\, (q^{-1}\Delta q)^{(i)}_{ik}\, p^{-1}_{ri}$$

$$(q^{-1}\Delta q)^{(r)}_{ij} = \Delta \ln \bar{\lambda}_{ri} + q^{-1(r)}_{ij} (p^{-1}\Delta p)^{(r)}_{ri}\, q^{(r)}_{i} - (p^{-1}\Delta p)^{(r)}_{ri} + p_{ri}\, (q^{-1}\Delta q)^{(i)}_{ij}\, p^{-1}_{ri}$$

By subtracting these two equations from each other, the transformation formula for the spatial $\Delta\Pi$-quantity, defined in (2.21), is obtained :

$$\Delta\Pi^{(r)}_{jik} = q^{-1(r)}_{ik} (p^{-1}\Delta p)^{(r)}_{ri}\, q^{(r)}_{ik} - q^{-1(r)}_{ij} (p^{-1}\Delta p)_{ri}\, q^{(r)}_{ij} + p_{ri}\, \Delta\Pi^{(i)}_{jik}\, p^{-1}_{ri} \tag{3.17}$$

3.3 Polar coordinate functions.

We consider (3.11) : (in this section stochastic variates are underlined).

$$(q^{-1}\underline{\Delta} q)^{(r)}_{ik} = \Delta \underline{\ln \bar{\lambda}}_{ri} + q^{-1(r)}_{ik} (p^{-1}\underline{\Delta} p)^{(r)}_{ri}\, q^{(r)}_{ik} - (p^{-1}\underline{\Delta} p)_{ri} + p_{ri}\, (q^{-1}\underline{\Delta} q)^{(i)}_{ik}\, p^{-1}_{ri}$$

In view of (2.20) here :

$$\left.\begin{aligned} (q^{-1}\underline{\Delta q})^{(i)}_{ik} &= \Delta\underline{\ln s}_{ik} + e'_{ik}\,[\sin\zeta\,\underline{\Delta r}]_{ik} + e''_{ik}\,\underline{\Delta\zeta}_{ik} \\ \text{with :}\quad e'_{ik} &= 0 - i\cos r\cos\zeta - j\sin r\cos\zeta - k\sin\zeta \\ e''_{ik} &= 0 - i\sin r + j\cos r + 0 \end{aligned}\right\} \qquad (3.18)$$

Because :

$$q^{(i)}_{ik} = 0 + i\,s\cos r\sin\zeta + j\,s\sin r\sin\zeta - k\,s\cos\zeta$$

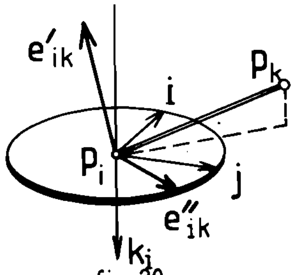


fig. 20

applies :

$$Sc\{e'_{ik}\,q_{ik}\} = 0.$$

$$Sc\{e''_{ik}\,q_{ik}\} = 0.$$

$$Sc\{e'_{ik}\,e''_{ik}\} = 0.$$

The vectors q_{ik} , e'_{ik} and e''_{ik} thus constitute a rectangular trihedral ; e''_{ik} lies in the plane $\perp k_i$.

Furthermore, according to (1.25) :

$$q^{-1}_{ik}(p^{-1}\Delta p)_{ri}\,q_{ik} - (p^{-1}\Delta p)_{ri} = -2\left[\text{the component of } (p^{-1}\Delta p)_{ri} \perp q_{ik}\right]$$

Summarizing, the following therefore applies :

$$Sc\{(q^{-1}\underline{\Delta q})^{(r)}_{ik}\} = \Delta\underline{\ln\bar{\lambda}}_{ri} + \Delta\underline{\ln s}_{ik}\,.$$

$$Vc\{(q^{-1}\underline{\Delta q})^{(r)}_{ik}\} \perp q_{ik}$$

The vector part of $(q^{-1}\underline{\Delta q})^{(r)}_{ik}$ can therefore be decomposed into any pair of vectors, perpendicular to q_{ik}. (3.19)

Now assume that :

$$r^{r}_{ik} \quad \text{and} \quad \zeta^{r}_{ik}$$

are the polar coordinates of the vector $P_i P_k$, relative to the unit vectors of the (r)-system :

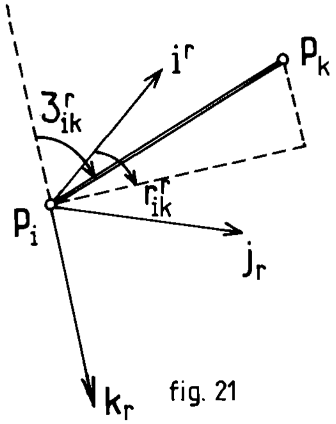


fig. 21

$$q^{(r)}_{ik} = 0 + i\,s\cos r^r\sin\zeta^r + j\,s\sin r^r\sin\zeta^r - k\,s\cos\zeta^r$$

Assume further, in analogy with (3.18) :

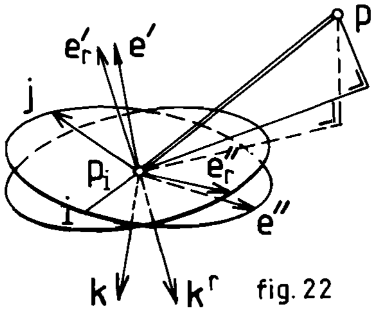


fig. 22

$$e'_{r;ik} = 0 - i\cos r^r \cos\zeta^r - j\sin r^r\cos\zeta^r - k\sin\zeta^r$$

$$e''_{r;ik} = 0 - i\sin r^r + j\cos r^r$$

then the following applies again :

$$\left.\begin{aligned} Sc\{e'_{r;ik}\,q_{ik}\} &= 0\,. \\ Sc\{e''_{r;ik}\,q_{ik}\} &= 0\,. \\ Sc\{e'_{r;ik}\,e''_{r;ik}\} &= 0\,. \end{aligned}\right\} \qquad (3.20)$$

Remark : in general : $e'_{r;ik} \neq e'_{ik}$ and $e''_{r;ik} \neq e''_{ik}$

because : $e''_{r;ik}$ lies in the plane $\perp$ k_r.
e''_{ik} lies in the plane $\perp$ k_i

Therefore, the vectors $e'_{r;ik}$ and $e''_{r;ik}$ constitute together with q_{ik} a rectangular trihedral. In view of (3.19) this means that $Vc\{(q^{-1}\underline{\Delta}q)_{ik}\}$ can be decomposed into :

$$Vc\{(q^{-1}\underline{\Delta}q)^{(r)}_{ik}\} = e'_{r;ik}\,\underline{R} + e''_{r;ik}\,\underline{Z} \qquad (3.21)$$

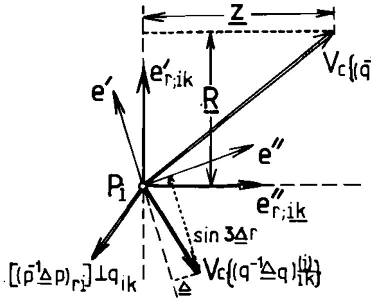


fig. 23

Here $\underline{R}$ and $\underline{Z}$ are scalar functions of the difference quantities, occuring in the vector part of $(q^{-1}\underline{\Delta}q)_{ik}$; in order to obtain an expression analogous to (3.18), we use the following designations :

$$\left.\begin{aligned} \underline{R} &= \sin\zeta^r_{ik}\,\underline{\Delta}r^r_{ik} \\ \underline{Z} &= \underline{\Delta}\zeta^r_{ik} \end{aligned}\right\} \text{"polar coordinate functions".} \qquad (3.22)$$

so that :

$$\boxed{(q^{-1}\underline{\Delta}q)^{(r)}_{ik} = \underline{\Delta\ln\bar\lambda}_{ri} + \underline{\Delta\ln s}_{ik} + e'^{(r)}_{r;ik}\sin\zeta^r_{ik}\underline{\Delta}r^r_{ik} + e''^{(r)}_{r;ik}\,\underline{\Delta}\zeta^r_{ik}}\,. \qquad (3.23)$$

This quaternion equation gives the relationships between the four components of $(q\,\underline{^{-1}\Delta\,g})^{(r)}_{ik}$ on the one hand, and the three "polar coordinate functions" $(\underline{\Delta\ln s}_{ik} + \underline{\Delta\ln\lambda}_{ri}$ being the third) on the other.
Using isomorphic matrices this becomes :

$$\begin{pmatrix} Sc\{(q^{-1}\underline{\Delta}q)^{(r)}_{ik}\} \\ Vi\{\quad''\quad\} \\ Vj\{\quad''\quad\} \\ Vk\{\quad''\quad\} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos r^r\cos\zeta^r & -\sin r^r \\ 0 & -\sin r^r\cos\zeta^r & \cos r^r \\ 0 & -\sin\zeta^r & 0 \end{pmatrix} \begin{pmatrix} \underline{\Delta\ln s}_{ik} + \underline{\Delta\ln\bar\lambda}_{ri} \\ \sin\zeta^r\,\underline{\Delta}r^r_{ik} \\ \underline{\Delta}\zeta^r_{ik} \end{pmatrix} \qquad (3.24)$$

From this, the definitions of the three polar coordinate functions can be derived :

$$\boxed{\begin{array}{lr}
\Delta \underline{\ln \bar{\lambda}}_{ri} + \Delta \underline{\ln s}_{ik} = Sc\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} \cdot & \underline{1} \\
\left[\sin\zeta\, \underline{\Delta r}\right]_{ik}^{r} = \frac{-\cos r^r}{\cos\zeta^r} Vi\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} - \frac{\sin r^r}{\cos\zeta^r} Vj\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} & \underline{2} \\
\underline{\Delta\zeta}_{ik}^{r} = -\sin r^r Vi\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} + \cos r^r Vj\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} \cdot & \underline{3}
\end{array}} \qquad (3.24')$$

It also follows from (3.24) that there is a linear dependency between the three vector components of $(q^{-1}\Delta q)_{ik}^{(r)}$:

$$\cos r^r \sin\zeta^r \; \underline{Vi} \; + \; \sin r^r \sin\zeta^r \; \underline{Vj} \; - \; \cos\zeta^r \; \underline{Vk} \; = 0 .$$

or, after conversion of r_{ik}^{r} and ζ_{ik}^{r} into rectangular coordinate differences X_{ik}^{r}, Y_{ik}^{r} and Z_{ik}^{r}:

$$\boxed{X_{ik}^{r} \; Vi\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} + Y_{ik}^{r} \; Vj\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} + Z_{ik}^{r} \; Vk\{(q^{-1}\underline{\Delta q})_{ik}^{(r)}\} = 0 .} \qquad (3.25)$$

Remark : (3.25) can also be written as :

$$Sc\{ q_{ik}^{(r)} \; (q^{-1}\underline{\Delta q})_{ik}^{(r)} \} = 0$$

or: $Sc\{ \underline{\Delta q}_{ik}^{(r)} \} = 0 .$

which results directly from the definition of q_{ik}.

3.4 Units of length and orientations.

3.4.1
The computation of $\bar{\lambda}_k$ and θ_k

We consider the side $P_i P_k$ of a network ; suppose the following observation variates are measured :

"terrestrial" : s_{ik}, r_{ik}, ζ_{ik}, s_{ki}, r_{ki}, ζ_{ki}

"astronomical" : φ_i, λ_{ik}, φ_k

Also assume that the transformation quantities :

$\bar{\lambda}_i$: unit of length of the i-system

θ_i : horizontal orientation of the i-system

are somehow known.

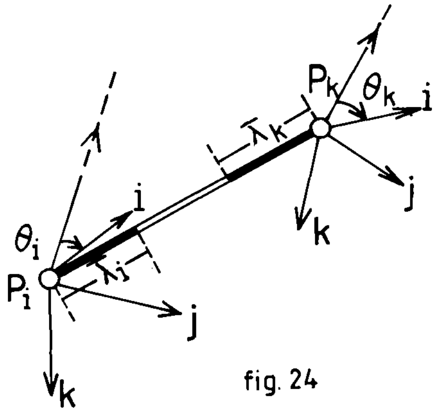


fig. 24

From (2.12') it follows immediately :

$$\bar{\lambda}_k \, q_{ik}^{(k')} = \bar{\lambda}_i \, p_{k'i} \, q_{ik}^{(i)} \, p_{k'i}^{-1} \tag{3.26}$$

with : $p_{k'i} = p(\theta_k = 0 , \varphi_k , \lambda_{ik} , \varphi_i , \theta_i)$

$q_{ik} = q(s_{ik} , r_{ik} , \mathfrak{z}_{ik})$.

The vector in the opposite direction is :

$$\bar{\lambda}_k \, q_{ki}^{(k')} = \bar{\lambda}_k \, p_{k'k} \, q_{ki}^{(k)} \, p_{k'k}^{-1} \tag{3.27}$$

with, see (2.27^1) : $p_{k'k} = k \sin \tfrac{1}{2} \theta_k$

therefore :

$$\bar{\lambda}_k \, q_{ki}^{(k')} = \bar{\lambda}_k \, s_{ki} \left[i \cos[r_{ki} + \theta_k] \sin \mathfrak{z}_{ki} + j \sin[r_{ki} + \theta_k] \sin \mathfrak{z}_{ki} - k \cos \mathfrak{z}_{ki} \right] \tag{3.27'}$$

In (3.26) and (3.27) the left-hand members only differ in sign ; therefore the quaternion equation can be established by means of the right-hand members :

$$\boxed{\bar{\lambda}_i \, p_{k'i} \, q_{ik}^{(i)} \, p_{k'i}^{-1} = - \bar{\lambda}_k \, p_{k'k} \, q_{ki}^{(k')} \, p_{k'k}^{-1}} \tag{3.28}$$

the rank of which in view of (3.25), is three.

From (3.28) we deduce two scalar equations, in order to determine, from this, the unknown quantities θ_k and $\bar{\lambda}_k$ (whilst underlining the stochastic quantities)

a) : $\sqrt{N\{\text{left hand member}\}} = \sqrt{N\{\text{right hand member}\}}$

hence : $\bar{\lambda}_i \, \underline{s}_{ik} = \bar{\lambda}_k \, \underline{s}_{ki}$

or :

$$\boxed{\underline{\bar{\lambda}}_k = \frac{\underline{s}_{ik}}{\underline{s}_{ki}} \underline{\bar{\lambda}}_i} \tag{3.29}$$

b) : $\arctan \dfrac{Vj\{\text{"left"}\}}{Vi\{\text{"left"}\}} = \arctan \dfrac{Vj\{\text{"right"}\}}{Vi\{\text{"right"}\}} =$

$= \underline{r}_{ki} + \theta_k \qquad (\text{if } Vi > 0).$

$= \underline{r}_{ki} + \theta_k + \pi \; (\text{if } Vi < 0).$

therefore :

$$\boxed{\underline{\theta}_k = \arctan \frac{Vi\{\underline{p}_{k'i} \, \underline{q}_{ik}^{(i)} \, \underline{p}_{k'i}^{-1}\}}{Vj\{ \quad \text{"} \quad \}} - \underline{r}_{ki} \quad [+\pi \;\; \text{if} \;\; Vi < 0]} \tag{3.30}$$

The units of length and the orientations of the local systems can be successively computed from (3.29) and (3.30), provided that a sufficient number of observation quantities have been measured. For this purpose, one unit of length, $\overline{\lambda}^{o}$, and one orientation must be known (the nature of these quantities is discussed in greater detail in section 4.2).

3.4.2
Difference quantities of $\overline{\lambda}_k$ and θ_k.

The expressions (3.29) and (3.30) for $\underline{\overline{\lambda}}_k$ and $\underline{\theta}_k$ respectively can be differentiated in two ways :

a) direct differentiation of (3.29) and (3.30) ; from this follows :

$$\Delta \underline{\ln \overline{\lambda}}_k = \Delta \underline{\ln \overline{\lambda}}_i + \Delta \underline{\ln s}_{ik} - \Delta \underline{\ln s}_{ki} \tag{3.31}$$

$$\underline{\Delta \theta}_k = \Delta \left[\arctan \frac{Vj\{\underline{q}_{ik}^{(k')}\}}{Vi\{\;{}''\;\}} \right] - \underline{\Delta r}_{ki} \tag{3.32}$$

b) first differentiating (3.28); from the difference equation, the difference quantities $\underline{\Delta \theta}_k$ and $\Delta \underline{\ln \overline{\lambda}}_k$ can then be solved.

We now apply method b) and start on the left-hand members of (3.26) and (3.27) ; they are equal to each other with opposite signs :

$$\overline{\lambda}_k \, \underline{q}_{ik}^{(k')} = - \overline{\lambda}_k \, \underline{q}_{ki}^{(k')}$$

The difference equation deduced from this reads, after division by $\overline{\lambda}_k$:

$$\underline{\Delta q}_{ik}^{(k')} = - \underline{\Delta q}_{ki}^{(k')} \tag{3.33}$$

and this passes, after premultiplication by q_{ik}^{-1} and $-q_{ki}^{-1}$ respectively, into :

$$(q^{-1} \underline{\Delta q})_{ik}^{(k')} = (q^{-1} \underline{\Delta q})_{ki}^{(k')} \tag{3.34}$$

The left-hand member is, according to (3.23) :

$$(q^{-1} \underline{\Delta q})_{ik}^{(k')} = \Delta \underline{\ln \overline{\lambda}}_{ki} + \Delta \underline{\ln s}_{ik} + e'^{(k')}_{k';ik} \left[\sin \zeta \, \underline{\Delta r} \right]_{ik}^{k'} + e''^{(k')}_{k;ik} \, \underline{\Delta \zeta}_{ik}^{k'} \tag{3.35}$$

With, according to the definition (2.13) :

$$\Delta \underline{\ln \overline{\lambda}}_{ki} = \Delta \underline{\ln \overline{\lambda}}_i - \Delta \underline{\ln \overline{\lambda}}_k$$

The right-hand member of (3.34) is, see also (3.11) :

$$(q^{-1} \underline{\Delta q})_{ki}^{(k')} = p_{k'k} \left[q_{ki}^{-1\,(k)} (p^{-1} \underline{\Delta p})_{k'k} \, q_{ki}^{(k)} - (p^{-1} \underline{\Delta p})_{k'k} + (q^{-1} \underline{\Delta q})_{ki}^{(k)} \right] p_{k'k}^{-1}$$

According to (2.30¹) here :

$$(p^{-1} \underline{\Delta p})_{k'k} = k \; \tfrac{1}{2} \, \underline{\Delta \theta}_k \; . \quad ; \quad (\Delta \underline{\ln \overline{\lambda}}_{k'k} \equiv 0) \, .$$

so, also taking into consideration (3.14) and (3.15) this becomes :

$$= p_{k'k}\left[(q^{-1}\underline{\Delta q})^{(k)}_{ki} \text{ in which } \underline{\Delta r}_{ki} \text{ is replaced by } \underline{\Delta r}_{ki}+\underline{\Delta \theta}_{k}\right] p^{-1}_{k'k}$$

In accordance with (3.18), this may be written as :

$$= p_{k'k}\left[\Delta \underline{\ln s}_{ki} + e'^{(k)}_{ki} \sin \mathfrak{z}_{ki} \left[\underline{\Delta r}_{ki} + \underline{\theta}_{k}\right] + e''^{(k)}_{ki}\, \underline{\Delta \mathfrak{z}}_{ki}\right] p^{-1}_{k'k} \cdot$$

$$= \underline{\Delta \ln s}_{ik} + e'^{(k')}_{ki} \sin \mathfrak{z}^{k}_{ki} \left[\underline{\Delta r}_{ki} + \underline{\Delta \theta}_{k}\right] + e''^{(k')}_{ki}\, \underline{\Delta \mathfrak{z}}_{ki} \tag{3.36}$$

Since the k-vectors of the systems (k) and (k') are equal to each other, the following applies according to the definitions (3.20) :

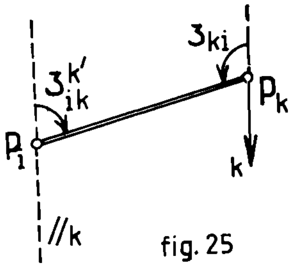


fig. 25

$$e'_{k';ik} \text{ (in 3.35) } = e'_{ik} = e'_{ki} \text{ (in 3.36)}.$$

$$e''_{k';ik} \text{ (in 3.35) } = e''_{ik} = - e''_{ki} \text{ (in 3.36)}.$$

and also :

$$\sin \mathfrak{z}^{k'}_{ik} \text{ (in 3.35) } = \sin \mathfrak{z}^{k}_{ik} = \sin \mathfrak{z}^{k}_{ki} \text{ (in 3.36)}$$

With this (3.36), which is the right-hand member of (3.34), can be converted into :

$$(q^{-1}\underline{\Delta q})^{(k')}_{ki} = \Delta \underline{\ln s}_{ki} + e'^{(k')}_{k';ik} \sin \mathfrak{z}^{k'}_{ik} \left[\underline{\Delta r}_{ki} + \underline{\Delta \theta}_{k}\right] - e''^{(k')}_{k';ik}\, \underline{\Delta \mathfrak{z}}_{ki} \tag{3.37}$$

By now substracting the left-hand member of (3.34), see (3.35), and the right-hand member of (3.34), see (3.37), from each other, we obtain a zero-mean variate :

$$\boxed{\begin{aligned} (q^{-1}\underline{\Delta q})^{(k')}_{ik} - (q^{-1}\underline{\Delta q})^{(k')}_{ki} = 0 = \Delta \underline{\ln \bar{\lambda}}_{i} - \Delta \underline{\ln \bar{\lambda}}_{k} + \Delta \underline{\ln s}_{ik} - \Delta \underline{\ln s}_{ki} \;+& \\ + e'^{(k')}_{k';ik} \sin \mathfrak{z}^{k'}_{ik} \left[\underline{\Delta r}^{k'}_{ik} - \underline{\Delta r}_{ki} - \underline{\Delta \theta}_{k}\right] +& \\ + e''^{(k')}_{k';ik} \left[\underline{\Delta \mathfrak{z}}^{k'}_{ik} + \underline{\Delta \mathfrak{z}}_{ki}\right]& \end{aligned}} \tag{3.38}$$

The next difference quantity follows from the scalar component of (3.38) :

$$\Delta \underline{\ln \bar{\lambda}}_{k} = \Delta \underline{\ln \bar{\lambda}}_{i} + \Delta \underline{\ln s}_{ik} - \Delta \underline{\ln s}_{ki}$$

More important than the units of length are the "length factors" defined in (2.13) (quotients) ; the length factor of (i)- and (k)-systems is :

$$\boxed{\Delta \underline{\ln \bar{\lambda}}_{ki} = \Delta \underline{\ln s}_{ki} - \Delta \underline{\ln s}_{ik}} \tag{3.39}$$

Because $e'_{k';ik}$ and $e''_{k';ik}$ are both $\neq 0$ and not parallel to each other, two independent scalar equations can be deduced from the vector component of (3.38) :

1: $\underline{\Delta r}_{ik}^{k'} - \underline{\Delta r}_{ki} - \underline{\Delta \theta}_k = 0$

from this it follows :

$$\boxed{\underline{\Delta \theta}_k = \underline{\Delta r}_{ik}^{k'} - \underline{\Delta r}_{ki}} \qquad (3.40)$$

with, see (3.24') :

$$\underline{\Delta r}_{ik}^{k'} = \left[\frac{-\cos r}{\sin \zeta \cos \zeta}\right]_{ik}^{k'} V_i\{(q^{-1}\underline{\Delta q})_{ik}^{(k')}\} - \left[\frac{\sin r}{\sin \zeta \cos \zeta}\right]_{ik}^{k'} V_j\{(q^{-1}\underline{\Delta q})_{ik}^{(k')}\}$$
$$= R\left(\underline{\Delta r}_{ik}, \underline{\Delta \zeta}_{ik}, \underline{\Delta \theta}_i, \underline{\Delta \varphi}_i, \underline{\Delta \lambda}_{ik}, \underline{\Delta \varphi}_k\right).$$

The complete elaboration of (3.32) gives the same result.

2: $\underline{\Delta \zeta}_{ik}^{k'} + \underline{\Delta \zeta}_{ki} = 0$ (3.41)

with, see again (3.24') :

$$\underline{\Delta \zeta}_{ik}^{k'} = -\sin r_{ik}^{k'} \; V_i\{(q^{-1}\underline{\Delta q})_{ik}^{(k')}\} + \cos r_{ik}^{k'} \; V_j\{(q^{-1}\underline{\Delta q})_{ik}\}$$

3.4.3
The rank of $(q^{-1}\underline{\Delta q})_{ik} - (q^{-1}\underline{\Delta q})_{ki}$

A network side $P_i P_k$ can thus be used for the computation of length factors $\underline{\bar{\lambda}}_{ik}$ and/or orientations $\underline{\theta}_k$; one then starts from an initial unit of length $\bar{\lambda}^o$ and an initial orientation θ^o (these need not be defined in the same local system ; see fig. 26)

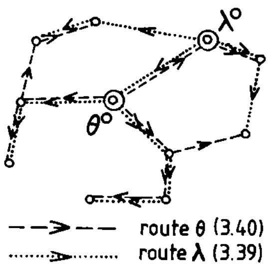


fig. 26

The zero-mean variate (3.38) assumes, depending on the use of side $P_i P_k$, one of the four following forms :

I Side $P_i P_k$ is neither used for the computation of $\underline{\bar{\lambda}}_{ik}$ nor for the computation of $\underline{\theta}_i$ from $\underline{\theta}_k$, or reverse. Then, the following is obtained :

$$\boxed{\begin{aligned} (q^{-1}\underline{\Delta q})_{ik}^{(r)} - (q^{-1}\underline{\Delta q})_{ki}^{(r)} &= \Delta \underline{\ln \bar{\lambda}}_i - \Delta \underline{\ln \bar{\lambda}}_k + \Delta \underline{\ln s}_{ik} - \Delta \underline{\ln s}_{ki} + \\ &+ e_{k';ik}^{\prime\,(r)} \sin \zeta_{ik}^{k'} \left[\underline{\Delta r}_{ik}^{k'} - \underline{\Delta r}_{ki} - \underline{\Delta \theta}_k\right] + \\ \text{rank} = 3 \qquad &+ e_{k';ik}^{\prime\prime\,(r)} \left[\underline{\Delta \zeta}_{ik}^{k'} + \underline{\Delta \zeta}_{ki}\right] . \end{aligned}} \qquad (3.42^{I})$$

II Side $P_i P_k$ is used for the computation of the length factor $\underline{\bar{\lambda}}_{ik}$; than (3.39) is substituted in (3.38), which means :

$$\boxed{\begin{array}{ll} (q^{-1}\underline{\Delta q})^{(r)}_{ik} - (q^{-1}\underline{\Delta q})^{(r)}_{ki} = & e'^{(r)}_{k;ik} \sin z^{k'}_{ik} \left[\underline{\Delta r}^{k'}_{ik} - \underline{\Delta r}_{ki} - \underline{\Delta \theta}_k\right] + \\ \text{rank} = 2\,. & + e''^{(r)}_{k';ik} \left[\underline{\Delta z}^{k'}_{ik} + \underline{\Delta z}_{ki}\right] \end{array}} \tag{3.42^{II}}$$

III Side $P_i P_k$ is used for the computation of $\underline{\theta}_k$ from $\underline{\theta}_i$; than (3.40) is substituted in (3.38) :

$$\boxed{\begin{array}{ll} (q^{-1}\underline{\Delta q})^{(r)}_{ik} - (q^{-1}\underline{\Delta q})^{(r)}_{ki} = & \underline{\Delta \ln \bar{\lambda}}_i - \underline{\Delta \ln \bar{\lambda}}_k + \underline{\Delta \ln s}_{ik} - \underline{\Delta \ln s}_{ki} + \\ \text{rank} = 2 & + e''^{(r)}_{k';ik} \left[\underline{\Delta z}^{k'}_{ik} + \underline{\Delta z}_{ki}\right]\,. \end{array}} \tag{3.42^{III}}$$

IV Side $P_i P_k$ is used both for the computation of $\underline{\bar{\lambda}}_{ik}$ and $\underline{\theta}_k$; now we obtain :

$$\boxed{\begin{array}{l} (q^{-1}\underline{\Delta q})^{(r)}_{ik} - (q^{-1}\underline{\Delta q})^{(r)}_{ki} = e''^{(r)}_{k';ik} \left[\underline{\Delta z}^{k'}_{ik} + \underline{\Delta z}_{ki}\right]\,. \\ \text{rank} = 1 \end{array}} \tag{3.42^{IV}}$$

From Chapter 5 it will appear that (3.42^{IV}) will occur as a condition equation in each side of a completely measured network.

3.4.4
Networks with parallel k-vectors.

As a special case, we now consider a network, in which the k-vectors of all the local systems are parallel to each other. In (2.34) it has been demonstrated that in such a network, the orientations occur as difference quantities in the "astronomical rotation quaternions" $(p^{-1}\Delta\, p)$.
This is therefore also the case in the zero-mean variate (3.38). In view of (3.15), the coefficients of $\underline{\Delta\theta}_i$ and $\underline{\Delta r}_{ik}$ in $(q^{-1}\underline{\Delta}\, q)^{(k')}_{ik}$ are equal to each other; therefore, see (3.40) :

$$\underline{\Delta r}^{k'}_{ik} = \left[\underline{\Delta r}_{ik} + \underline{\Delta \theta}_i\right] + \dots\dots \textit{other terms}$$

Thus, in a network with parallel k-vectors, (3.40) passes into the simple form :

$$\underline{\Delta\theta}_k = \underline{\Delta\theta}_i + \underline{\Delta r}_{ik} + \cdots\cdots - \underline{\Delta r}_{ki}\;. \tag{3.43}$$

Chapter 4

SIMILARITIES AND DIFFERENCES BETWEEN THE TWO-DIMENSIONAL AND THE THREE-DIMENSIONAL MODEL.

4.1 Introduction.

In the Polygon Theory in the Complex Plane [2] , the Π-quantity :

$$\underline{\Pi}_{jik} = \ln \frac{\underline{z}_{ik}}{\underline{z}_{ij}} \quad ; \quad \underline{\Delta\Pi}_{jik} = \Delta \underline{\ln z}_{ik} - \Delta \underline{\ln z}_{ij}$$

plays a fundamental role in the relation between the measuring procedure and the function model for the adjustment. This Π-quantity is fully invariant with respect to similarity transformations in R_2, owing to which orientations and length factors of the instrumental (local) systems do not occur in the conditions of the adjustment model, if they are composed from Π-quantities.
In the polygon theory for three-dimensional space, the Q-quantity :

$$\underline{\mathcal{Q}}^{(i)}_{jik} = \underline{q}_{ik}\,\underline{q}^{-1}_{ij} \quad ; \quad \underline{\Delta\Pi}^{(i)}_{jik} = (q^{-1}\underline{\Delta q})^{(i)}_{ik} - (q^{-1}\underline{\Delta q})^{(i)}_{ij} \tag{4.1}$$

is used for the construction of conditions. It is, however, not invariant relative to rotations :

$$\underline{\mathcal{Q}}^{(r)}_{jik} = \underline{p}_{ri}\,\underline{\mathcal{Q}}^{(i)}_{jik}\,\underline{p}^{-1}_{ri} \neq \underline{\mathcal{Q}}^{(i)}_{jik}$$

and, see (3.17) :

$$\underline{\Delta\Pi}^{(r)}_{jik} = p_{ri}\left[\underline{\Delta\Pi}^{(i)}_{jik} + q^{-1}_{ik}(p^{-1}\underline{\Delta p})_{ri}\,q_{ik} - q^{-1}_{ij}(p^{-1}\underline{\Delta p})_{ri}\,q_{ij}\right] p^{-1}_{ri}$$

The "three dimensional" $\underline{\Delta\Pi}_{jik}$-quantity is therefore indeed invariant, if :

$$q^{-1}_{ik}(p^{-1}\underline{\Delta p})_{ri}\,q_{ik} - q^{-1}_{ij}(p^{-1}\underline{\Delta p})_{ri}\,q_{ij} = 0$$

so, if : either a) : $q_{ij} \,//\, q_{ik}$ ("stretched" quaternion $\mathcal{Q}$)

or b) : $(p^{-1}\underline{\Delta p})_{ri} \perp q_{ij}$ and $(p^{-1}\underline{\Delta p})_{ri} \perp q_{ik}$,

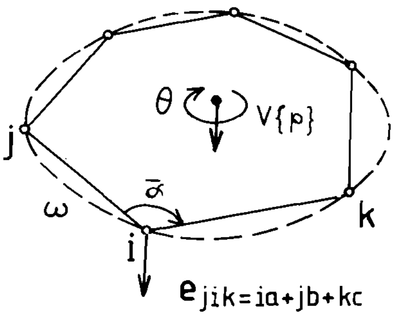
or: $(p^{-1}\underline{\Delta p})_{ri} \,//\, e_{jik}$

Situation b) is encountered, if the network lies entirely in one plane ω , and the following applies to all rotation quaternions :

(suppose : $\underline{p} = \cos \tfrac{1}{2}\underline{\theta} + i\,a\sin\tfrac{1}{2}\underline{\theta} + j\,b\sin\tfrac{1}{2}\underline{\theta} + k\,c\sin\tfrac{1}{2}\underline{\theta}$)

$$\begin{cases} Vc\{p\} \perp \omega \\ \Delta a = \Delta b = \Delta c = 0 \end{cases}$$

(only the angle of rotation is stochastic, the axis is not).



Then :

$$\mathcal{G} = v\cos\bar{\alpha} + v\sin\bar{\alpha}\,[\,i\,a + j\,b + k\,c\,]$$

$$(p^{-1}\underline{\Delta}p)_{ri} = [\,i\,a + j\,b + k\,c\,]\,\tfrac{1}{2}\,\underline{\Delta\theta}_{ri}$$

The fact that the $\underline{\Delta\Pi}$-quantities (4.1) are not generally invariant, leads to three important structural differences between the systems of condition equations in the two-dimensional model on the one hand and in the three-dimensional model on the other, to which we shall briefly refer in this introduction and analyse subsequently in sections 4.2, 4.3 and 4.4.

a)

In R_3 orientations of local systems do occur in condition equations. The orientations can be computed as functions of observation variates according to (3.30). If in a network not only terrestrial but also "astronomical" observation variates occur, it will be necessary for at least one azimuth to be measured; otherwise it will not be possible to connect the other astronomical quantities (longitudes and latitudes) with the terrestrial ones, because the rotation quaternions p would not be fully defined then : see (2.27). But in section 4.2 it will be demonstrated that this initial azimuth does not furnish a contribution to the rank of the system of condition equations : in fact it creates a linear dependency between the horizontal directions r at the station where this azimuth was measured.
It will also become apparent that, if the k-unit vectors of all local systems are parallel to each other, the coefficient of $\underline{\Delta A}$ (the azimuth) will become zero in all linearized condition equations. This corresponds to the existance of a linear dependency between the orientations in the adjustment model of the method of observation equations, so that the rank of the system remains unchanged (since the rank is equal to the number of observation variates minus the number of unknowns). From section 4.5 it will become apparent that this is important in the "transition" of the R_2-model to the R_3-model.

b)

The Q_{jik}-quantities must all be transformed from their "own" local system (i) to one common system (r). Subsequently, conditions may be established for the adjustment model of the method of condition equations. In section 4.3 it will be shown that the adjustment corrections obtained from this -consequently, also estimators $\underline{X}^i$ – and weight coefficients (G^{ij}) of observation variates (not yet of coordinate quantities) are independent of the choice of the (r)-system, i.e. one of the (i)-systems.

$$\underline{X}^{i;r} = \underline{X}^{i} \quad ; \quad G^{ij;r} = G^{ij}$$

c)

In the complex plane S-coordinates $\underline{z}^{(rs)}$ can be computed from estimators $\underline{X}^i$ of observation variates by :

$$\underline{z}^{(rs)}_{ik} = (-1)^n e^{[\underline{\Pi}_{rs.} + \cdots + \underline{\Pi}_{.ik}]}[z^o_s - z^o_r] \tag{4.2}$$

with : $\underline{\Delta\Pi}_{jik} = \Delta\underline{\ln v}_{jik} + i\,\underline{\Delta\alpha}_{jik}$

after the choice of two, non-stochastic pairs of coordinates z_r^o and z_s^o ($z_r^o - z_s^o \neq 0$); backgrounds of this line of thought are to be found in W. Baarda's theory of S-transformations [3] .
In the analogous formula for the three dimensional model, the $Q^{(r)}$-quantities have been defined in one of the local systems, the (r)-system (see b) and they are not invariant (see a). The introduction of operationally defined coordinates, an "S-system" (R) must therefore be made by means of a (difference) similarity transformation $\underline{\bar{\lambda}}_{Rr}$; $\underline{p}_{Rr}$:

$$\underline{q}_{ik}^{(R)} = (-1)^n \, \underline{p}_{Rr} \, \underline{\bar{\bar{\varphi}}}_{jik}^{(r)} \, \underline{\bar{\bar{\varphi}}}_{.ji}^{(r)} \cdots\cdots \underline{\bar{\bar{\varphi}}}_{rs.}^{(r)} \, \underline{p}_{Rr}^{-1} \, \underline{q}_{rs}^{(R)} \qquad (4.3)$$

$$\left.\begin{aligned} \text{with : } \underline{\bar{\bar{\varphi}}}^{(r)} &= \varphi^{r}(\ldots,\underline{X}^{i},\ldots) \\ \underline{q}_{rs}^{(R)} &= \underline{\bar{\lambda}}_{Rr} \underline{p}_{Rr} \, \underline{\bar{\bar{q}}}_{rs}^{(r)} \, \underline{p}_{Rr}^{-1} \end{aligned}\right\} \; \underline{X}^{i} \; ; \; \underline{\bar{\lambda}}_{Rr} \; : \text{ estimators} \qquad (4.4)$$

Notation : see page 31.

This is the "first basis equation"; it has three independent components. By adding to this one component of, for example,the vector q_{rt} , a system of four "basis equations" is obtained. After differentiating -see section 4.4- $\Delta \, \underline{\ln \bar{\lambda}}_{Rr}$ and the three independent components of $\Delta \underline{p}_{Rr}$ can be solved from these (i.e. expressed in differences $\Delta \underline{X}^i$ of estimators of observation variates in the vectors q_{rs} and q_{rt}).

Owing to this (4.4) becomes :

$$\underline{q}_{rs}^{(R)} = [q_s^o - q_r^o] = q_{rs}^o$$

and 4.3 becomes :

$$\underline{q}_{ik}^{(R)} = (-1)^n \, \underline{p}_{Rr} \, \underline{\bar{\bar{\varphi}}}_{jik}^{(r)} \, \underline{\bar{\bar{\varphi}}}_{.ji}^{(r)} \cdots\cdots \bar{\bar{\varphi}}_{rs.}^{(r)} \, \underline{p}_{Rr}^{-1} [q_s^o - q_r^o] \; . \qquad (4.5)$$

Compare (4.2) !

When applying the adjustment method of observation equations, the transformation $\bar{\lambda}_{Rr}$; p_{Rr} is represented in the observation equations by four unknowns $\underline{Y}^{\alpha}$:

$$\underline{\ln \bar{\lambda}}_{Rr} \equiv \underline{Y}^1 \; ; \; Vi\{\underline{p}_{Rr}\} \equiv \underline{Y}^2 \; ; \; Vj\{\underline{p}_{Rr}\} \equiv \underline{Y}^3 \; ; \; Vk\{\underline{p}_{Rr}\} \equiv \underline{Y}^4 \; .$$

In section 4.5 it will be shown that the functional model for the adjustment of a two-dimensional network of closed polygons, despite the discussed structural differences a) and c) (sections 4.2 and 4.4) is a "special case" of the "three-dimensional" model ; by removing the zenith angles and Z-coordinates, the system of condition equations for R_3 automatically" becomes R_2 , well known from [2] . By "automatic" is meant here that the other differences between the R_3 -model and the R_2 -model (numbers and types of quantities and condition equations) correspond directly to the algebraic properties of R_3 and R_2 : 3 and 2 components respectively in the "coordinate condition"; 4 + 2, respectively 2 + 2 parameters in a complete similarity transformation.
This is illustrated in the list (4.53) of observation variates, unknowns and conditions

4.2 The first unit of length and the first orientation

In section 3.4 it was shown how the units of length $\bar{\lambda}_i$ and the orientations Θ_i of the local systems can be expressed as functions of observation variates $\underline{x}^i$, by solution from a series of equations (3.28).
For this, however, an initial unit of length $\bar{\lambda}^o$, resp. orientation $\overset{o}{\Theta}$ must be known. Let us assume, for the time being, that these are stochastic quantities, then :

$$(3.29) \rightarrow \underline{\bar{\lambda}}_i = \bar{\lambda}_i(\ldots,\underline{x}^i,\ldots,\underline{\bar{\lambda}}^o) . \qquad (4.6^a)$$

$$(3.30) \rightarrow \underline{\theta}_i = \theta_i(\ldots,\underline{x}^i,\ldots,\underline{\theta}^o) . \qquad (4.6^b)$$

In section 3.4 it was already stated that $\bar{\lambda}^o$ and Θ^o may be located in different stations ; on this occasion, we shall see that they also have quite different functions :

$\bar{\lambda}^o$ is a non-stochastic factor which is only relevant for the computation technique ; it is chosen such that the numerical values of the lengths of the sides, $\sqrt{N\{\bar{q}\}}$, are given the order of magnitude 1. (see also the scheme on page 31.

By θ^o the "astronomical" observation variates (longitudes and latitudes) are connected with the "terrestrial" horizontal directions, zenith angles, and distances. It should be determined from azimuth measurements in one of the local systems, P_a , and is then a stochastic variate.

4.2.1

The first unit of length $\bar{\lambda}^o$.

We consider (2.10) applied to the side $P_r P_u$:

$$\underline{\bar{\lambda}}_r \underline{s}_{ru} = \sqrt{N\{\underline{\bar{q}}_{ru}\}}$$

If q_{ru} is a vector of average length, one achieves by choosing :

$$\bar{\lambda}^o = \bar{\lambda}_r = \frac{1}{s_{ru}} \qquad (f \text{ on page 31}: f = s_{ru}) \qquad (4.7)$$

that :

$$\sqrt{N\{\bar{q}_{ru}\}} = 1 .$$

and the lengths of all other sides of the network:

$$\sqrt{N\{\bar{q}_{ik}\}} \approx 1 .$$

4.2.2

The first orientation.

According to (2.24) the orientation of a local system is the angle in the horizontal plane between astronomical north and the i-vector of the local system.

The first orientation θ^o can thus be determined by measuring an azimuth $\underline{A}$ in one of the stations, P_a . Then θ^o is :

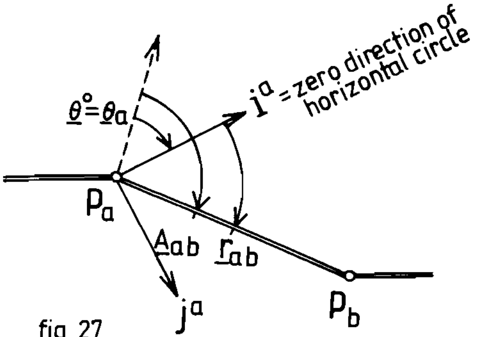


fig. 27

$$\boxed{\underline{\theta}^o \equiv \underline{\theta}_a = \underline{A}_{ab} - \underline{r}_{ab}} \tag{4.8}$$

In appendix 1 it is proved, that in each condition equation of the types N, V, Z and A (see chapter 5) the coefficient of $\Delta\underline{\theta}^o$ (i.e. $\Delta\underline{\theta}_a$) equals the sum of the coefficients of all horizontal directions in P_a . Using Baarda's notation [4] for a condition equation with m observation variates x^i :

$$\Delta\underline{y}^p = (u^p_i)(\Delta\underline{x}^i) \qquad i = 1 \ldots\ldots m$$

and assuming that there are three directions in P_a :

$$\Delta\underline{r}_{a1} \equiv \Delta\underline{x}^1$$

$$\Delta\underline{r}_{a2} \equiv \Delta\underline{x}^2$$

$$\Delta\underline{r}_{a3} \equiv \Delta\underline{x}^3$$

the conclusion of appendix 1 reads :

$$\Delta\underline{y}^p = u^p_1\, \Delta\underline{r}_{a1} + u^p_2\, \Delta\underline{r}_{a2} + u^p_3\, \Delta\underline{r}_{a3} + \left[u^p_1 + u^p_2 + u^p_3\right] \Delta\underline{\theta}_a + \cdots u^p_j\, \Delta\underline{x}^j \ldots\ldots \quad (j = 4 \ldots\ldots m) \tag{4.9}$$

Because $\Delta\underline{\theta}_a$ is not an observation variate, we now substitute the difference equation of (4.8) into (4.9) ; the azimuth be measured in side $P_a P_3$:

$$\Delta\underline{\theta}_a = \Delta\underline{A}_{a3} - \Delta\underline{r}_{a3}$$

and (4.9) becomes :

$$\boxed{\Delta\underline{y}^p = \underbrace{u^p_1\, \Delta\underline{r}_{a1} + u^p_2\, \Delta\underline{r}_{a2} + \left[-u^p_1 - u^p_2\right]}_{!} \Delta\underline{r}_{a3} + \left[u^p_1 + u^p_2 + u^p_3\right] \Delta\underline{A} + \cdots u^p_j\, \Delta\underline{x}^j \ldots} \tag{4.10}$$

There arises a linear dependency between the directions in P_a. The number of observation variates can be reduced by one, by passing from n directions to n-1 angles in P_a :

$$\Delta\underline{\alpha}_{jai} = \Delta\underline{r}_{ai} - \Delta\underline{r}_{aj} \tag{4.11}$$

then (4.10) becomes :

$$\Delta\underline{y}^p = u^p_1\, \Delta\underline{\alpha}_{3a1} + u^p_2\, \Delta\underline{\alpha}_{3a2} + \left[u^p_1 + u^p_2 + u^p_3\right] \Delta\underline{A}_{a3} + \cdots + u^p_j\, \Delta\underline{x}^j \ldots\ldots \tag{4.12}$$

4.2.3
The position of the first orientation in a network with parallel k-unit-vectors.

As a special case of spatial networks with astronomically measured rotations, we now consider a network, in which all k-vectors ("first axes" of the local systems) are parallel to each other. In section 2.3.3 it was shown that in this situation the orientations $\underline{\Theta}_i$ in the rotation quaternions p_{ri} exclusively occur as difference quantities. This is, therefore, also the case in the zero-mean variates discussed in the appendix, from which the condition equations are composed : (t_{ik} and u_{ji} are quaternions)

$$\underline{V}_{ik}^{(r)} = (q^{-1}\underline{\Delta q})_{ik}^{(r)} - (q^{-1}\underline{\Delta q})_{ki}^{(r)} = \cdots + t_{ik}[\underline{\Delta\theta}_k - \underline{\Delta\theta}_i] \qquad (4.13^a)$$

$$\underline{W}_{1\ldots n}^{(r)} = \sum \underline{\Delta q}_{ik}^{(r)} = \cdots\cdots + u_{ik}[\underline{\Delta\theta}_k - \underline{\Delta\theta}_i] \,.\ \text{(closed polygon)} \qquad (4.13^b)$$

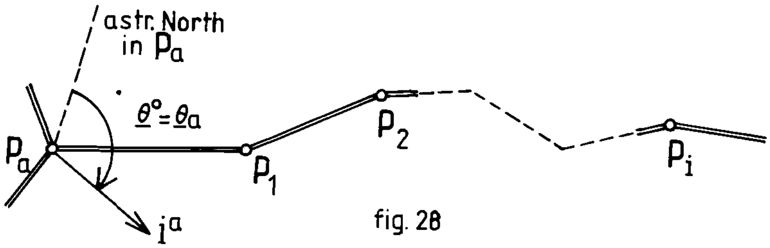


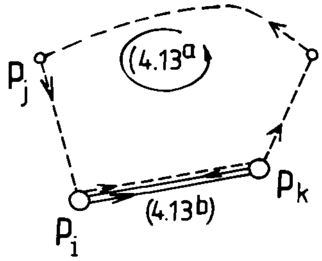
fig. 28

Moreover, the difference formula (3.40) for the orientation passes into the simple form (3.43) :

$$\underline{\Delta\theta}_k = -\underline{\Delta r}_{ki} + \underline{\Delta\theta}_i + \underline{\Delta r}_{ik} \,.$$

so that the differences of the orientations then read as follows :

$$\begin{aligned} \underline{\Delta\theta}_1 &= \underline{\Delta\theta}^o + \underline{\Delta r}_{a1} - \underline{\Delta r}_{1a} \,. \\ \underline{\Delta\theta}_2 &= \underline{\Delta\theta}_1 + \underline{\Delta r}_{12} - \underline{\Delta r}_{21} \,. \\ &= \underline{\Delta\theta}^o + \underline{\Delta r}_{a1} - \underline{\Delta r}_{1a} + \underline{\Delta r}_{12} - \underline{\Delta r}_{21} \,. \\ &\vdots \\ \underline{\Delta\theta}_i &= \underline{\Delta\theta}^o + \quad - \quad + \quad - \quad \cdots\cdots \end{aligned}$$



This means that in the zero-mean variates (4.13^a) and (4.13^b) and therefore also in the condition equations, now the coefficient of $\underline{\Delta\theta}^o$ equals zero. In view of (4.9), the coefficient of $\Delta \underline{A}$ in the condition equations then also equals zero :

$$\boxed{\frac{\partial y^p}{\partial A} = 0} \qquad (4.14)$$

The azimuth must, therefore, be deleted in the condition model as observation variate ; in doing so, the rank of the system of condition equations is b, as in (4.11).

Also in the observation equations (of $\Delta \underline{r}_{ik}$), the orientations now occur as difference quantities ; since they act here as "unknowns", this would lead to a linear dependence,

and result in singularity of the matrix of the normal equations.
In order to prevent this, we pass to the difference quantities $\underline{t}_i$; for example :

$$\boxed{\underline{t}_i = \underline{\theta}_i - \underline{\theta}_a \; ; \quad \theta_a \text{ is deleted as unknown.}} \tag{4.15}$$

Conclusion.
In a network with parallel "first axes", the observation variate (azimuth) is deleted in both adjustment models, see (4.14) and in the method of observation equations, moreover, one of the orientation unknowns see (4.15).

Schematically :

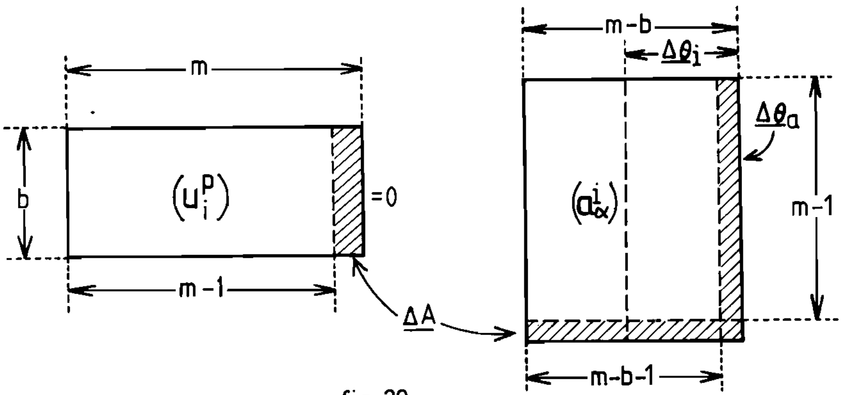


fig. 29

4.3 Estimators and weight coefficients of observation variates are invariant.

In a spatial polygon network, the following types of observation variates are measured :

$\underline{s}_{ik}$: distance measures
$\underline{r}_{ik}$: horizontal directions
$\underline{\mathfrak{z}}_{ik}$: zenith angles
$\underline{\lambda}_{ik}$: astr. longitude differences
φ_i : astr. latitudes

From $\underline{s}_{ik}$, $\underline{r}_{ik}$ en $\underline{\mathfrak{z}}_{ik}$ the quaternion :

$$\underline{q}_{ik}^{(i)} = 0 + i\, \underline{s}_{ik} \cos \underline{r}_{ik} \sin \underline{\mathfrak{z}}_{ik} + j\, \underline{s}_{ik} \sin \underline{r}_{ik} \sin \underline{\mathfrak{z}}_{ik} - k\, \underline{s}_{ik} \cos \underline{\mathfrak{z}}_{ik} \tag{4.16}$$

see (2.9) can be computed in the local system (i).
For carrying out transformations of local systems, the orientations $\underline{\theta}_i$ are required.
Provided that a sufficient number of observation variates have been measured in the network considered, they can be computed as functions of observation

variates by (3.30) :

$$\underline{\theta}_i = \theta_i(\cdots,\underline{x}^i,\cdots,\underline{\theta}^\circ)\,.$$

In view of (4.8) here :

$$\underline{\theta}^\circ = \underline{A}_{ab} - \underline{r}_{ab}$$

is also a function of observation variates $\underline{x}^i$, so we can put :

$$\underline{\theta}_i = \theta_i(\cdots,\underline{x}_i,\cdots) \tag{4.17}$$

Then the rotations between the local systems can be expressed in the quaternions :

$$\begin{aligned} \underline{p}_{ji} &= p_{ji}\left(\theta_j(\cdot\cdot,\underline{x},\cdots),\underline{\varphi}_j,\underline{\lambda}_{ji},\underline{\varphi}_i,\theta_i(\cdot\cdot,\underline{x}^i,\cdots)\right) = \\ &= p_{ji}(\cdots,\underline{x}^i,\cdots)\,. \end{aligned} \tag{4.18}$$

Furthermore, from the distance measures $\underline{s}$, the "length factors " $\underline{\bar{\lambda}}_{ik}$ can be computed via network sides in which $\underline{s}$ are measured in both directions :

$$(3.29),(2.13):\ \underline{\bar{\lambda}}_{ji} = \frac{\underline{s}_{ji}}{\underline{s}_{ij}} = \bar{\lambda}_{ji}(\cdot\cdot,\underline{x}^i,\cdot\cdot) \tag{4.19}$$

We now choose one of the local systems, the (r)-system and transform all quaternions (4.16) to this system through a "chain" of rotations and length factors :

$$\begin{aligned} \underline{p}_{ri} &= \underline{p}_{ra}\,\underline{p}_{a\cdot}\cdots\underline{p}_{\cdot k}\,\underline{p}_{ki} \\ \underline{\bar{\lambda}}_{ri} &= \underline{\bar{\lambda}}_{rb}\,\underline{\bar{\lambda}}_{b\cdot}\cdots\underline{\bar{\lambda}}_{\cdot j}\,\underline{\bar{\lambda}}_{ji} \\ \underline{\overset{(r)}{q}}_{ik} &= \underline{\bar{\lambda}}_{ri}\,\underline{p}_{ri}\,\underline{\overset{(i)}{q}}_{ik}\,\underline{p}_{ri}^{-1}\,. \end{aligned} \tag{4.20}$$

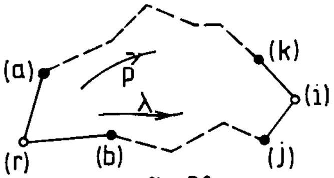


fig. 30

In view of (4.16), (4.18) and (4.19), this may be put as :

$$\underline{\overset{(r)}{q}}_{ik} = \overset{r}{q}_{ik}(\cdots,\underline{x}^i,\cdots)\,. \tag{4.21}$$

or, introducing means $\tilde{X}^i$of observation variates :

$$\overset{(r)}{\tilde{q}}_{ik} = \overset{r}{q}_{ik}(\cdots,\tilde{X}^i,\cdots) \tag{4.21'}$$

Furthermore :

$$\begin{aligned} \underline{\overset{(r)}{\bar{\bar{\mathcal{G}}}}}_{jik} &= \underline{p}_{ri}\,\underline{\overset{(i)}{\bar{\bar{\mathcal{G}}}}}_{jik}\,\underline{p}_{ri}^{-1} = \\ &= \overset{r}{\mathcal{G}}_{jik}(\cdots,\underline{X}^i,\cdots)\,. \end{aligned} \tag{4.22}$$

and : $\overset{(r)}{\tilde{\mathcal{G}}}_{jik} = \overset{r}{\mathcal{G}}_{jik}(\cdot\cdot,\tilde{X}^i,\cdots)\,.$

(4.22')

With the quantities (4.21') and (4.22'), being functions exclusively of observation variates, we can deduce conditions. In the form of ρ quaternion equations reduced to zero, with means of zero-mean variates Y^{ρ} these conditions are :

$$\tilde{Y}^{\rho(r)} = Y^{\rho}(\ldots,\tilde{\mathcal{Q}}^{(r)}_{jik},\ldots,\tilde{q}^{(r)}_{\ell m},\ldots) = 0 . \tag{4.23}$$

By substitution of observations $\underline{x}^{i}$ (4.21) and (4.22), misclosures $\underline{y}^{\rho(r)}$ are obtained :

$$\underline{y}^{\rho(r)} = Y^{\rho}(\ldots,\underline{\mathcal{Q}}^{(r)}_{jik},\ldots,\underline{q}^{(r)}_{\ell m},\ldots) \tag{4.24}$$

Differentiation leads to :

$$\begin{aligned}\underline{\Delta y}^{\rho(r)} = \sum_{i=1}^{m} \frac{\partial Sc\{y^{\rho(r)}\}}{\partial x^{i}}\, \underline{\Delta x}^{i} + i \sum_{i=1}^{m} \frac{\partial Vi\{y^{\rho(r)}\}}{\partial x^{i}}\, \underline{\Delta x}^{i} +\\ + j \sum_{i=1}^{m} \frac{\partial Vj\{y^{\rho(r)}\}}{\partial x^{i}}\, \underline{\Delta x}^{i} + k \sum_{i=1}^{m} \frac{\partial Vk\{y^{\rho(r)}\}}{\partial x^{i}}\, \underline{\Delta x}^{i} .\end{aligned} \tag{4.25}$$

When choosing another local system, different from the (r)-system, e.g. the (w)-system, the series of operations (4.20) - (4.25) leads to quaternion condition equations :

$$\underline{y}^{\rho(w)} \; ; \; \underline{\Delta y}^{\rho(w)}$$

Now, the following applies :

$$\boxed{\underline{y}^{\rho(w)} = \underline{\lambda}_{wr}\, \underline{p}_{wr}\, \underline{y}^{\rho(r)}\, \underline{p}^{-1}_{wr}} \tag{4.26}$$

hence :

$$\underline{\Delta y}^{\rho(w)} = \lambda_{wr}\, p_{wr} \left[\underline{\Delta y}^{\rho(r)} + y^{\rho(r)}\, \underline{\Delta \ln \lambda}_{wr} + (p^{-1}\underline{\Delta} p)_{wr}\, y^{\rho(r)} - y^{\rho(r)} (p^{-1}\underline{\Delta} p)_{wr} \right] p^{-1}_{wr}$$

Because $\underline{y}^{\rho(r)} \approx 0$, the three products of difference quantities and $\underline{y}^{\rho(r)}$ therein may be neglected; hence :

$$\boxed{\underline{\Delta y}^{\rho(w)} = \lambda_{wr}\, p_{wr}\, \underline{\Delta y}^{\rho(r)}\, p^{-1}_{wr}} \tag{4.27}$$

This means that the system of condition equations $\underline{\Delta y}^{\rho(w)}$ is linearly dependent on the system $\underline{\Delta y}^{\rho(r)}$, the same dependency prevailing between the respective misclosures (4.26)
The result of adjustment by the method of condition equations is therefore independent of the choice of an (r)-system, apart from effects of the second order ;

$$\begin{aligned}&\text{Estimators :} && (\underline{x}^{i})^{w} = (\underline{x}^{i})^{r} \rightarrow (\underline{x}^{i})\\ &\text{Weight coefficients :} && (G^{ij})^{w} = (G^{ij})^{r} \rightarrow (G^{ij})\end{aligned} \tag{4.28}$$

4.4 The introduction of S-coordinates.

According to section 4.3, "invariant" estimators $\underline{X}^i$ and weight coefficients G^{ij} are obtained from least-squares adjustment by the method of observation equations, The substitution of the estimators in the functions (4.16) - (4.21) inclusive gives :

$$\underline{\overline{\overline{q}}}^{(r)}_{ik} = q^{r}_{ik}(\cdots,\underline{X}^i,\cdots) \tag{4.28'}$$

for all the sides of the network.
The three vector components of these quaternions can be considered as coordinate differences in the (r)-system, i.e. one of the local systems.
The transition to an "operationally defined" coordinate system (R)—an "S-coordinate system in the terminology of [3] — is now effected using the similarity transformation :

$$\underline{q}^{(R)}_{ik} = \underline{\overline{\lambda}}_{Rr}\,\underline{p}_{Rr}\,\underline{\overline{\overline{q}}}^{(i)}_{ik}\,\underline{p}^{-1}_{Rr} \tag{4.29}$$

The transformation (4.29) has four parameters, viz. $\overline{\lambda}_{Rr}$ and the three independent components of $\underline{p}_{Rr}$; this is the correct number for a similarity transformation of coordinate differences in R_3.
The (R)-system, apart from the translation, can thus be operationally defined by considering four coordinate differences, or functions thereof, as non-stochastic quantities q^o ; we adopt the choice made by Baarda in several manuscripts, about 1970 :

a) : $$\boxed{\underline{q}^{(R)}_{rs} = \overset{o}{q}_{rs}} \tag{4.30}$$

i.e. : all three components of vector q_{rs} are non-stochastic. For $\overset{o}{q}_{rs}$ three arbitrary numbers may be chosen, provided $N\{q_{rs}\} \neq 0$.

b) : $$\tfrac{1}{2}\left[\underline{q}^{(R)}_{rt} + \overset{o}{e}{}^{-1}_{tsr}\,\underline{q}^{(R)}_{rt}\,\overset{o}{e}_{tsr}\right] = \tfrac{1}{2}\left[\overset{o}{q}_{rt} + \overset{o}{e}{}^{-1}_{tsr}\,\overset{o}{q}_{rt}\,\overset{o}{e}_{tsr}\right]$$

i.e. : of the vector q_{rt} , the component perpendicular to the plane through P_r P_s and P_t is non-stochastic.
Because : $\overset{o}{e}_{trs} \perp \overset{o}{q}_{rt}$ we have :

$$\overset{o}{e}{}^{-1}_{trs}\,\overset{o}{q}_{rt} = -\overset{o}{q}_{rt}\,\overset{o}{e}_{trs}$$

and the right-hand member of b) is zero.

Therefore b) becomes :

$$\boxed{\tfrac{1}{2}\left[\underline{q}^{(R)}_{rt} + \overset{o}{e}{}^{-1}_{tsr}\,\underline{q}^{(R)}_{rt}\,\overset{o}{e}_{tsr}\right] = 0.} \tag{4.31}$$

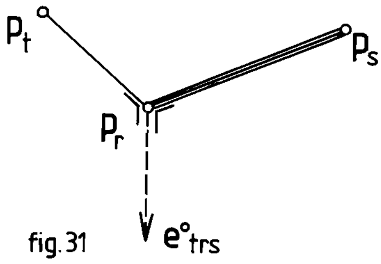


fig.31

Remark :
Only one component of vector q_{rt} is used for the definition of the (R)-system ; $\overset{o}{q}_{rt}$ does not explicitely occur in (4.31), therefore. It is, however, possible to choose numbers for $\overset{o}{q}_{rt}$, allowing $\overset{o}{e}_{trs}$ to be computed as follows :

$$\overset{o}{e}_{trs} = \sqrt{\frac{N\{\overset{o}{q}_{rt}\}}{N\{\overset{o}{q}_{rs}\}}}\;\overset{o}{q}_{rs}\,\overset{o}{q}{}^{-1}_{rt}$$

A clearer procedure consists in choosing $\overset{o}{e}_{trs}$ itself :

$$\overset{o}{e}_{trs} = 0 + i\, a^o + j\, b^o + k\, c^o$$

Now it becomes obvious that, in accordance with the fact that (4.31) represents one component, only one number needs to be chosen, since between a) , b) and c) there are two relationships :
-first $\overset{o}{e}_{trs}$ must be perpendicular to :

$$\overset{o}{q}_{rs} = 0 + i\, X^o + j\, Y^o + k\, Z^o$$

hence : $a^o X^o + b^o Y^o + c^o Z^o = 0.$

-secondly, $\overset{o}{e}_{trs}$ is a unit vector, so :

$$a^{o\,2} + b^{o\,2} + c^{o\,2} = 1 .$$

We now substitute (4.29), applied to the vectors q_{rs} and q_{rt} , which are measured sides of the network, in the left-hand members of (4.30) and (4.31) respectively :

$$\boxed{\begin{array}{l} \underline{\bar{\lambda}}_{Rr}\, \underline{p}_{Rr}\, \underline{\bar{\bar{q}}}^{(r)}_{rs}\, \underline{p}^{-1}_{Rr} = \overset{o}{q}_{rs} \\ \underline{\bar{\lambda}}_{Rr}\left[\underline{p}_{Rr}\, \underline{\bar{\bar{q}}}^{(r)}_{rt}\, \underline{p}^{-1}_{Rr} + e^{o-1}_{tsr}\, \underline{p}_{Rr}\, \underline{\bar{\bar{q}}}^{(r)}_{rt}\, \underline{p}^{-1}_{Rr}\, \overset{o}{e}_{tsr} \right] = 0 \end{array}} \qquad (4.32)$$

These quaternion equations constitute, on condition that q_{rs} is not $//q_{rt}$, four independent equations in scalars, the so-called "basis equations". From these the four parameters of the (stochastic) transformation $\{\underline{\bar{\lambda}}_{Rr} ; \underline{p}_{Rr}\}$ can be solved, i.e. expressed in components of :

$$\underline{\bar{\bar{q}}}^{(r)}_{rs}(\ldots,\underline{x}^i,\ldots) \quad \text{and} \quad \underline{\bar{\bar{q}}}^{(r)}_{rt}(\ldots,\underline{x}^i,\ldots)$$

Hence :

$$\left.\begin{array}{l} \underline{\bar{\lambda}}_{Rr} = \bar{\lambda}_{Rr}(\ldots,\underline{x}^i,\ldots) \\ \\ \underline{p}_{Rr} = p_{Rr}(\ldots,\underline{x}^i,\ldots) \end{array}\right\} \ \underline{x}^i \text{ are observation variates in } q_{rs} \text{ and } q_{rt} \qquad (4.33)$$

When using the adjustment method of observation equations, the transformation $\{\underline{\bar{\lambda}}_{Rr} ; \underline{p}_{Rr}\}$ is applied in the computation of S-coordinates $\underline{X}^r$, from observation variates $\underline{X}^i$, according to (4.5) of (4.29).

In (4.29) the following applies :

$$\underline{\bar{\bar{q}}}^{(r)}_{ik} = q^{r}_{ik}(\ldots,\underline{x}^i,\ldots) : \quad \text{see } (4.28') .$$

$$\underline{\bar{\lambda}}_{Rr} = \bar{\lambda}_{Rr}(\ldots,\underline{x}^i,\ldots) : \quad \text{see } (4.33) .$$

$$\underline{p}_{Rr} = p_{Rr}(\ldots,\underline{x}^i,\ldots) : \quad \text{see } (4.33) .$$

Let :

$$\underline{q}_{ik}^{(R)} = 0 + i\,\underline{X}^{r1} + j\,\underline{X}^{r2} + k\,\underline{X}^{r3} = q_{ik}^{(R)}(\ldots,\underline{X}^{r},\ldots) \tag{4.34}$$

notation : $\underline{x}_{ik} = \underline{X}^{r1}$; $\underline{y}_{ik} = \underline{X}^{r2}$; $\underline{z}_{ik} = \underline{X}^{r3}$

then (4.29) becomes :

$$q_{ik}^{R}(\ldots,\underline{X}^{r},\ldots) = \lambda_{Rr}(\ldots,\underline{X}^{i},\ldots)\, p_{Rr}(\ldots,\underline{X}^{i},\ldots)\, q_{ik}^{r}(\ldots,\underline{X}^{i},\ldots)\,[\,p_{Rr}(\ldots,\underline{X}^{i},\ldots)]^{-1}. \tag{4.35}$$

Via a difference equation of (4.35), the weight coefficients for the S-coordinates $\underline{X}^{r}$ can then be computed : see (4.43)

$$(G^{rs})^{R} = (T^{r}_{i})(G^{ij})(T^{s}_{j})^{*} \tag{4.36}$$

In the method of condition equations, the transformation $\{\underline{p}_{Rr};\underline{\bar{\lambda}}_{Rr}\}$ is directly entered in the observation equations in the form of four "unknowns" $\underline{Y}^{\alpha}$
From (4.29) follows directly :

$$\underline{\bar{\bar{q}}}_{ik}^{(r)} = \frac{1}{\underline{\bar{\lambda}}_{Rr}}\, \underline{p}_{Rr}^{-1}\, \underline{q}_{ik}^{(R)}\, \underline{p}_{Rr} \tag{4.37}$$

Here again, see (4.28') :

$$\underline{\bar{\bar{q}}}_{ik}^{(r)} = q_{ik}^{r}(\ldots,\underline{X}^{i},\ldots)$$

with : $\underline{X}^{i} = \underline{s}_{ik}+\underline{\varepsilon}\ ,\ \underline{r}_{ik}+\underline{\varepsilon}\ ,\ \mathfrak{z}_{ik}+\underline{\varepsilon}$

The S-coordinates $\underline{q}^{(R)}$ are now unknowns $\underline{Y}^{\alpha}$; let :

$$\underline{q}_{ik}^{(R)} = 0 + i\,\underline{Y}^{\alpha 1} + j\,\underline{Y}^{\alpha 2} + k\,\underline{Y}^{\alpha 3} \tag{4.38}$$

Assume also :

$$\underline{\bar{\lambda}}_{Rr} = \underline{Y}^{1}$$

$$\underline{p}_{Rr} = \sqrt{1-\underline{I}_{Rr}^{2}-\underline{J}_{Rr}^{2}-\underline{k}_{Rr}^{2}} + i\,\underline{I}_{Rr} + j\,\underline{J}_{Rr} + k\,\underline{k}_{Rr} \tag{4.39}$$

with : $\underline{I}_{Rr} = \underline{Y}^{2}$; $\underline{J}_{Rr} = \underline{Y}^{3}$; $\underline{k}_{Rr} = \underline{Y}^{4}$

it will then be possible to deduce observation equations for $\underline{s}_{ik}$, $\underline{r}_{ik}$ and $\mathfrak{z}_{ik}$ from the difference equation of (4.36), after some manipulations (see section 5.6)
The adjustment results in least-squares corrections and weight-coefficients :

$\underline{\Delta Y}^{\alpha}$; $(\bar{g}^{\alpha\beta})$.

The solution of $\Delta \ln \underline{\bar{\lambda}}_{Rr}$ and $\Delta\, \underline{p}_{Rr}$ from the basis equations

The four numerical values for q^{o}_{rs} and e^{o}_{trs} (or : q^{o}_{rt}) may be arbitrarily chosen, provided $N\{q^{o}_{rs}\} \neq 0$
However, we make a deliberate choice, using the observations $\underline{x}^{i}$:

$$\bar{q}^{o}_{rs} = \bar{\lambda}^{o}\, q_{rs}(\ldots,x^{i},\ldots) \quad ; \quad x^{i} = s_{rs}\ ,\ r_{rs}\ ,\ \mathfrak{z}_{rs}\ .$$

$$\bar{q}^{o}_{rt} = \bar{\lambda}^{o}\, q_{rt}(\ldots,x^{i},\ldots) \quad ; \quad x^{i} = s_{rt}\ ,\ r_{rt}\ ,\ \mathfrak{z}_{rt}\ .$$

Hence :

$$\bar{\bar{q}}^{(r)}_{rs} \approx \bar{q}^{\,0}_{rs} \quad ; \quad \bar{\bar{q}}^{(r)}_{rt} \approx \bar{q}^{\,0}_{rt}$$

and we can choose the following approximate values for $\underline{\bar{\lambda}}_{Rr}$ and $\underline{p}_{Rr}$:

$$\bar{\lambda}^{\circ}_{Rr} = 1 \, .$$

$$p^{\circ}_{Rr} = 1 + i o + j o + k o$$

We differentiate the basis equations (4.32) :

$$q^{(R)}_{rs} \, \Delta \underline{\ln \bar{\lambda}}_{Rr} + \underline{\Delta p}_{Rr} \, q^{(R)}_{rs} - q^{(R)}_{rs} \, \underline{\Delta p}_{Rr} + \underline{\Delta \bar{\bar{q}}}^{(r)}_{rs} = 0 \, .$$

$$\tfrac{1}{2}\left[q^{(R)}_{rt} \, \Delta \underline{\ln \bar{\lambda}}_{Rr} + \underline{\Delta p}_{Rr} \, q^{(R)}_{rt} - q^{(R)}_{rt} \, \underline{\Delta p}_{Rr} + \underline{\Delta \bar{\bar{q}}}^{(r)}_{rt} \right] + \tfrac{1}{2} \, e^{-1}_{tsr} \left[\;\text{"}\; \right] e_{tsr} = 0$$

Here $\underline{\Delta p}_{Rr}$ is, see (4.39) :

$$\underline{\Delta p}_{Rr} = 0 + i \, \underline{\Delta I}_{Rr} + j \, \underline{\Delta J}_{Rr} + k \, \underline{\Delta k}_{Rr} \, .$$

Premultiplication by $q^{-1(R)}_{rs}$ resp. $q^{-1(R)}_{rt}$ gives, whilst deleting the upper indices : $(q^{(R)} = q^{0} = q)$

$$\Delta \underline{\ln \bar{\lambda}}_{Rr} + q^{-1}_{rs} \, \underline{\Delta p}_{Rr} \, q_{rs} - \underline{\Delta p}_{Rr} + (q^{-1} \underline{\Delta \bar{\bar{q}}})^{(r)}_{rs} = 0 \, .$$

$$\tfrac{1}{2}\left[\Delta \underline{\ln \bar{\lambda}}_{Rr} + q^{-1}_{rt} \, \underline{\Delta p}_{Rr} \, q_{rt} - \underline{\Delta p}_{Rr} + (q^{-1} \underline{\Delta \bar{\bar{q}}})^{(r)}_{rt} \right] - \tfrac{1}{2} \, e^{-1}_{tsr} \left[\;\text{"}\; \right] e_{tsr} = 0 \, . \qquad (4.40)$$

Let :

$$\bar{q}^{\circ}_{rs} = 0 + iX + jY + kZ \quad ; \quad X^2 + Y^2 + Z^2 = S^2$$

$$\bar{q}^{\circ}_{rt} = 0 + ix + jy + kz \quad ; \quad x^2 + y^2 + z^2 = s^2$$

$$e^{\circ}_{tsr} = 0 + ia + jb + kc \quad ; \quad a^2 + b^2 + c^2 = 1 \, .$$

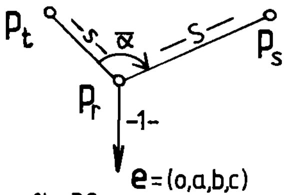


fig. 32

then the equations (4.40) in matrix notation will read :

$$\begin{pmatrix} \Delta \underline{\ln \bar{\lambda}}_{Rr} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{2}{S^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -Y^2-Z^2 & XY & XZ \\ 0 & XY & -X^2-Z^2 & YZ \\ 0 & XZ & YZ & -X^2-Y^2 \end{pmatrix} \begin{pmatrix} 0 \\ \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta k}_{Rr} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{XZ}{S^2} & \frac{-Y}{\sqrt{X^2+Y^2}} \\ 0 & \frac{YZ}{S^2} & \frac{X}{\sqrt{X^2+Y^2}} \\ 0 & \frac{-X^2-Y^2}{S^2} & 0 \end{pmatrix} \begin{pmatrix} \Delta \underline{\ln s}_{rs} \\ \underline{\Delta r}_{rs} \\ \underline{\Delta \mathfrak{z}}_{rs} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (4.40^{a})$$

$$\underset{\uparrow \; \frac{1}{2}\left[(1) - (\bar{e})(e)^{*}\right]}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b^2+c^2 & -ab & -ac \\ 0 & -ab & a^2+c^2 & -bc \\ 0 & -ac & -bc & a^2+b^2 \end{pmatrix}} \left[\begin{pmatrix} \Delta \underline{\ln \bar{\lambda}}_{Rr} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{2}{s^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -y^2-z^2 & xy & xz \\ 0 & xy & -x^2-z^2 & yz \\ 0 & xz & yz & -x^2-y^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta k}_{Rr} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{xz}{s^2} & \frac{-y}{\sqrt{x^2+y^2}} \\ 0 & \frac{yz}{s^2} & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & \frac{-x^2-y^2}{s^2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \Delta \underline{\ln s}_{rt} \\ \underline{\Delta r}_{rt} \\ \underline{\Delta \mathfrak{z}}_{rt} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (4.40^{b})$$

Between the three vector components of (4.40^a), there is the linear dependency (3.25) :

$$X\ Vi\{(4.40^a)\} + Y\ Vj\{(4.40^a)\} + Z\ Vk\{(4.40^a)\} = 0$$

<u>The system (4.40^a) thus has the rank : 3.</u>

From the scalar component of (4.40^b) it becomes apparent that $\underline{\Delta \ln s}_{rt}$ does not occur in the transformation. Between the three vector components of (4.40^b) there are two linear dependencies :

$$a\ Vi\{(4.40^b)\} + b\ Vj\{(4.40^b)\} + c\ Vk\{(4.40^b)\} = 0\,.$$

$$x\ Vi\{(4.40^b)\} + y\ Vj\{(4.40^b)\} + z\ Vk\{(4.40^b)\} = 0\,.$$

(the first resulting from the first matrix ; the second is analogous to the dependency (3.25)

<u>The system (4.40^b) thus has the rank : 1.</u>

From (4.40^a) it follows immediately :

$$\boxed{\underline{\Delta \ln \bar{\lambda}}_{Rr} = -\,\underline{\Delta \ln s}_{rs}} \tag{4.41}$$

The three components cf $\underline{\Delta p}_{Rr}$ may be solved from two components of (4.40^a) and the one independent component of (4.40).
Via manipulations, such as :

$$Ss\sin\bar{\alpha} = \frac{-Yz+Zy}{a} = \frac{-Zx+Xz}{b} = \frac{-Xy+Yx}{c}\ ;$$

$$Ss\cos\bar{\alpha} = Xx+Yy+Zz$$

this results in :

$$\boxed{\begin{aligned}
\underline{\Delta p}_{Rr} &= 0 + i\ \underline{\Delta I}_{Rr} + j\ \underline{\Delta J}_{Rr} + k\ \underline{\Delta k}_{Rr}\ .\\
\underline{\Delta I}_{Rr} &= \frac{(ay-bx)\,X}{2Ss\sin\bar{\alpha}}\left[\underline{\Delta r}_{rs} - \underline{\Delta r}_{rt}\right] - \frac{cX\cos\bar{\alpha}+Y\sin\bar{\alpha}}{2\sin\bar{\alpha}\sqrt{X^2+Y^2}}\,\underline{\Delta \zeta}_{rs} + \frac{csX}{2\sin\bar{\alpha}\,S\sqrt{x^2+y^2}}\,\underline{\Delta \zeta}_{rt}\\
\underline{\Delta J}_{Rr} &= \frac{(ay-bx)\,Y}{2Ss\sin\bar{\alpha}}\left[\underline{\Delta r}_{rs} - \underline{\Delta r}_{rt}\right] - \frac{cY\cos\bar{\alpha}-X\sin\bar{\alpha}}{2\sin\bar{\alpha}\sqrt{X^2+Y^2}}\,\underline{\Delta \zeta}_{rs} + \frac{csY}{2\sin\bar{\alpha}\,S\sqrt{x^2+y^2}}\,\underline{\Delta \zeta}_{rt}\\
\underline{\Delta k}_{Rr} &= -\tfrac{1}{2}\,\underline{\Delta r}_{rs} + \frac{(ay-bx)\,Z}{2Ss\sin\bar{\alpha}}\left[\underline{\Delta r}_{rs} - \underline{\Delta r}_{rt}\right] - \frac{cZ\cos\bar{\alpha}}{2\sin\bar{\alpha}\sqrt{X^2+Y^2}}\,\underline{\Delta \zeta}_{rs} + \frac{csZ}{2\sin\bar{\alpha}\,S\sqrt{x^2+y^2}}\,\underline{\Delta \zeta}_{rt}
\end{aligned}}$$

(4.42)

The expressions (4.41) and (4.42) for $\underline{\Delta \ln \bar{\lambda}}_{Rr}$ and $\underline{\Delta p}_{Rr}$ can now be substituted in the difference equation of (4.35) :

$$\underline{\Delta q}_{ik}^{(R)} = \underline{\Delta \bar{\bar{q}}}_{ik}^{(r)} + \bar{q}_{ik}\ \underline{\Delta \ln \bar{\lambda}}_{Rr} + \underline{\Delta p}_{Rr}\,\bar{q}_{ik} - \bar{q}_{ik}\,\underline{\Delta p}_{Rr} \tag{4.43}$$

from which the weight coefficients (4.36) are computed.

The S-coordinates, when using the method of condition equations, read :

$$\underline{\overset{=}{q}}{}_{ik}^{(r)} = q_{ik}^{r}(\cdot\cdot,\underline{x}^{i},\cdots) \quad ; \quad q_{rs}^{r}(\cdot\cdot,\underline{x}^{i},\cdots). \tag{4.44^1}$$

From the method of observation equations, the coordinate quantities are directly obtained ; see (4.38) :

$$\underline{q}_{ik}^{(R)} = 0 + i\,\underline{Y}^{\alpha_1} + j\,\underline{Y}^{\alpha_2} + k\,\underline{Y}^{\alpha_3} \;;\quad \underline{q}_{rs}^{(R)} \tag{4.44^2}$$

Amongst these the coordinate quantities in the computational base, $q_{rs}^{(R)}$ and one component of $q_{rt}^{(R)}$ are non-stochastic. They will therefore, not have adjustment corrections, so :

$$q_{rs}^{(R)} = \bar{q}_{rs}^{o} = q_{rs}^{r}(\cdot\cdot,x^{i},\cdots)$$

The coordinate quantities (4.44^1) and (4.44^2) constitute conformal systems; they are mutually transformed according to (4.29), (4.35). The numerical values for $\underline{\bar{\lambda}}_{Rr}$ and $\underline{p}_{Rr}$ follow from the basis equations, in which we now substitute the coordinate quantities (4.44^1) and (4.44^2) :

$$\begin{aligned} q_{rs}^{(R)} &= \bar{q}_{rs}^{r}(\cdot\cdot,x^{i},\cdots) = \\ &= \bar{\lambda}_{Rr}\; p_{Rr}\; \overset{=}{q}{}_{rs}^{r}(\cdot\cdot,x^{i},\cdots)\; p_{Rr}^{-1} \end{aligned}$$

From this it becomes apparent that numerical values for $\underline{\bar{\lambda}}_{Rr}$ and $\underline{p}_{Rr}$ follow from the least-squares corrections $\underline{\varepsilon}^{i}$ of observation variates in the basis equations :

$$\begin{aligned} \underline{\ln \bar{\lambda}}_{Rr} &= \frac{-\underline{\varepsilon}[s_{rs}]}{s_{rs}} \\ \underline{p}_{Rr} &= 1 + i\;\Delta I_{Rr}(\cdot\cdot,\underline{\varepsilon}^{i},\cdots) + j\;\Delta J_{Rr}(\cdot\cdot,\underline{\varepsilon}^{i},\cdots) + k\;\Delta K_{Rr}(\cdot\cdot,\underline{\varepsilon}^{i},\cdots). \end{aligned} \tag{4.45}$$

Relationship with the general S-transformation.

In [17] M. Molenaar gives a formula for the general S-transformation for three-dimensional coordinate systems :

$$\begin{aligned} \Delta\underline{q}_{i}^{(rs;t)} &= \Delta\underline{q}_{i}^{(a)} - \tfrac{1}{2}\left[\mathcal{Q}_{sri}\,\Delta\underline{q}_{rs}^{(a)} + \Delta\underline{q}_{rs}^{(a)}\,\mathcal{Q}_{sri}^{T}\right] + \\ &\quad + \left[\mathcal{Q}_{sri} - \mathcal{Q}_{sri}^{T}\right]\left[\mathcal{Q}_{srt} - \mathcal{Q}_{srt}^{T}\right]^{-1}\left[\underbrace{\left\{\Delta\underline{q}_{rt}^{(a)} - \tfrac{1}{2}\left(\mathcal{Q}_{srt}\,\Delta\underline{q}_{rs}^{(a)} + \Delta\underline{q}_{rs}\,\mathcal{Q}_{srt}^{T}\right)\right\}}_{} \right. \\ &\quad \left. + \left[\mathcal{Q}_{srt} - \mathcal{Q}_{srt}^{T}\right]^{-1}\left\{\;\text{''}\;\right\}\left[\mathcal{Q}_{srt} - \mathcal{Q}_{srt}^{T}\right]\right] \\ \text{symb. notation:} \\ &= \Delta\underline{q}_{i}^{(a)} + M_{i}\left(\Delta\underline{q}_{rs}^{(a)}, \Delta\underline{q}_{rt}^{(a)}\right). \end{aligned} \tag{4.46}$$

Molenaar, as in this study, has started from the basis equations (4.32); Molenaar's (4.46) is therefore essentially the same transformation as (4.43), with :

$$(R) = (rs\,;t) \quad ; \quad (r) = (a).$$

The right-hand member of (4.46) is, however, composed of differences of orthogonal coordinates, the right-hand member of (4.43) of differences of observation variates, i.e. of polar coordinates, in the basis vectors q_{rs} and q_{rt} . In order to make the formulae comparable, we pass in (4.43), i.e. (4.41) and (4.42) from polar coordinates to orthogonal coordinates as follows :

$$\begin{pmatrix} \underline{\Delta r}_{rs} \\ \underline{\Delta J}_{rs} \\ \underline{\Delta r}_{rt} \\ \underline{\Delta J}_{rt} \\ \underline{\Delta \ln s}_{rs} \end{pmatrix} = \begin{pmatrix} \frac{-Y}{X^2+Y^2} & \frac{X}{X^2+Y^2} & 0 & 0 & 0 & 0 \\ \frac{-ZX}{S^2\sqrt{X^2+Y^2}} & \frac{-ZY}{S^2\sqrt{X^2+Y^2}} & \frac{X^2+Y^2}{S^2\sqrt{X^2+Y^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 0 & \frac{-zx}{s^2\sqrt{x^2+y^2}} & \frac{-zy}{s^2\sqrt{x^2+y^2}} & \frac{x^2+y^2}{s^2\sqrt{x^2+y^2}} \\ \frac{X}{S^2} & \frac{Y}{S^2} & \frac{Z}{S^2} & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{\Delta X}_{rs} \\ \underline{\Delta Y}_{rs} \\ \underline{\Delta Z}_{rs} \\ \underline{\Delta x}_{rt} \\ \underline{\Delta y}_{rt} \\ \underline{\Delta z}_{rt} \end{pmatrix}$$

We substitute this in (4.41) and (4.42) and obtain :

$$\underline{\Delta I}_{Rr} = \frac{-Xa\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta X}_{rs} + \frac{Z\sin\bar{\alpha} - Xb\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Y}_{rs} - \frac{Y\sin\bar{\alpha} + Xc\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Z}_{rs} +$$
$$+ \frac{Xa}{2Ss\sin\bar{\alpha}}\,\underline{\Delta x}_{rt} + \frac{Xb}{2Ss\sin\bar{\alpha}}\,\underline{\Delta y}_{rt} + \frac{Xc}{2Ss\sin\bar{\alpha}}\,\underline{\Delta z}_{rt} .$$

$$\underline{\Delta J}_{Rr} = \frac{-Ya\cos\bar{\alpha} - Z\sin\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta X}_{rs} - \frac{Yb\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Y}_{rs} + \frac{X\sin\bar{\alpha} - Yc\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Z}_{rs} +$$
$$+ \frac{Ya}{2Ss\sin\bar{\alpha}}\,\underline{\Delta x}_{rt} + \frac{Yb}{2Ss\sin\bar{\alpha}}\,\underline{\Delta y}_{rt} + \frac{Yc}{2Ss\sin\bar{\alpha}}\,\underline{\Delta z}_{rt} .$$

$$\underline{\Delta k}_{Rr} = \frac{Y\sin\bar{\alpha} - Za\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta X}_{rs} - \frac{X\sin\bar{\alpha} + Zb\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Y}_{rs} - \frac{Zc\cos\bar{\alpha}}{2S^2\sin\bar{\alpha}}\,\underline{\Delta Z}_{rs} +$$
$$+ \frac{Za}{2Ss\sin\bar{\alpha}}\,\underline{\Delta x}_{rt} + \frac{Zb}{2Ss\sin\bar{\alpha}}\,\underline{\Delta y}_{rt} + \frac{Zc}{2Ss\sin\bar{\alpha}}\,\underline{\Delta z}_{rt} .$$

and :

$$\underline{\Delta \ln \bar{\lambda}}_{Rr} = \frac{-X}{S^2}\,\underline{\Delta X}_{rs} - \frac{Y}{S^2}\,\underline{\Delta Y}_{rs} - \frac{Z}{S^2}\,\underline{\Delta Z}_{rs} .$$

After, assuming as third basis equation, in accordance with Molenaar :

$$\underline{\Delta q}_r^{(R)} = 0$$

we can substitute these expressions for $\underline{\Delta \ln \bar{\lambda}}_{Rr}$ and $\underline{\Delta p}_{Rr}$ in (4.43) :

$$\underline{\Delta q}_i^{(R)} = \underline{\Delta q}_{ri}^{(R)} = \underline{\Delta \bar{\bar{q}}}_{ri}^{(R)} + \bar{q}_{ri}\,\underline{\Delta \ln \bar{\lambda}}_{Rr} + \underline{\Delta p}_{Rr}\,\bar{q}_{ri} - \bar{q}_{ri}\,\underline{\Delta p}_{Rr} .$$

Then, it becomes apparent that :

$$\boxed{M_i\,(\underline{\Delta q}_{rs}^{(a)}, \underline{\Delta q}_{rt}^{(a)}) = \bar{q}_{ri}\,\underline{\Delta \ln \bar{\lambda}}_{Rr} + \underline{\Delta p}_{Rr}\,\bar{q}_{ri} - \bar{q}_{ri}\,\underline{\Delta p}_{Rr}} \qquad (4.47)$$

An important difference between (4.43) and Molenaar's (4.46) is that in (4.43) the basis vectors q_{rs} and q_{rt} must be measured sides of the network; in q_{rs} must be measured : distance measure, direction and zenith angle; in q_{rt} : direction and zenith angle.
Contrary to this, q_{rs} and q_{rt} in (4.46) may be arbitrary connections between points of the network.

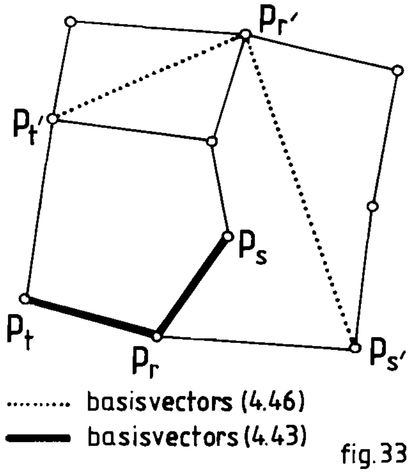


fig.33

4.5 Observation variates, conditions and unknowns.

Starting from a specified vector of observation variates $\underline{x}^{i}$ this section will present a provisional consideration of the numbers of conditions and unknowns in the function model of a closed polygon with astronomically measured rotations between the "local systems". The relationship between the function model of a spatial network and that of a network in the complex plane will be described via two intermediate forms.

We specify the vector of m observation variates as follows :
(one closed polygon)

2 n	directions r
2 n	distance measures s
2 n	zenith angles ζ
n	astr. differences of longitude λ
n	astr. latitudes φ
1	azimuth A

m= 8 n + 1

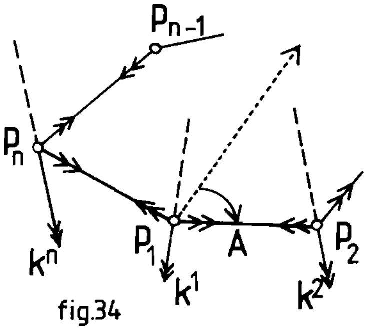


fig.34

(4.48)

A preliminary determination of the rank of the condition model

We consider the fully measured closed polygon of n points P_1 P_2 ...P_n and choose the local system of P_1 as (r)-system.
Now the following series of computations, following the sides of the polygon, can be carried out :

$$\begin{tikzcd}
\underline{q}^{(1)}_{1n} = q^{1}(\underline{s}_{1n}, \underline{r}_{1n}, \underline{\zeta}_{1n}) \arrow[rr, dotted] & & \underline{x}^{1*}_{1n}, \underline{y}^{1*}_{1n}, \underline{z}^{1*}_{1n} \\
\underline{q}^{(1)}_{12} = q^{1}(\underline{s}_{12}, \underline{r}_{12}, \underline{\zeta}_{12}) \arrow[rr, dotted] \arrow[r] & \underline{q}^{(2')}_{12} \arrow[dd, "2'"] & \underline{x}^{1}_{12}, \underline{y}^{1}_{12}, \underline{z}^{1}_{12} \\
\underline{\theta}^{0} \equiv \underline{\theta}_{1} = \underline{A}_{12} - \underline{r}_{12} \arrow[d] \arrow[ru] & & \\
\underline{p}_{2'1} = p(0, \underline{\varphi}_{2}, -\underline{\lambda}_{12}, \underline{\varphi}_{1}, \underline{\theta}_{1}) \arrow[ruu] & \underline{\theta}_{2} = \underline{r}^{2'}_{12} - \underline{r}_{21}(+\pi) \arrow[dl] & \\
\underline{p}_{12} = p(\underline{\theta}_{1}, \underline{\varphi}_{1}, \underline{\lambda}_{12}, \underline{\varphi}_{2}, \underline{\theta}_{2}) \arrow[r] & \underline{q}^{(1)}_{23} \arrow[r, dotted] & \underline{x}^{1}_{23}, \underline{y}^{1}_{23}, \underline{z}^{1}_{23} \\
\underline{q}^{(2)}_{23} = q^{2}(\underline{s}_{23}, \underline{r}_{23}, \underline{\zeta}_{23}) \arrow[ru] \arrow[r] & \underline{q}^{(3')}_{23} \arrow[d] & \\
\underline{p}_{3'2} = p(0, \underline{\varphi}_{3}, -\underline{\lambda}_{23}, \underline{\varphi}_{2}, \underline{\theta}_{2}) \arrow[ru] & \underline{r}^{3'}_{23} & \\
\vdots & &
\end{tikzcd}$$

$$\underline{\theta}_n = \overset{\downarrow}{\underline{r}}{}^{\,n-1\,'}_{n-1,n} - \underline{r}_{n1} + \pi .$$

$$\downarrow$$

$$\left.\begin{aligned} \underline{p}_{1n} &= p(\underline{\theta}_1 , \underline{\varphi}_1 , [\underline{\lambda}_{12} + \cdots + \underline{\lambda}_{n-1,n}], \underline{\varphi}_n , \underline{\theta}_n) \\ \underline{q}_{n1} &= q^n (\underline{s}_{n1} , \underline{r}_{n1} , \underline{\mathfrak{z}}_{n1}) \end{aligned}\right\} \underline{q}^{(1)}_{n1} : \ldots\ldots \underline{x}^1_{n1} , \underline{y}^1_{n1} , \underline{z}^1_{n1}$$

Now, six conditions have come into being :

$$\left.\begin{aligned} x^1_{12} + x^1_{23} + \cdots + x^1_{n1} &= 0 \\ y^1_{12} + y^1_{23} + \cdots + y^1_{n1} &= 0 \\ z^1_{12} + z^1_{23} + \cdots + z^1_{n1} &= 0 \end{aligned}\right\} \text{" coordinate condition"} \tag{4.49}$$

$$\left.\begin{aligned} x^*_{1n} &= -x^1_{n1} \\ y^*_{1n} &= -y^1_{n1} \\ z^*_{1n} &= -z^1_{n1} \end{aligned}\right\} \text{"polygon condition"} \tag{4.50}$$

This notation has been adapted from Baarda's [2] , and it anticipates Chapter 5. "The coordinate condition" cannot be linearly dependent on the "polygon condition", because there are more observation variates in the polygon condition than in the coordinate condition : viz. $\underline{s}_{1n}$, $\underline{r}_{1n}$ and $\underline{\mathfrak{z}}_{1n}$.

There remain n observation variates, which have not yet been used in this computation :

n - 1 zenith angles $(\underline{\mathfrak{z}}_{21} , \underline{\mathfrak{z}}_{32} \cdots\cdots \underline{\mathfrak{z}}_{n,n-1})$.
1 difference of longitude $(\underline{\lambda}_{n1})$.

These n variates may be expressed in the form of n condition equations as a function of the m-n other observation variates (this is discussed in greater detail in Chapter 5). Consequently, now n + 6 conditions have been found by the procedure of computation alongside the polygon ; if, by way of precaution, we assume that, possibly, conditions may have been overlooked, this number is a lower limit for the rank b of the condition model :

$$b \geqslant n + 6 \tag{4.51}$$

By analysing linear dependencies in a set of n + 12 conditions, an upper limit, which also equals n + 6, shall be determined in section 5.4.

Unknowns

In case directions and distance measures are used as observation variates, the following "unknowns" are frequently chosen in the function model in the complex plane for adjustment according to the method of observation equations :

$2n - 4$	coordinates
n	orientations $\underline{\theta}_i$
n	length factors $\underline{\bar{\lambda}}_{ik}$
$4n - 4$	

The number of non-stochastic coordinates (four) equal the number of parameters of a similarity transformation in R_2 ; this is connected with the "operational definition" of coordinate quantities. ("S-coordinates"). In analogy there are in a spatial network of n points : 3 n - 7 stochastic coordinate unknowns (the (R)-system ; see section 4.4).
In R_2, the relation between the S-coordinate system (R) and the n local systems is established by n pairs of quantities :

$\underline{\theta}_{Ri}$: orientations

$\underline{\ln \bar{\lambda}}_{Ri}$: logarithm of length-factors

In section 4.4 we have seen that in R_3, the relationship between the S-coordinate system (R) and the first local system (r) is established by the four variates of the "basis transformation"; see (4.39) :

$\underline{\ln \bar{\lambda}}_{Rr}$; $\underline{I}_{Rr}$; $\underline{J}_{Rr}$; $\underline{K}_{Rr}$

As in R_2, the other n - 1 length factors $\bar{\lambda}_{ri}$ can now be defined as unknowns ; with regard to the orientation unknowns there arises, however, a difference between R_2 and R_3 ; so as to enable all local systems to be rotated astronomically, n orientations $\underline{\theta}_i$ are required ; moreover, n astronomic latitudes and n astronomical longitudes or n - 1 astronomical longitude differences. Summarizing, in a fully measured spatial network of n points, the following unknowns occur :

$3n - 7$	S-coordinates
1	"first" length-factor
3	parameters of the basis transformation
$n - 1$	other length-factors
n	orientations
n	astronomical latitudes
$n - 1$	astronomical longitude differences
$7n - 5$	

$$7n - 5 \qquad (4.52)$$

This number agrees with the number of observation variates m = 8 n + 1 (4.48) and the provisionally determined number of conditions b = n + 6 (4.51), since :

$$8n + 1 - (n + 6) = 7n - 5$$

The transition from R 3 to R 2

We shall now describe the relationship between the three-dimensional function model and the two-dimensional model through a transition via two intermediate forms :

1. A spatial network with parallel k-unit vectors ("first axes" of local systems), as described in section 4.2
2. The same spatial network as above, where no astronomical quantities (latitudes φ_i and longitude differences λ_{ik}) occur, i.e. networks as used for e.g. trigonometric levelling.

In (4.53) the numbers of observation variates, unknowns and conditions are stated in columns for the general R_3 model, the two intermediate forms and the R_2 model; regarding the types of conditions, we must anticipate Chapter 5 here.

Observation variates :	R_3				R_2
	general	k^i //	without φ and λ		
directions	2n	2n	2n		2n
distance measures	2n	2n	2n		2n
zenith angles	2n	2n	2n	a	
differences of longitude	n	n			
latitudes	n	n			
azimuth	1 *	-			
	8n + 1	8n	6n		4n
Unknowns :				c	
S-coordinates	3n - 7	3n - 7	3n - 7	d	2n - 4
first rotation	3	3	3	e	1
first length factor	1	1	1		1
other orientations	n *	n - 1	n - 1		n - 1
other length factors	n - 1	n - 1	n - 1		n - 1
differences of longitude	n - 1	n - 1			
latitudes	n	n			
	7n - 5	7n - 6	5n - 5		4n - 4
Conditions :					
Coordinate condition	3		3	f	2
Polygon condition	2		2		2
Z-conditions	n		n	b	
Sum of longitude diff.	1				
	n + 6		n + 5		4

(4.53)

The differences * between the "general" R model and the model of "parallel k-vectors" are analysed in section 4.2; the azimuth obtains coefficients = 0 see (4.14), between the orientations there arises a dependency, see (4.15).

The differences between the second and the third model are trivial ; all "astronomical" observation variates and unknowns disappear and so does the condition "sum of longitude differences".

The differences between the third and fourth (the R_2 model) are either trivial :

a - zenith angles are not entered in R_2 as observation variates in the function model.
b - the Z-conditions arise in the R_3 model through the measurement of zenith angles in both directions, therefore they do not occur in the R_2 model.

or they follow directly from the algebraic properties of R_3 and R_2 :

c - three, resp. two coordinates per point.
d - seven, resp. four parameters in a similarity transformation.
e - a rotation has three parameters, resp. one.
f - the network or coordinate condition has three, resp. two components.

This shows that the R_2 model is a "special case" of the R_3 model.

Chapter 5.

THE ADJUSTMENT MODEL

5.1 Introduction.

In this Chapter, the theory described will be applied for the construction of a system of functional relations (conditions) of observation variates, with a view to the application of the adjustment theory, as standardized by Baarda [4] .
These functional relations apply to means of stochastic quantities.

$$[4]\ (9.1):\quad (\tilde{Y}^\rho) = (Y^\rho(\ldots,\tilde{X}^i,\ldots)) = (0)\,. \tag{5.1}$$

"adjustment model of condition equations"

$$[4]\ (9.2):\quad (\tilde{X}^i) = (X^i(\ldots,\tilde{Y}^\alpha,\ldots)) = (0)\,. \tag{5.2}$$

"adjustment model of observation equations"

Here :

X^i : observation variates, $i = 1 \ldots\ldots m$

Y^ρ : zero-mean variates $\rho = 1 \ldots\ldots b$

Y^α : unknowns $\alpha = 1 \ldots\ldots m-b$

The introduction of the stochastic observation variates $\underline{x}^i$ furnishes "misclosures" :

$$(\underline{y}^\rho) = (Y^\rho(\ldots,\underline{x}^i,\ldots)) \tag{5.3}$$

The estimators to be obtained, $\underline{X}^i$ and $\underline{Y}^\alpha$, should comply with (5.1) and (5.2) :

$$(0) = (Y^\rho(\ldots,\underline{X}^i,\ldots)) \tag{5.4^a}$$

$$(\underline{X}^i) = (X^i(\ldots,\underline{Y}^\alpha,\ldots)) \tag{5.4^b}$$

In view of the linearization of the functional relations (5.1)and (5.2), a complete set of approximate values X^i_0 , Y^α_0 must be chosen, also complying with (5.1).

Hence :

$$(Y^\rho(\ldots,X^i_0,\ldots)) = (0) = (Y^\rho_0) \tag{5.5^a}$$

$$(X^i_0) = (X^i(\ldots,Y^\alpha_0,\ldots)) \tag{5.5^b}$$

Now (5.3) can be linearized by expansion in a Tailor series of

$$(\underline{y}^\rho - Y^\rho_0) = (u^\rho_i)(\underline{x}^i - X^i_0)$$

or, whilst neglecting terms of the second and higher orders :

$$\boxed{\begin{aligned} &(\underline{\Delta y}^{\rho}) = (u^{\rho}_{i})(\underline{\Delta x}^{i}) \\ &\text{with : } (u^{\rho}_{i}) = \left(\frac{\partial Y^{\rho}}{\partial X^{l}}\right)_{X^{i}_{o}} \end{aligned}} \tag{5.6}$$

See also (17.21) in [4] .
It also follows from (5.4^b) with (5.5^b) :

$$(\underline{X}^{i} - X^{i}_{o}) = (a^{i}_{\alpha})(\underline{Y}^{\alpha} - Y^{\alpha}_{o})$$

or :

$$\boxed{\begin{aligned} &(\underline{\Delta X}^{i}) = (a^{i}_{\alpha})(\underline{\Delta Y}^{\alpha}) \\ &\text{with : } (a^{i}_{\alpha}) = \left(\frac{\partial X^{i}}{\partial Y^{\alpha}}\right)_{X^{i}_{o};Y^{\alpha}_{o}} \end{aligned}} \tag{5.7}$$

See also (17.23) in [4] .

<u>Relation with the Polygon theory in the complex plane.</u>

In a manuscript by Baarda [8] , dating back as far as 1962-64, an elegant structural agreement was found between the quaternion relations elaborated there, and the relations in complex numbers, as they are known from the two-dimensional polygon theory [2] . Accordingly, the choice and the notation of the quaternion relations in section 5.2 are adapted from them. The Π-quantity plays a central part in the structural relation between the two- and three-dimensional polygon theories :

-two-dimensional : . . . . $\underline{\Delta\Pi}_{jik} = \underline{\Delta\Lambda}_{ik} - \underline{\Delta\Lambda}_{ij}$

(complex number, see [2] (2.2.17) with : $\mathrm{Re}\{\underline{\Delta\Pi}_{jik}\} = \Delta\underline{\ln v}_{jik}$

-three-dimensional : . . . . $\Delta\Pi^{(i)}_{jik} = (q^{-1}\underline{\Delta q})^{(i)}_{ik} - (q^{-1}\underline{\Delta q})^{(i)}_{ij}$

(quaternion, see (2.21) with: $\mathrm{Sc}\{\underline{\Delta\Pi}^{(i)}_{jik}\} = \underline{\Delta\ln v}_{jik}$

From section 5.4 it will become apparent that not only the quaternion conditions and the $\underline{\Delta\Pi}$-quantities, but also the dependencies between the conditions roughly present the same structure as those in the two-dimensional theory.
In the three-dimensional theory it will, however, be necessary to introduce more types of conditions, owing to which the overall system of dependencies becomes more complicated.

In order not to obscure the subject matter unnecessarily, the sections 5.2, 5.3 and 5.4 will be restricted to the discussion of a network, consisting of one closed polygon, $P_1 P_2 \ldots\ldots P_n$, with complete measurement according to (4.48).

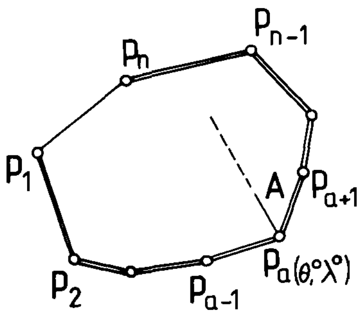


fig. 35

Orientations $\underline{\theta}_i$ andlength factors $\underline{\bar{\lambda}}_{ik}$ are computed according to (3.29/30), via the sides P_a, P_{a+1}, ...P_n and via $P_a P_{a-1}$ ......P_1. This means that only on the side $P_n P_1$ the quantity :

$$(q^{-1}\underline{\Delta}q)_{ik} - (q^{-1}\underline{\Delta}q)_{ki}$$

assumes the form of $(3.42)^{I}$. On all other sides, it assumes the form $(3.42)^{IV}$, i.e. :

$$(q^{-1}\underline{\Delta}q)^{(r)}_{ik} - (q^{-1}\underline{\Delta}q)^{(r)}_{ki} = e''^{(r)}_{k;ik} \left[\underline{\Delta\zeta}^{k}_{ik} + \underline{\Delta\zeta}_{ki} \right] \cdot \qquad (5.8)$$

In section 5.5 an example of a case, differing from this pattern, will be discussed.

Remark on the notation.

In the following sections, one starts tacitly (i.e. by omitting the indices (r) and ^{o}) from the assumption that in all (quaternion) difference equations, the coefficients are computed according to (5.5) in one of the local (instrumental) systems, i.e. the (r)-system : thus, for example :

$$q_{ik} = \bar{q}^{o\,(r)}_{ik} = \bar{\lambda}^{o}_{ri}\, \overset{o}{p}_{ri}\, q^{o(i)}_{ik}\, \overset{o}{p}{}^{-1}_{ri} \cdot$$

5.2 Conditions.

We now again apply the procedure indicated in [8] , viz. subsequent computation of the sides in a closed polygon with n points, starting in side $P_n P_1$:

$$\begin{aligned}
&q_{n1} \;; \\
&q_{12} = -\mathcal{G}_1\, q_{n1} \cdot \\
&q_{23} = (-1)^2 \mathcal{G}_2 \mathcal{G}_1\, q_{n1} \\
&\quad\vdots \\
&q_{n-1,n} = (-1)^{n-1} \mathcal{G}_{n-1}\, \mathcal{G}_{n-2} \cdots \mathcal{G}_2 \mathcal{G}_1\, q_{n1} \cdot
\end{aligned} \qquad (5.9)$$

By the summation of these equations, a zero vector is obtained in the left-hand member :

$$q_{nn} = o = \left[1 - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{G}_1 - \cdots + (-1)^{n-1} \mathcal{G}_{n-1} \mathcal{G}_{n-2} \cdots \mathcal{G}_2\, \mathcal{G}_1 \right] q_{n1} \cdot \qquad (5.10^a)$$

Through postmultiplication by q^{-1}_{n1} $(\neq 0)$
this becomes :

$$o = \left[1 - \mathcal{G}_1 + \mathcal{G}_2 \mathcal{G}_1 - \cdots + (-1)^{n-1} \mathcal{G}_{n-1} \mathcal{G}_{n-2} \cdots \mathcal{G}_3 \mathcal{G}_2 \mathcal{G}_1 \right] \qquad (5.10^b)$$

The series (5.9) may be continued by :

$$q_{n1} = (-1)^n \mathcal{Q}_n \mathcal{Q}_{n-1} \cdots \mathcal{Q}_2 \mathcal{Q}_1 \; q_{n1}$$

or, after reducing to zero :

$$0 = -q_{n1} + (-1)^n \mathcal{Q}_n \mathcal{Q}_{n-1} \cdots \mathcal{Q}_2 \mathcal{Q}_1 \; q_{n1} \qquad (5.11^a)$$

or, through postmultiplication by q_{n1}^{-1} :

$$0 = -1 + (-1)^n \mathcal{Q}_n \mathcal{Q}_{n-1} \cdots \mathcal{Q}_2 \mathcal{Q}_1 \; . \qquad (5.11^b)$$

Thus, we see the conditions known from [2] , viz. "coordinate condition" and "polygon condition" come into being, both in a form with dimension length : (5.10^a) and (5.11^a) resp. and in a dimensionless form (5.10^b) and (5.11^b) resp.

How to choose from these ?

The dimensionless forms would seem to deserve preference, because they were composed from exclusively dimensionless observation variates (length ratios and angles). In the difference equations of (5.10^a) and (5.11^a), however, the non-dimensionless factor Δq_{n1} obtains zero coefficients. This means that in the difference equation there are only dimensionless observation variates. The choice may thus be based on other reasons.

From section 5.3 it will become apparent that for the coordinate condition, the non-dimensionless shape (5.10^a) deserves preference.

Therefore :

$$0 = \left[1 - \mathcal{Q}_1 + \mathcal{Q}_2 \mathcal{Q}_1 - \mathcal{Q}_3 \mathcal{Q}_2 \mathcal{Q}_1 + \cdots + (-1)^n \mathcal{Q}_{n-1} \mathcal{Q}_{n-2} \cdots \mathcal{Q}_2 \mathcal{Q}_1\right] q_{n1} \qquad (5.12)$$

"Coordinate condition" $N_{(n),1,2\ldots n-1}$; See: [2] : (4.2.2) .

As far as the polygon condition is concerned, it will become apparent, see (5.25), that in the dimensionless form (5.11^b), the distance ratios occur in the scalar component and all other observation variates occur in the vector components of the difference equations. Owing to this elegant nature, we consequently choose the dimensionless form for the polygon condition ; therefore :

$$0 = -1 + (-1)^n \mathcal{Q}_n \mathcal{Q}_{n-1} \mathcal{Q}_{n-2} \cdots \mathcal{Q}_2 \mathcal{Q}_1 \; . \qquad (5.13)$$

"Polygon condition" $V_{1,2,\ldots n}$; See [2] : (4.3.1) .

Other conditions.

From the preliminary consideration of section 4.5 it becomes apparent that, contrary to the plane polygon theory, the number of linearly independent conditions is also dependent on the number of sides of the network (this is caused by the zenith angles).

Therefore, for each fully measured side, an extra condition must be established ; in principle, each of the three vector components of

$$o = q_{ik}^{(r)} + q_{ki}^{(r)}$$

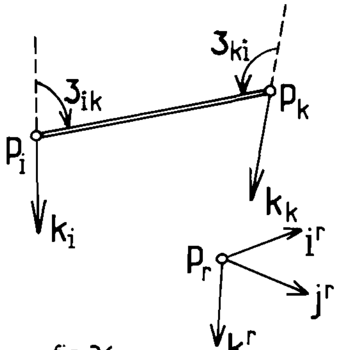


fig. 36

may be used for this purpose. If, however, the network is of limited size (in comparison with the circumference of the earth), the k-unit vectors of the local systems are approximately parallel to each other and to those of the (r)-system.

The zenith angles then have small coefficients in :

$$Vi\{\Delta q_{ik}^{(r)} + \Delta q_{ki}^{(r)}\} \quad \text{and} \quad Vj\{\Delta q_{ik}^{(r)} + \Delta q_{ki}^{(r)}\}$$

Therefore, we choose as condition on each side with two zenith angles :

$$\boxed{\begin{array}{l} o = Vk\{q_{ik}^{(r)} + q_{ki}^{(r)}\} \\ \text{"}Z_{ik}\text{ - condition"} \end{array}} \tag{5.14}$$

Between the astronomical rotations p_{ik} , which are composed from difference quantities (longitude differences λ_{ik}) and also the latitudes φ_i , there exists a "condition of rotation" :

$$\boxed{\begin{array}{l} o = -1 + p_{i,i+1}\, p_{i+1,i+2} \cdots p_{i-1,i} \\ \text{"Condition of rotation" : } R_{i\ldots i} \end{array}} \tag{5.15}$$

In section 4.1 it was shown that, in a spatial network with astronomical rotations, one azimuth must be measured so as to connect the astronomical observation variates with the "terrestrial" ones. The coefficients of this first azimuth are small in all condition equations and equal to zero, if the k-unit vectors are parallel to each other ; see (4.14).

The addition to the network of each next azimuth results in the creation of a condition, in which both azimuths have a large coefficient (i.e. approx. =1). If the first azimuth is measured on side $P_a P_b$ and a second on side $P_i P_k$, this condition can be adapted from the quaternion equation :(see fig. 37)

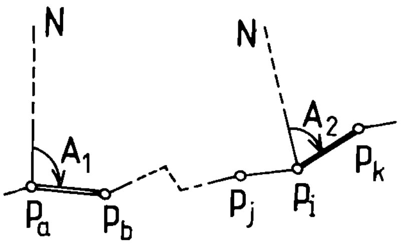


fig. 37

$$q_{ik;A} = (-1)^{m} \mathcal{G}_i \mathcal{G}_j \cdots \mathcal{G}_b \, q_{ab;A} \quad \hookrightarrow \text{number of } \mathcal{G}\text{-quantities} \tag{5.16}$$

Here, in the quaternions $q_{ik;A}$ and $q_{ab;A}$ the azimuth takes the place of the direction, thus :

$$\begin{aligned} q_{ik;A} &= \overset{(r)}{q}_{ik;A} = \bar{\lambda}_{ri}\, p_{ri'}\, \overset{(i')}{q}_{ik}\, p_{ri'}^{-1} \\ \text{with: } p_{ri'} &= p_{ri}\left(\theta_r, \varphi_r, \lambda_{r....i}, \varphi_i, \theta=0\right) \\ \overset{(i')}{q}_{ik} &= q_{ik}\left(s_{ik}, A_{ik}, \mathfrak{J}_{ik}\right). \end{aligned} \qquad (5.16')$$

Now :

$$\overset{(r)}{q}_{ab} = \bar{\lambda}_{ra}\, p_{ra'}\, q\left(s_{ab}, [r_{ab}+\theta_a], \mathfrak{J}_{ab}\right) p_{ra'}^{-1}$$

According to (4.9), Θ_a in the condition model is, however, replaced by $A_{ab} - r_{ab}$, therefore :

$$\overset{(r)}{q}_{ab} = \bar{\lambda}_{ra}\, p_{ra'}\, q\left(s_{ab}, A_{ab}, \mathfrak{J}_{ab}\right) p_{ra'}^{-1} \equiv q_{ab;A} \qquad (5.16'')$$

Through premultiplication by q_{ik}^{-1} , reduction to zero and substitution of (5.16"), (5.16) passes into :

$$\boxed{\begin{aligned} &0 = -1 + (-1)^m\, q_{ik;A}^{-1}\, \mathcal{G}_{jik}\, \mathcal{G}_{.ji} \cdots \mathcal{G}_{ab}.\, q_{ab} \\ &\text{"Azimuth condition" : } A_{a...i} \,. \end{aligned}} \qquad (5.17)$$

Relations between coordinate and polygon conditions.

In section 5.4 it will be shown that of the Z-, R- and A-conditions, only one component is independent. Regarding the N- and V-conditions the situation is different. Also in order to establish links with the polygon theory in the complex plane, in which the N- and V-conditions take a central place, we now first consider the relations (dependencies) between the N- and V-conditions.

The coordinate condition (5.12) contains n-1 of the n $\mathcal{G}$ — quantities. As in the two-dimensional theory there are, consequently, in a fully measured closed polygon of n points n different coordinate conditions. Because quaternion algebra is non-commutative relative to multiplication, here—contrary to the two-dimensional theory—also the n polygon relations (obtainable from cyclic changing of the factors) are different !

We now introduce zero-mean variates, see (5.1) ; we use the characters N and V :

$$\left.\begin{aligned} Y^{\rho} &= N_{(n)} = \left[1 - \mathcal{G}_1 + \mathcal{G}_2\mathcal{G}_1 - \cdots + (-1)^{n-1}\mathcal{G}_{n-1}\mathcal{G}_{n-2}\cdots\mathcal{G}_2\mathcal{G}_1\right] q_{n1} \\ \text{,,} &= N_{(1)} = \left[1 - \mathcal{G}_2 + \mathcal{G}_3\mathcal{G}_2 - \cdots + (-1)^{n-1}\mathcal{G}_{n}\mathcal{G}_{n-1}\cdots\mathcal{G}_3\mathcal{G}_2\right] q_{12} \\ \text{,,} &= N_{(2)} = \left[1 - \mathcal{G}_3 + \mathcal{G}_4\mathcal{G}_3 - \cdots + (-1)^{n-1}\mathcal{G}_{1}\mathcal{G}_{n}\cdots\mathcal{G}_4\mathcal{G}_3\right] q_{23} \end{aligned}\right\} \qquad (5.18)$$

$$\left.\begin{aligned} \text{,,} &= V_{1...n} = -1 + (-1)^n\,\mathcal{G}_n\mathcal{G}_{n-1}\cdots\mathcal{G}_2\mathcal{G}_1 \\ \text{,,} &= V_{2...1} = -1 + (-1)^n\,\mathcal{G}_1\mathcal{G}_n\cdots\mathcal{G}_3\mathcal{G}_2\,. \end{aligned}\right\} \qquad (5.19)$$

Now (5.19) directly supplies a relation between two "consecutive" polygon conditions :

$$V_{2\ldots 1} = \mathcal{G}_1 V_{1\ldots n} \mathcal{G}_1^{-1}$$

Likewise, if the polygon conditions do not directly succeed each other :

$$\boxed{V_{i\ldots i-1} = \mathcal{G}_{i-1} \mathcal{G}_{i-2} \cdots \mathcal{G}_1 \; V_{1\ldots n} \; \mathcal{G}_1^{-1} \mathcal{G}_2^{-1} \cdots \mathcal{G}_{i-2}^{-1} \mathcal{G}_{i-1}^{-1}} \qquad (5.20)$$

This means that of the n polygon conditions, only one is independent.
From (5.18) follow the relations :

$$N_{(2)} - N_{(1)} = q_{21} - (-1)^n \mathcal{G}_1 \mathcal{G}_n \cdots \mathcal{G}_3 \mathcal{G}_2 \, q_{21} \, . \qquad (5.21^a)$$

$$N_{(1)} - N_{(n)} = q_{1n} - (-1)^n \mathcal{G}_n \mathcal{G}_{n-1} \cdots \mathcal{G}_2 \underbrace{\mathcal{G}_1 \, q_{1n}} \qquad (5.21^b)$$

Therefore also :

$$\mathcal{G}_1 \left[N_{(1)} - N_{(n)} \right] = q_{12} - (-1)^n \mathcal{G}_1 \mathcal{G}_n \cdots \mathcal{G}_2 \, q_{12} =$$

$$(5.21^a): \quad = - \left[N_{(2)} - N_{(1)} \right] .$$

In consequence, there exists the following relation between each three coordinate conditions :

$$\boxed{N_{(i)} - N_{(i-1)} = - \mathcal{G}_{i-1} \left[N_{(i-1)} - N_{(i-2)} \right] .} \qquad (5.22)$$

This means, that of the n coordinate conditions, only two are independent.
From (5.21) follows :

$$\left[N_{(1)} - N_{(n)} \right] q_{1n}^{-1} = 1 - (-1)^n \mathcal{G}_n \mathcal{G}_{n-1} \cdots \mathcal{G}_2 \mathcal{G}_1 =$$

$$(5.19): \quad = - V_{1\ldots n}$$

In consequence, there exists the following relation between two coordinate conditions and one polygon condition :

$$\boxed{N_{(i)} - N_{(i-1)} = V_{i\ldots i-1} \, q_{i-1,i}^{-1}} \qquad (5.23)$$

Conclusion :

The relations (5.20), (5.22) and (5.23) lead to the conclusion that two independent coordinate and polygon conditions can be established in a fully measured closed polygon, viz. :

either : two network conditions

or : one network condition and one polygon condition

5.3 Linearization of conditions.

After the introduction of approximate values, complying with (5.5), the conditions N, V, Z, R and A are linearized by expansion in a Taylor series ; if the approximate values are good enough, the terms of the zero and first order will suffice.
In all the difference equations, the terms $\underline{\Delta\varphi}_i$ will be replaced by $\underline{\Delta\Pi}$, according to :

$$(2.21): \quad \underline{\Delta\varphi}_{jik} \rightarrow q_{ik}\, \underline{\Delta\Pi}_{jik}\, q_{ij}^{-1}$$

By substitution of stochastic observation variates, the conditions mentioned in the previous chapters are now transformed into "condition equations".

The coordinate condition equation.

We differentiate the equations (5.9) :

$$\underline{\Delta q}_{n1} = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = q_{n1}(q^{-1}\underline{\Delta q})_{n1}$$

$$\underline{\Delta q}_{12} = -\,\underline{\Delta\varphi}_1\, q_{n1} - \varphi_1\, \underline{\Delta q}_{n1} =$$

$$= -\,q_{12}\, \underline{\Delta\Pi}_1\, q_{1n}^{-1}\, q_{n1} - q_{12}\, q_{1n}^{-1}\, \underline{\Delta q}_{n1} = \cdot\cdot\; q_{12}\, \underline{\Delta\Pi}_1 + q_{12}(q^{-1}\underline{\Delta q})_{n1}$$

$$\underline{\Delta q}_{23} = \cdot\cdot \text{ likewise} \rightarrow \qquad q_{23}\, \underline{\Delta\Pi}_2 + q_{23}\, \underline{\Delta\Pi}_1 + q_{23}(q^{-1}\underline{\Delta q})_{n1}.$$

$$\vdots$$

$$\underline{\Delta q}_{n-1,n} = q_{n-1,n}\, \underline{\Delta\Pi}_{n-1} + \cdots\cdots\cdots + q_{n-1,n}\, \underline{\Delta\Pi}_2 + q_{n-1,n}\, \underline{\Delta\Pi}_1 + q_{n-1,n}(q^{-1}\underline{\Delta q})_{n1}.$$

As after (5.9) we add these equations and thus obtain the difference equation of (5.10) / (5.12) As in (5.18/19) we use $\underline{N}_{(n)}$ as zero-mean variate in the left hand member ; introducing observations $\underline{\Delta x}^i$ in the right hand member, $\underline{N}_{(n)}$ becomes the "misclosure":

$$\boxed{\underline{\Delta N}_{(n)} = q_{n-1,n}\, \underline{\Delta\Pi}_{n-1} + \cdots\cdots\cdots + q_{2n}\, \underline{\Delta\Pi}_2 + q_{1n}\, \underline{\Delta\Pi}_1} \qquad (5.24)$$

or :

$$\underline{\Delta N}_{(n)} = \sum_{i=1}^{n-1} q_{in}\, \underline{\Delta\Pi}_{jik}$$

$$\text{with:} \quad \underline{\Delta\Pi}_{jik} = \Pi_{jik}\,(\ldots, \underline{\Delta x}^i, \ldots).$$

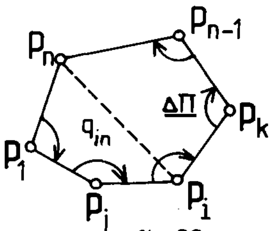


fig.38

The difference equation of the quaternion coordinate condition is thus identical to that of the two-dimensional coordinate condition equation, see (17.1.5) in [2] .

The polygon condition equation.

Differentiation of (5.13), after the introduction of the misclosure $\underline{\Delta V}_{1\ldots n}$ as $\underline{\Delta Y}^p$–quantity, gives :

$$\underline{\Delta V}_{1\ldots n} = (-1)^{n}\, q_{n1}\, \underline{\Delta\Pi}_{n}\, q^{-1}_{n,n-1}\, \varphi_{n-1}\, \varphi_{n-2} \cdots \varphi_{1} +$$

$$+ (-1)^{n}\, \varphi_{n}\, q_{n-1,n}\, \underline{\Delta\Pi}_{n-1}\, q^{-1}_{n-1,n-2}\, \varphi_{n-2} \cdots \varphi_{1} +$$

$$\vdots$$

$$+ (-1)^{n}\, \varphi_{n}\, \varphi_{n-1} \cdots \varphi_{3}\, \varphi_{2}\, q_{12}\, \underline{\Delta\Pi}_{1}\, q^{-1}_{1n} =$$

$$= (-1)^{n}\, q_{n1} \left[(-1)^{n-1} \underline{\Delta\Pi}_{n} + (-1)^{n-1} \underline{\Delta\Pi}_{n-1} + \cdots + (-1)^{n-1} \underline{\Delta\Pi}_{1} \right] q^{-1'}_{1n} .$$

therefore :

$$\boxed{\underline{\Delta V}_{1\ldots n} = q_{n1} \left[\underline{\Delta\Pi}_{1} + \underline{\Delta\Pi}_{2} + \cdots\cdots + \underline{\Delta\Pi}_{n-1} + \underline{\Delta\Pi}_{n} \right] q^{-1}_{n1}} \tag{5.25}$$

Because : $\mathrm{Sc}\{\underline{\Delta\Pi}_{i}\} = \Delta \underline{\ln v_{i}}$:

$$\downarrow$$

$$\mathrm{Sc}\{\underline{\Delta V}_{1\ldots n}\} = \sum_{i=1}^{n} \Delta \underline{\ln v}_{i} \tag{5.25'}$$

Here, too, the strong resemblance with the polygon condition equation in the complex plane is striking; see (17.2.2) in [2] .

The Z-condition equation

The difference equation of (5.14) reads :

$$\boxed{\underline{\Delta Z}_{ik} = \mathrm{Vk}\,\{\underline{\Delta q}_{ik} + \underline{\Delta q}_{ki}\}} \tag{5.26}$$

We expand this by the other components to :

$$i\, \underline{\Delta X}_{ik} + j\, \underline{\Delta Y}_{ik} + k\, \underline{\Delta Z}_{ik} = \underline{\Delta q}_{ik} + \underline{\Delta q}_{ki} =$$

$$= q_{ik} \left[(q^{-1} \underline{\Delta q})_{ik} - (q^{-1} \underline{\Delta q})_{ki} \right] ;$$

According to (5.8) this expression becomes on all sides except P_nP_1 :

$$= q_{ik}\, e''_{k;ik} \left[\underline{\Delta\mathfrak{J}}^{k}_{ik} + \underline{\Delta\mathfrak{J}}_{ki} \right] =$$

$$= -\sqrt{N\{q_{ik}\}}\; e'_{k;ik} \left[\underline{\Delta\mathfrak{J}}^{k}_{ik} + \underline{\Delta\mathfrak{J}}_{ki} \right] =$$

$$= \ell_{ik} \left[0 + i\,a + j\,b + k\,c \right] \left[\underline{\Delta\mathfrak{J}}^{k}_{ik} + \underline{\Delta\mathfrak{J}}_{ki} \right] .$$

Consequently :

$$\underline{\Delta X}_{ik} = \ell_{ik}\, a \left[\underline{\Delta\mathfrak{J}}^{k}_{ik} + \underline{\Delta\mathfrak{J}}_{ki} \right] .$$

$$\underline{\Delta Y}_{ik} = \ell_{ik}\, b \left[\quad '' \quad \right] .$$

$$\underline{\Delta Z}_{ik} = \ell_{ik}\, c \left[\quad '' \quad \right] .$$

Remark :

$(a, b, c) \perp q_{ik}$.

$a \approx 0$; $b \approx 0$; $c \approx -1$

To all sides, except P_nP_1, the following therefore applies :

$$\underline{\Delta q}_{ik} + \underline{\Delta q}_{ki} = [o + i\frac{a}{c} + j\frac{b}{c} + k]\ \underline{\Delta Z}_{ik}\ .$$

i.e. the i- and the j-component depend on the k-component, which is the Z-condition.

By premultiplication by q_{ik}^{-1} , the left-hand member of (5.8) is obtained again :

$$\boxed{(q^{-1}\underline{\Delta q})_{ik} - (q^{-1}\underline{\Delta q})_{ki} = [o + i\, f_{ik} + j\, g_{ik} + k\, h_{ik}]\ \underline{\Delta Z}_{ik}}$$

$\downarrow$ on all networksides exept P_nP_1 .

$$\begin{aligned} f_{ik} &= \frac{1}{\ell_{ik}}\left[-Vj\{q_{ik}\} + \frac{b}{c}\, Vk\{q_{ik}\}\right] . \\ g_{ik} &= \frac{1}{\ell_{ik}}\left[\ Vi\{q_{ik}\} - \frac{a}{c}\, Vk\{q_{ik}\}\right] . \\ h_{ik} &= \frac{1}{\ell_{ik}}\left[-\frac{b}{c}\ Vi\{q_{ik}\} + \frac{a}{c}\ Vj\{q_{ik}\}\right] \end{aligned} \tag{5.27}$$

The R-condition equation.

The difference equation of (5.15) reads :
see also (3.7)

$$\underline{\Delta R}_{i\,\ldots\ldots\, i} = p_{i,i+1}\,(p^{-1}\underline{\Delta}p)_{i,i+1}\ p^{-1}_{i,i+1} + p_{i,i+2}\,(p^{-1}\underline{\Delta}p)_{i+1,i+2}\ p^{-1}_{i,i+2} + \cdots\cdots + (p^{-1}\underline{\Delta}p)_{i-1,i}$$

After rotation to the (r)-system :

$$\boxed{\underline{\Delta R}^{(r)}_{i\ldots i} = p_{r,i+1}\,(p^{-1}\underline{\Delta}p)_{i,i+1}\, p^{-1}_{r,i+1} + p_{r,i+2}(p^{-1}\underline{\Delta}p)_{i+1,i+2}\, p^{-1}_{r,i+2} + \cdots + p_{ri}\,(p^{-1}\underline{\Delta}p)_{i-1,i}\ p^{-1}_{ri}} \tag{5.27'}$$

The A-condition equation.

The difference equation of (5.17) reads :

$$\begin{aligned} \underline{\Delta A}_{a\ldots i} &= (-1)^m(-1)\ q^{-1}_{ik}\,\underline{\Delta q}_{ik;A}\ q_{ik}\,\mathcal{G}_i\cdots\mathcal{G}_b\ q_{ab}\ + \\ &\quad + (-1)^m\, q^{-1}_{ik}\, q_{ik}\ \underline{\Delta \Pi}_i\ q^{-1}_{ij}\ \mathcal{G}_i\cdots\cdots\mathcal{G}_b\ q_{ab} + \cdots\cdots \\ &\quad \cdots\cdots + (-1)^m\, q^{-1}_{ik}\ \mathcal{G}_i\cdots\cdots\mathcal{G}_b\ \underline{\Delta q}_{ab} = \\ &= (-1)^{2m+1}(q^{-1}\underline{\Delta q})_{ik;A} + (-1)^{2m}\underline{\Delta \Pi}_i + \cdots + (-1)^{2m}(q^{-1}\underline{\Delta q})_{ab}\ . \end{aligned}$$

therefore :

$$\boxed{\underline{\Delta A}_{a\ldots i} = -\,(q^{-1}\underline{\Delta q})_{ik;A} + \underline{\Delta \Pi}_i + \underline{\Delta \Pi}_j + \cdots\cdots\cdots + \underline{\Delta \Pi}_b + (q^{-1}\Delta q)_{ab}} \tag{5.28}$$

5.4 Dependencies/Selection of condition equations.

In this section we shall analyse the linear dependencies between the condition equations described in section 5.3, or between components of these condition equations.

As a continuation of (5.23), we start with the dependency between two coordinate condition equations and a polygon condition equation.

5.4.1
The dependency between network and polygon conditions

From (5.24) follow :

$$\underline{\Delta N}_{(n)} = q_{1n}\,\underline{\Delta \Pi}_1 + q_{2n}\,\underline{\Delta \Pi}_2 + \cdots + q_{n-1,n}\,\underline{\Delta \Pi}_{n-1}\ .$$

$$\underline{\Delta N}_{(1)} = \qquad q_{21}\,\underline{\Delta \Pi}_2 + \cdots + q_{n-1,1}\,\underline{\Delta \Pi}_{n-1} + q_{n1}\,\underline{\Delta \Pi}_n$$

therefore :

$$\begin{aligned}\underline{\Delta N}_{(1)} - \underline{\Delta N}_{(n)} &= q_{n1}\left[\underline{\Delta \Pi}_1 + \underline{\Delta \Pi}_2 + \cdots + \underline{\Delta \Pi}_n\right] = \\ {\scriptstyle (5.25)} \qquad &= \underline{\Delta V}_{1\ldots n}\, q_{n1}\ .\end{aligned} \qquad (5.29)$$

Note : This result is also obtained from the direct differentiation of (5.23) :

$$\underline{\Delta N}_{(1)} - \underline{\Delta N}_{(n)} = \underline{\Delta V}_{1\ldots n}\, q_{n1} + V_{1\ldots n}\,\underline{\Delta q}_{n1}$$

in which V = 0

(compare [2] : (17.3.2)

5.4.2
Dependencies within the R-condition

According to the general difference formula (3.7) for an open chain of astronomical rotations with longitude differences, (5.27') can be reduced to :

$$\begin{aligned}\underline{\Delta R}^{(r)}_{i\ldots i} = p_{ri}\Big[\, & p_{ii'}\,(p^{-1}\underline{\Delta} p)_{ii'}\, p^{-1}_{ii'} + p_{ii''}(p^{-1}\underline{\Delta} p)_{i'i''}\, p^{-1}_{ii''} + \\ & - p_{ii''}\, k\, p^{-1}_{ii''}\, \tfrac{1}{2}\left[\underline{\Delta\lambda}_{i,i+1} + \underline{\Delta\lambda}_{i+1,i+2} + \cdots + \underline{\Delta\lambda}_{i-1,i}\right] + \\ & + p_{ii'}\,(p^{-1}\underline{\Delta} p)_{i''i'}\, p^{-1}_{ii'} + (p^{-1}\underline{\Delta} p)_{i'i}\Big]\, p^{-1}_{ri}\end{aligned}$$

Here the terms with orientations and latitudes of the "intermediate" systems have been deleted. (see figure 19)
Because we are faced here with a closed chain, (i, i + 1, ... i - 1), also the terms with $\underline{\Delta\theta}_i$ and $\underline{\Delta\varphi}_i$ cancel each other in pairs :
since :

$$\begin{aligned}(2.30^5) : \quad p_{ii'}\,(p^{-1}\underline{\Delta} p)_{ii'}\, p^{-1}_{ii'} &= - p_{ii'}\, k\, p^{-1}_{ii'}\, \tfrac{1}{2}\,\underline{\Delta\theta}_i = \\ &= - k\, \tfrac{1}{2}\,\underline{\Delta\theta}_i \\ &\quad (p_{ii'} /\!/ k\ !\ )\ .\end{aligned}$$

$$(2.30^1) : \qquad (p^{-1}\underline{\Delta} p)_{i'i} = k\, \tfrac{1}{2}\,\underline{\Delta\theta}_i\ .$$

(likewise the terms with $\underline{\Delta\varphi}_i$, i.e. $(p^{-1}\underline{\Delta} p)_{i'i}$ and $(p^{-1}\underline{\Delta} p)_{i''i'}$ cancel each other)
Therefore, in the R-conditions equation only the terms of the differences of longitude remain :

$$\underline{\Delta R}_{i\ldots\ldots i} = - p_{ri''}\, k\, p_{ri''}^{-1}\, \tfrac{1}{2} \left[\underline{\Delta\lambda}_{i,i+1} + \cdots\cdots + \underline{\Delta\lambda}_{i-1,i} \right]$$

If : $p_{ri''} = d + ia + jb + kc$, this is :

$$\boxed{\underline{\Delta R}_{i\ldots\ldots i} = -\tfrac{1}{2}\left[0 + i\, 2(db+ac) + j\, 2(-da+bc) + k(d^2 - a^2 - b^2 - c^2)\right]\left[\Sigma\, \underline{\Delta\lambda}_{i,i+1}\right]} \tag{5.30}$$

Because $Sc\left\{ R_{i\ldots\ldots i}\right\} \doteq 0$, this means that the scalar component of the R-condition equation has been fulfilled identically :

$$Sc\left\{ \underline{\Delta R}_{i\ldots i}\right\} \equiv 0$$

and also that there are two dependencies between the three vector components :

$$\begin{aligned} Vi\left\{ \underline{\Delta R}_{i\ldots i}\right\} &= \frac{2(db+ac)}{d^2 - a^2 - b^2 + c^2}\; Vk\left\{ \underline{\Delta R}_{i\ldots i}\right\} \\ Vj\left\{ \underline{\Delta R}_{i\ldots i}\right\} &= \frac{2(-da+bc)}{d^2 - a^2 - b^2 + c^2}\; Vk\left\{ \underline{\Delta R}_{i\ldots i}\right\} \end{aligned} \tag{5.31}$$

Out of the four components of the R-condition, only one is non-identical and non-dependent. In a network, in which the k-vectors of the local systems are approximately parallel, a and b are approximately = 0 and $d^2 + c^2 \approx 1$. This means that the difference quantities $\underline{\Delta\lambda}$ only have large coefficients in the k-component of the R-condition equation (5.30)

5.4.3
The dependencies between the components of the A-.and the Z-condition equations.

We split up the $\underline{\Delta\Pi}$ —quantities according to (2.21) into $(q^{-1}\underline{\Delta q})$-quantities ; $\underline{A}_{a\ldots\ldots i}$ is now used as misclosure of the A-condition equation :

$$\underline{\Delta A}_{a\ldots i} = -(q^{-1}\underline{\Delta} q)_{ik;A} + \overbrace{(q^{-1}\underline{\Delta} q)_{ik} \underbrace{- (q^{-1}\underline{\Delta} q)_{ij} + \overbrace{(q^{-1}\underline{\Delta} q)_{ji}}^{=\underline{\Delta\Pi}_j\,..}}_{(5.8)}}^{=\underline{\Delta\Pi}_i} - \cdots - \underbrace{\overbrace{(q^{-1}\underline{\Delta} q)_{ba}}^{..=\underline{\Delta\Pi}_b} + (q^{-1}\underline{\Delta} q)_{ab}}_{(5.8)}$$

If all sides of the trajectory $P_a P_b \ldots P_j P_i$ comply with (5.8), this can be rewritten by substitution of (5.27) as :

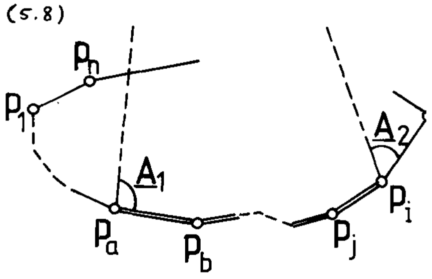


fig.39

$$\underline{\Delta A}_{a\ldots i} = -(q^{-1}\underline{\Delta} q)_{ik;A} + (q^{-1}\underline{\Delta} q)_{ik} + \sum_{m,n}\left[i\, f_{mn} + j\, g_{mn} + k\, h_{mn}\right] \underline{\Delta Z}_{mn}\,. \tag{5.32}$$

Applying (3.23) this becomes with (i') instead of (r) :

$$(q^{-1}\underline{\Delta q})^{(r)}_{ik} = \Delta \underline{\ln \bar{\lambda}}_{ri} + \Delta \underline{\ln s}_{ik} + p_{ri'}\left[e'^{(i')}_{ik} \sin \mathfrak{Z}_{ik} (\underline{\Delta r}_{ik} + \underline{\Delta \theta}_i) + e''^{(i')}_{ik} \underline{\Delta \mathfrak{Z}}_{ik}\right] p^{-1}_{ri'} + \ldots \text{terms with } \underline{\Delta p}_{ri} \ldots$$

and, with (5.16) :

$$(q^{-1}\underline{\Delta q})_{ik;A} = \Delta \underline{\ln \bar{\lambda}}_{ri} + \Delta \underline{\ln s}_{ik} + p_{ri'}\left[e'^{(i')}_{ik} \sin \mathfrak{Z}_{ik}\, \underline{\Delta A}_2 + e''^{(i')}_{ik} \underline{\Delta \mathfrak{Z}}_{ik}\right] p^{-1}_{ri'} + \ldots \text{terms with } \underline{\Delta p}_{ri} \ldots$$

In these formulae, the terms with Δp_{ri} are equal.

Remark : Contrary to (3.38), the $(q^{-1}\underline{\Delta q})$-quantities in (5.32) are pointing in the same direction ; therefore, the terms with $\underline{\Delta p}_{ri}$ are annulled in the difference.

Thus, (5.32) passes into .

$$\boxed{\underline{\Delta A}_{a\ldots i} = e'^{(r)}_{ik} \sin \mathfrak{Z}_{ik}\left[-\underline{\Delta A}_2 + \underline{\Delta r}_{ik} + \underline{\Delta \theta}_i\right] + \sum_{m,n}\left[i\, f_{mn} + j\, g_{mn} + k\, h_{mn}\right] \underline{\Delta Z}_{mn}} \qquad (5.33)$$

The second azimuth therefore only occurs, through the quantity $(q^{-1}\underline{\Delta q})_{ik;A}$ in the A-condition equation. From (5.33) it becomes clear that the scalar component of the A-condition equation has been fulfilled identically :

$$Sc\,\{\underline{\Delta A}_{a\ldots i}\} \equiv 0 \,.$$

If $e'^{(r)}_{ik} = 0 + iA + jB + kC$, the three vector components of the A-condition equation are :

$$Vi\,\{\underline{\Delta A}_{a\ldots i}\} = A \sin \mathfrak{Z}_{ik}\left[-\underline{\Delta A}_2 + \underline{\Delta r}_{ik} + \underline{\Delta \theta}_i\right] + \sum_{m,n} f_{mn}\, \underline{\Delta Z}_{mn} \,.$$

$$Vj\,\{\underline{\Delta A}_{a\ldots i}\} = B \sin \mathfrak{Z}_{ik}\left[-\underline{\Delta A}_2 + \underline{\Delta r}_{ik} + \underline{\Delta \theta}_i\right] + \sum_{m,n} g_{mn}\, \underline{\Delta Z}_{mn} \,.$$

$$Vk\,\{\underline{\Delta A}_{a\ldots i}\} = C \sin \mathfrak{Z}_{ik}\left[-\underline{\Delta A}_2 + \underline{\Delta r}_{ik} + \underline{\Delta \theta}_i\right] + \sum_{m,n} h_{mn}\, \underline{\Delta Z}_{mn} \,.$$

Between these equations there are two dependencies :
(A and B ≈ 0 ; $C \approx -1$)

$$\left.\begin{aligned} Vi\,\{\underline{\Delta A}_{a\ldots i}\} &= \frac{A}{C}\, Vk\,\{\underline{\Delta A}_{a\ldots i}\} + \sum_{m,n}\left[f_{mn} - \frac{A}{C} h_{mn}\right] \underline{\Delta Z}_{mn} \,. \\ Vj\,\{\underline{\Delta A}_{a\ldots i}\} &= \frac{B}{C}\, Vk\,\{\underline{\Delta A}_{a\ldots i}\} + \sum_{m,n}\left[g_{mn} - \frac{B}{C} h_{mn}\right] \underline{\Delta Z}_{mn} \,. \end{aligned}\right\} \qquad (5.34)$$

Only one of the three components of the A-condition equation is, therefore, independent : the azimuths only have large coefficients in the k-component. If the network is approximately plane, then :

$$\underline{\Delta \theta}_i \approx \underline{\Delta A}_1 + \underline{\Delta \alpha}_b + \cdots + \underline{\Delta \alpha}_j - \underline{\Delta r}_{ij} \,.$$

and the h_{mn} are ≈ 0 ; $Vk\,\{e'\} \approx -1$; therefore :

$$Vk\,\{\underline{\Delta A}_{a\ldots i}\} \approx -\underline{\Delta A}_2 + \underline{\Delta A}_1 + \underline{\Delta \alpha}_b + \cdots + \underline{\Delta \alpha}_j + \underline{\Delta \alpha}_i \,. \qquad (5.35)$$

5.4.4
The scalar component of the coordinate conditions.

We now consider the network condition $N_{(n)}$, see (5.24) ; after splitting up the $\underline{\Delta\Pi}$–quantities, this becomes :

$$\underline{\Delta N}_{(n)} = q_{1n}\left[-(q^{-1}\underline{\Delta q})_{1n} + (q^{-1}\underline{\Delta q})_{12}\right] + q_{2n}\left[-(q^{-1}\underline{\Delta q})_{21} + (q^{-1}\underline{\Delta q})_{23}\right] + \cdots$$

$$\cdots + q_{n-1,n}\left[-(q^{-1}\underline{\Delta q})_{n-1,n-2} + (q^{-1}\underline{\Delta q})_{n-1,n}\right] .$$

In using $q_{2n} = q_{21} + q_{1n}$ etc., the following arrangement of the terms obtained :

$$\underline{\Delta N}_{(n)} = \underbrace{-\underline{\Delta q}_{1n} - \underline{\Delta q}_{21} - \cdots - \underline{\Delta q}_{n-1,n-2} + \underline{\Delta q}_{n-1,n}}_{\substack{a\\ \downarrow}} + \underbrace{\sum_{i=1}^{n-2} q_{in}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right]}_{\substack{b\\ \downarrow}} . \qquad (5.36)$$

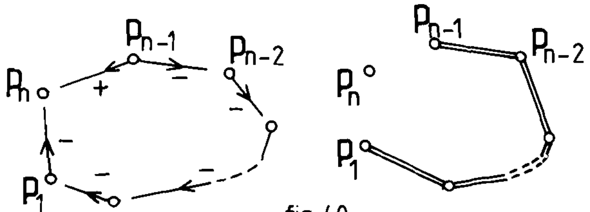


fig. 40

Of the "a-terms", the scalar component equals zero :

$$a : \quad Sc\{\underline{\Delta q}_{i,i'}\} = 0 .$$

Of the "b-terms" the scalar part is :

$$Sc\{q_{in}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right] = 0\; Sc\{[\;]\} - x_{in}\, Vi\{[\;]\} - y_{in}\, Vj\{[\;]\} - z_{in}\, Vk\{[\;]\} .$$

Because side P_nP_1 is absent from the b-terms, (5.27) applies in all the b-terms, thus :

$$b : \quad Sc\{q_{in}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right] = \left[-x_{in} f_{i,i+1} - y_{in}\, g_{i,i+1} - z_{in} h_{i,i+1}\right] \underline{\Delta Z}_{i,i+1} =$$

$$= \overset{\text{Suppose:}\nearrow}{w_{n,i,i+1}}\; \underline{\Delta Z}_{i,i+1} .$$

The scalar component of the coordinate condition equation is consequently dependent on n - 2 of the Z– condition equations ;

$$\boxed{Sc\{\underline{\Delta N}_{(n)}\} = w_{n12}\, \underline{\Delta Z}_{12} + w_{n23}\, \underline{\Delta Z}_{23} + \cdots + w_{n,n-2,n-1}\, \underline{\Delta Z}_{n-2,n-1} .} \qquad (5.37)$$

In the same way, the coordinate condition equation $N_{(1)}$ can be rewritten as :

$$\underline{\Delta N}_{(1)} = -\underline{\Delta q}_{21} - \underline{\Delta q}_{32} - \cdots - \underline{\Delta q}_{n,n-1} + \underline{\Delta q}_{n,1} + \underbrace{\sum_{i=2}^{n-1} q_{i1}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right]} . \qquad (5.38)$$

Also here, side P_nP_1 is absent from the "b-terms" : thus (5.27) can be used again, and (5.38) becomes :

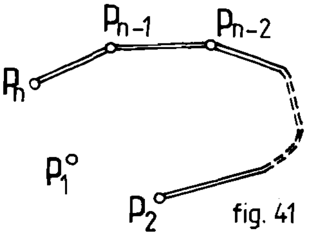


fig. 41

$$\boxed{Sc\left\{\underline{\Delta N}_{(1)}\right\} = w_{123}\,\underline{\Delta Z}_{23} + w_{134}\,\underline{\Delta Z}_{34} + \cdots + w_{1,n-1,n}\,\underline{\Delta Z}_{n-1,n}\,.} \qquad (5.39)$$

5.4.5
The k-component of the coordinate condition

Subsequently, we subtract (5.38) from (5.36) :

$$\begin{aligned} \underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)} = &- \underline{\Delta q}_{1n} + \underline{\Delta q}_{n-1,n} + \sum_{i=1}^{n-2} q_{in}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right] + \\ &\underbrace{-\left[\underline{\Delta q}_{n1} - \underline{\Delta q}_{n,n-1}\right]}_{\text{"a-terms"}} \underbrace{-\sum_{i=2}^{n-1} q_{i1}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right]}_{\text{"b-terms"}} . \end{aligned} \qquad (5.40)$$

(the other a-terms cancel each other).

The k-components of the "b-terms" are : see (5.27) :

$$Vk\left\{q_{in}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right]\right\} = \left[x_{in}\,g_{i,i+1} - y_{in}\,f_{i,i+1}\right]\underline{\Delta Z}_{i,i+1} =$$

$$\text{Suppose :} \longrightarrow \quad = V_{n,i,i+1}\;\underline{\Delta Z}_{i,i+1}\,.$$

$$Vk\left\{q_{i1}\left[(q^{-1}\underline{\Delta q})_{i,i+1} - (q^{-1}\underline{\Delta q})_{i+1,i}\right]\right\} = V_{1,i,i+1}\;\underline{\Delta Z}_{i,i+1}\,.$$

Thus, the k-component of (5.40) becomes :

$$\begin{aligned} Vk\left\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\right\} = &- \underline{\Delta Z}_{n1} + \underline{\Delta Z}_{n-1,n} + V_{n12}\,\underline{\Delta Z}_{12} + \cdots + V_{n,n-1,n-2}\,\underline{\Delta Z}_{n-1,n-2} + \\ &- V_{123}\,\underline{\Delta Z}_{23} - \cdots - V_{1,n-1,n}\,\underline{\Delta Z}_{n-1,n}\,. \end{aligned}$$

therefore :

$$\boxed{\begin{aligned} Vk\left\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\right\} = &- \underline{\Delta Z}_{n1} + V_{n12}\,\underline{\Delta Z}_{12} + \left[V_{n23} - V_{123}\right]\underline{\Delta Z}_{23} + \cdots\cdots \\ &\cdots + \left[V_{n,n-2,n-1} - V_{1,n-2,n-1}\right]\underline{\Delta Z}_{n-2,n-1} + \left[1 - V_{1,n-1,n}\right]\underline{\Delta Z}_{n-1,n} \end{aligned}} \qquad (5.41)$$

There is, consequently, a dependency between the k-components of the two coordinate condition equations and all Z-condition equations.

5.4.6
The NN and the NV-model; rank of the system.

Formula (5.29) signifies, as already concluded in (5.23), that in a fully measured closed polygon, there are either two independent coordinate conditions or one coordinate and one polygon condition. Consequently we can specify two condition models :

(5.42)

NN-model	number of components	NV-model	number of components
2 network conditions	8	1 network condition	4
n Z-conditions	n	1 polygon condition	4
1 R-condition	4	n Z-condition	n
		1 R-condition	4
	n + 12		n + 12

In this table the azimuth conditions, if any, have been left out of consideration : in fact, the added azimuths occur, according to (5.33), exclusively in the azimuth conditions. These are therefore, at any rate, independent of the other conditions, and therefore do not play a part in an analysis of dependencies between conditions.

In the preceding sections, the following six dependencies have been found :

$$\left.\begin{array}{lll}
(5.31) & : \ Sc\{\underline{\Delta R}\} \equiv 0 & 1 \\
(5.31) & : \ Vi\{\underline{\Delta R}\} \text{ and } Vk\{\underline{\Delta R}\} & 2 \\
(5.31) & : \ Vj\{\underline{\Delta R}\} \text{ and } Vk\{\underline{\Delta R}\} & 3 \\
(5.37) & : \ Sc\{\underline{\Delta N}_{(n)}\} \text{ and } n-2 \ \underline{\Delta Z}\text{'s} & 4 \\
(5.39) & : \ Sc\{\underline{\Delta N}_{(1)}\} \text{ and } n-2 \ \underline{\Delta Z}\text{'s} & 5 \\
(5.41) & : \ Vk\{\underline{\Delta N}_{(n)}\} \text{ and } Vk\{\underline{\Delta N}_{(1)}\} \text{ and } \underline{\Delta Z}_{n1} & 6
\end{array}\right\} \qquad (5.43)$$

Note : the latter only applies in the NN-model, owing to the existence of two coordinate conditions.

If, in this case too, we assume, by way of caution, that dependencies have been overlooked, this implies an upper limit of the rank of the NN-model :

$$b \leqslant \underset{(5.42)}{n+12} - \underset{(5.43)}{6} .$$

Because, however, in (4.51), a lower limit was found :

$$b \geqslant n+6$$

we now arrive at the conclusion :

$$\boxed{b = n+6} \qquad (5.44)$$

Specification of observations : (4.48).

The NV-model

The NV-model too, must contain six dependencies. The dependencies (5.43/1, 2, 3, 4) apply to both the NN and to the NV-model. The remaining two dependencies can be derived from those of the NN-model with the aid of (5.29). We write the dependencies (5.43/5 and 6) of the NN-model, as equations with the misclosures as variables, in the form of a matrix :

$$\begin{array}{l} (5.43^5) \rightarrow \\ (5.43^6) \rightarrow \end{array} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & w_{123} & \cdots\cdots & w_{1,n-2,n-1} & w_{1,n-1,n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & V_{n12} & [V_{n23}-V_{123}] & \cdots & [V_{n,n-2,n-1}-V_{1,n-2,n-1}] & [1-V_{1,n-1,n}] & -1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

$$\Big(\underbrace{\underline{Sc} \ \underline{Vi} \ \underline{Vj} \ \underline{Vk}}_{\underline{\Delta N}_{(n)}} \ \underbrace{\underline{Sc} \ \underline{Vi} \ \underline{Vj} \ \underline{Vk}}_{\underline{\Delta N}_{(1)}} \ \underline{\Delta Z}_{12} \ \underline{\Delta Z}_{23} \cdots\cdots \underline{\Delta Z}_{n-2,n-1} \ \ \underline{\Delta Z}_{n-1,n} \ \ \underline{\Delta Z}_{n,1} \ \underbrace{\underline{Sc} \ \underline{Vi} \ \underline{Vj} \ \underline{Vk}}_{\underline{\Delta V}_{1\ldots n}}\Big)^*$$

$$(5.45)$$

To this we add (5.29), also in the form of a matrix :

let : $q_{n1} = 0 + ix + jy + kz$:

$$\begin{array}{l}(5.21^1)\rightarrow\\(5.21^2)\rightarrow\\(5.21^3)\rightarrow\\(5.21^4)\rightarrow\end{array}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1&0&0&0&-1&0&0&0&0\cdots\cdots0&0&-x_{n1}&-y_{n1}&-z_{n1}\\0&1&0&0&0&-1&0&0&0\cdots\cdots0&x_{n1}&0&-z_{n1}&y_{n1}\\0&0&1&0&0&0&-1&0&0\cdots\cdots0&y_{n1}&z_{n1}&0&-x_{n1}\\0&0&0&1&0&0&0&-1&0\cdots\cdots0&z_{n1}&-y_{n1}&x_{n1}&0\end{pmatrix}\cdot$$

$$\Big(\underbrace{\underline{Sc}\;\underline{Vi}\;\underline{Vj}\;\underline{Vk}}_{\underline{\Delta N}_{(n)}}\;\underbrace{\underline{Sc}\;\underline{Vi}\;\underline{Vj}\;\underline{Vk}}_{\underline{\Delta N}_{(1)}}\;\underline{\Delta Z}_{12}\cdots\cdots\underline{\Delta Z}_{n1}\;\underbrace{\underline{Sc}\;\underline{Vi}\;\underline{Vj}\;\underline{Vk}}_{\underline{\Delta V}_{1\ldots n}}\Big)^{*}$$

From the system of six equations thus obtained, the four components of $\underline{\Delta N}_{(1)}$ can be eliminated ; this leads to the two equations (dependencies) sought between the conditions of the NV-model :

$(5.43^5)-(5.21^1)\rightarrow$

$$0=-Sc\{\underline{\Delta N}_{(n)}\}+w_{123}\,\underline{\Delta Z}_{23}+\cdots\cdots+w_{1,n-1,n}\,\underline{\Delta Z}_{n-1,n}+x_{n1}\,Vi\{\underline{\Delta V}_{1\ldots n}\}+$$
$$+y_{n1}\,Vj\{\underline{\Delta V}_{1\ldots n}\}+z_{n1}\,Vk\{\underline{\Delta V}_{1\ldots n}\}\cdot \tag{5.46}$$

$(5.43^6)+(5.21^4)\rightarrow$

$$0=v_{n12}\,\underline{\Delta Z}_{12}+[v_{n23}-v_{123}]\,\underline{\Delta Z}_{23}+\cdots+[v_{n,n-2,n-1}-v_{1,n-2,n-1}]\,\underline{\Delta Z}_{n-2,n-1}+$$
$$+[1-v_{1,n-1,n}]\,\underline{\Delta Z}_{n-1,n}-\underline{\Delta Z}_{n1}+z_{n1}\,Sc\{\underline{\Delta V}_{1..n}\}-y_{n1}\,Vi\{\underline{\Delta V}_{1\ldots n}\}+x_{n1}\,Vj\{\underline{\Delta V}_{1..n}\} \tag{5.47}$$

The overall system of dependencies within the NV-model thus consists of (5.43/1, 2, 3, 4) and (5.46), (5.47). Here, the i, j and k-components of the coordinate condition do not occur, whereas all four components of the polygon condition are present. This implies that the coordinate condition rather takes a position of its own. This substantiates for spatial networks Baarda's conclusion (see section 4.5 of [2]) that the coordinate condition "is the most fundamental condition".

5.4.7
Selection of conditions.

From the condition models NN and NV, six conditions must be eliminated. As a general criterion for this choice it applies that the matrix of weight coefficients of the remaining misclosures, ($g^{p\tau}$) in the terminology of [4] , must be as orthogonal as possible. From this requirement follows that those misclosures that have the largest coefficients in the dependency equations (5.31, 5.37, 5.39, 5.41 and 5.47) are those to be considered above all for elimination, because otherwise, the diagonal elements of ($g^{p\tau}$) would become very small.

As far as the R-condition is concerned, these are :

$Vi\{\underline{\Delta R}_{i\ldots i}\}$

$Vj\{\underline{\Delta R}_{i\ldots i}\}$

Moreover Sc $\{\underline{\Delta R}_{i..i}\}$ is deleted.

The Z-conditions have been introduced on account of the reciprocal measurement of zenith angles on the network sides ; the number of Z-conditions is therefore equal to the number of sides and it also re-occurs in the rank of the overall condition model. In principle, the elimination of one (or more) of the Z-conditions is possible, but this would lead to a less balanced structure of the condition model. Consequently, we can also eliminate from the NN-model :

see (5.37) : Sc $\{\underline{\Delta N}_{(n)}\}$

see (5.39) : Sc $\{\underline{\Delta N}_{(1)}\}$

see (5.41) : Vk $\{\underline{\Delta N}_{(n)}\}$ or Vk $\{\underline{\Delta N}_{(1)}\}$

Thus, the specification of the NN-model becomes :

$$\boxed{\underbrace{\cancel{\text{Sc}}\ \text{Vi}\ \ \text{Vj}\ \ \text{Vk}}_{N_{(n)}}\ \ \underbrace{\cancel{\text{Sc}}\ \text{Vi}\ \ \text{Vj}\ \ \cancel{\text{Vk}}}_{N_{(1)}}\ \ \underbrace{\cancel{\text{Sc}}\ \cancel{\text{Vi}}\ \ \cancel{\text{Vj}}\ \ \text{Vk}}_{R_{i,\dots i-1}}\ \ \begin{array}{l} Z_{12},\dots\dots\dots Z_{n1} \\ \boxed{b=n+6} \end{array}} \qquad (5.48)$$

In addition to three components of $\underline{\Delta R}$, in principle Sc $\{\underline{\Delta N}_{(n)}\}$ and each of the four components of $\underline{\Delta V}$ can be eliminated from the NV condition model. However, Sc $\{\underline{\Delta V}\}$ and Vk $\{\underline{\Delta V}\}$ have small coefficients in (5.47) and (5.46) respectively and are thus less suitable for being eliminated. Therefore, we eliminate : Sc $\{\underline{\Delta N}_{(n)}\}$, Vi $\{\underline{\Delta V}\}$ and Vj $\{\underline{\Delta V}\}$:

$$\boxed{\underbrace{\cancel{\text{Sc}}\ \text{Vi}\ \ \text{Vj}\ \ \text{Vk}}_{N_{(n)}}\ \ \underbrace{\text{Sc}\ \cancel{\text{Vi}}\ \ \cancel{\text{Vj}}\ \ \text{Vk}}_{V_{1\dots n}}\ \ \underbrace{\cancel{\text{Sc}}\ \cancel{\text{Vi}}\ \ \cancel{\text{Vj}}\ \ \text{Vk}}_{R_{i,\dots i-1}}\ \ \begin{array}{l} Z_{12}\dots\dots Z_{n1} \\ \boxed{b=n+6} \end{array}} \qquad (5.49)$$

This leads to complete agreement with the NV-model of the two-dimensional polygon theory :

- the coordinate condition there, is composed of two, and here of three "vector" components,
- the polygon condition in the two-dimensional model reads :

$$\mathrm{Re}\{\underline{\Delta V}\} = \sum^{n} \Delta\underline{\ln v}_{jik}$$

$$\mathrm{Im}\{\underline{\Delta V}\} = i\sum^{n}\left[\underline{\Delta r}_{ik} - \underline{\Delta r}_{ij}\right]$$

and, in the three-dimensional model :

$$\mathrm{Sc}\{\underline{\Delta V}\} = \sum^{n} \Delta\underline{\ln v}_{jik}$$

$$\mathrm{Vk}\{\underline{\Delta V}\} = \sum^{n}\left[a_1\,\underline{\Delta r}_{ik} - a_2\,\underline{\Delta r}_{ij} + \text{terms with small coefficients}\dots\right]$$

$$a_1 \approx 1$$
$$a_2 \approx 1$$

(the R- and Z-conditions do not occur in the two-dimensional theory).

5.5 Modification of the starting points.

5.5.1

Deviation from (5.8)

In the preceeding sections, we have started from the fact that (5.27) applies, viz. that one side $P_1 P_n$ is not used for the computation of orientations and length factors and that, moreover, those coordinate conditions are chosen, in which the φ-quantities (or: $\Delta\Pi$-quantities) of P_n and P_1 are absent. What are the consequences for the system dependencies, if we depart from this ? The number of possibilities amounts to $n^3 (n - 1)$, i.e. even in a triangle there are as many as 54 !

location of $N_{(a)}$: n
location of $N_{(b)}$: n - 1
location of θ^o : n
location of $\bar{\lambda}^o$: n

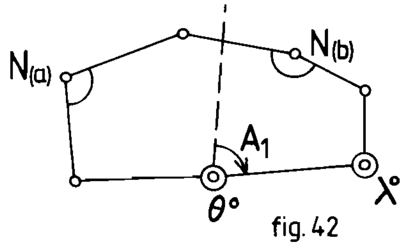


fig. 42

We should, therefore, restrict ourselves to the discussion of one single example :
Assume : the side $P_n P_1$ is used for the computation of the orientations. This means that one of the other sides, $P_1 P_m$, does nòt comply with $(3.42)^{IV}$, but with $(3.42)^{III}$:

$$(q^{-1}\underline{\Delta}q)_{\ell m}-(q^{-1}\underline{\Delta}q)_{m\ell} = \Delta\underline{\ln\bar{\lambda}}_{\ell} + \Delta\underline{\ln s}_{\ell m} - \Delta\underline{\ln\bar{\lambda}}_{m} - \Delta\underline{\ln s}_{m\ell} + e''_{m';\ell m}\left[\Delta\underline{\mathfrak{J}}^{m'}_{\ell m} + \Delta\underline{\mathfrak{J}}_{m\ell}\right] \cdot$$

However, now :

$$\Delta\underline{\ln\bar{\lambda}}_{\ell} = \Delta\underline{\ln\bar{\lambda}}_{m} + \left[\Delta\underline{\ln s}_{m,m+1} - \Delta\underline{\ln s}_{m+1,m}\right] + \cdots + \left[\Delta\underline{\ln s}_{\ell-1,\ell} - \Delta\underline{\ln s}_{\ell,\ell-1}\right] \cdot$$

So :

$$(q^{-1}\underline{\Delta}q)_{\ell m}-(q^{-1}\underline{\Delta}q)_{m\ell} = Sc\{\underline{\Delta V}\} + e''_{m';\ell m}\left[\Delta\underline{\mathfrak{J}}^{m'}_{\ell m} + \Delta\underline{\mathfrak{J}}_{m\ell}\right] =$$

$$= Sc\{\underline{\Delta V}\} + \left[o + i f_{\ell m} + j g_{\ell m} + k h_{\ell m}\right] \underline{\underline{\Delta Z}}_{\ell m}$$

The vector components of $[(q^{-1}\Delta q)_{lm} - (q^{-1}\Delta q)_{ml}]$ thus remain unmodified, but the scalar component was $=0$ and now becomes : Sc $\{\underline{\Delta V}\}$. With this, the dependency relations can now be adapted .

In the NN model :

(5.37) and (5.39) remain unchanged, since there only the vector components of $(q^{-1}\underline{\Delta} q)_{ik}$ - $(q^{-1}\underline{\Delta} q)_{ki}$ occur ; however, (5.41) changes now :

$$Vk\{q_{\ell n}[(q^{-1}\underline{\Delta}q)_{\ell m}-(q^{-1}\underline{\Delta}q)_{m\ell}]\} = z_{\ell n}\, Sc\{\underline{\Delta V}\} + v_{n\ell m}\, \underline{\underline{\Delta Z}}_{\ell m} \cdot$$

$$Vk\{q_{\ell 1}[(q^{-1}\underline{\Delta}q)_{\ell m}-(q^{-1}\underline{\Delta}q)_{m\ell}]\} = z_{\ell 1}\, Sc\{\underline{\Delta V}\} + v_{1\ell m}\, \underline{\underline{\Delta Z}}_{\ell m} \cdot$$

Consequently (5.41) becomes :

$$Vk\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\} = [z_{\ell_n} - z_{\ell_1}]\ Sc\{\underline{\Delta V}\} + \{\text{the terms as stated in (5.41)}\} \qquad (5.50)$$

Here we must replace Sc $\{\underline{\Delta V}\}$, which does not occur in the NN-model, by means of (5.29) by :

$$Sc\{\underline{\Delta V}_{1\ldots n}\} = Sc\{[\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}]\ q_{n1}^{-1}\} =$$

$$= \frac{-x_{n1}}{\ell_{n1}^2} Vi\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\} - \frac{y_{n1}}{\ell_{n1}^2} Vj\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\} - \frac{z_{n1}}{\ell_{n1}^2} Vk\{\underline{\Delta N}_{(n)} - \underline{\Delta N}_{(1)}\} .$$

<u>in the NV-model</u>: because (5.41) passes into (5.50), (5.47) passes into :

$$0 = \{\text{the terms stated in (5.47)}\} + [z_{\ell_n} - z_{\ell_1}]\ Sc\{\underline{\Delta V}_{1\ldots n}\}$$

5.5.2

Other types of observation variates

- Pseudo distances.

From (5.25') it becomes apparent that, as in the polygon theory in the complex plane, the scalar component of the V-condition has been fulfilled identically; so :

Pseudo distances → Sc $\{\underline{\Delta V}\} \equiv 0$

- Astronomical longitudes.

From (5.30) it becomes apparent that the same applies to the R-condition if, instead of astronomical differences of longitude, longitudes are measured :

astronomical longitudes → $\underline{\Delta R} \equiv 0$

Astronomical differences of latitude

When differences of latitude $\underline{\varphi}_{ik}$ are measured this does not lead to the creation of an additional condition, as already shown by (5.30). A condition of differences of "latitude"

$$\Sigma\, \underline{\varphi}_{ik} = 0$$

would only arise, if all the differences of longitude $\lambda_{ik} = 0$

5.3.3

More loops.

We apply the specification of observation variates (4.48) to a network, composed from more than one loop, which have all been fully measured.
So as to determine the number of observation variates, m, the number of sides z must be introduced in addition to the number of points n.
(4.48) then passes into :

2 z	directions
2 z	distance measures
2 z	zenith angles
z	astronomical differences of longitude
n	astronomical latitudes
+ 1	azimuth
m = 7 z + n + 1	

$$m = 7z + n + 1 \qquad (5.51)$$

According to (4.52), the number of unknowns, m - b only remains dependent on the number of points n .
In polygon networks there is a relation between the numbers of :

points : n
sides : z
loops : k

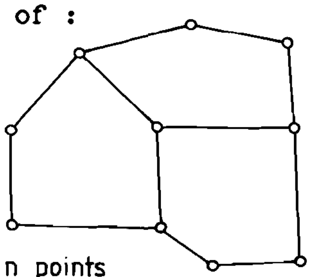


fig. 43

since, by extending a network by one loop, with p new points, the number of sides is increased by p + 1 :

$$\left.\begin{array}{l} n_{k+1} = n_k + p \\ z_{k+1} = z_k + p + 1 \end{array}\right\} \rightarrow (z-n)_{k+1} = (z-n)_k + 1 \qquad (5.53)$$

In a network consisting of one loop, the number of sides is equal to the number of points, therefore : $(z-n)_1 = 0$, therefore, with (5.53):

$$(z-n)_2 = 1 \;;\quad (z-n)_3 = 2 \;;\ \text{etc.}$$

or:

$$(z-n)_k = k-1 \qquad (5.54)$$

or : $n = z - k + 1$

Herewith (5.51) becomes :

$$m = 8z - k + 2$$

(5.52):

$$m-b = 7z - 7k + 2 \rightarrow b = m - 7z + 7k - 2 \;;$$

hence : $\boxed{b = z + 6k}$
(this agrees with (5.44) for one loop : k = 1, z→n).

The condition models NN and NV comprise z pieces of Z-conditions.
(they occur per side and not per point !) In view of (5.54) each loop contains 6 pieces of the other types of conditions, also in a polygon network consisting of more than one loop.

5.6 The adjustment model of observation equations.

5.6.1
Unknowns

In (5.44) was demonstrated that the rank of the condition model is :

$$b = n + 6$$

in a closed polygon with n points, when there are $8n + 1$ observation variates according to (4.48).

This means that there are 7n - 5 unknowns Y^{α} :

$$\boxed{\begin{array}{l} Y^{\alpha} \; ; \quad \alpha = 1 \ldots\ldots 7n-5 \\ \text{"unknowns", see (4.52).} \end{array}} \tag{5.58}$$

The coordinates are 3n−7 S-coordinates in the (R)-system, in accordance with (4.35). They may be introduced in observation equations by means of the quaternions :

$$\tilde{q}_{ik}^{(R)} = 0 + i\left[\tilde{X}_k^R - \tilde{X}_i^R\right] + j\left[\tilde{Y}_k^R - \tilde{Y}_i^R\right] + k\left[\tilde{Z}_k^R - \tilde{Z}_i^R\right] . \tag{5.59}$$

$$\tilde{X}_i^R , \tilde{Y}_i^R , \tilde{Z}_i^R \quad : \quad 3n-7 \quad \text{"S-coordinates"} \quad (\text{means}) .$$

The astronomical latitudes φ_i and differences in longitude λ_{ik} are both observation variate and unknown. Together with the observation unknowns Θ_i , they constitute rotation quaternions p_{ri} :

$$\tilde{p}_{ri} = p_{ri}\left(\tilde{\theta}_r , \tilde{\varphi}_r , \tilde{\lambda}_{r \ldots i} , \tilde{\varphi}_i , \tilde{\theta}_i\right)$$

$$\left.\begin{array}{l} \text{n orientation unknowns} \\[1ex] \text{n latitude unknowns} \\[1ex] \text{n - 1 longitude unknowns} \end{array}\right\} 3n - 1 \tag{5.60}$$

The relation with the (R)-system is established by the basis transformation p_{Rr} ; see (4.39) :

$$\tilde{\lambda}_{Rr}$$

$$\tilde{p}_{Rr} = \tilde{S}_{Rr} + i\,\tilde{I}_{Rr} + j\,\tilde{J}_{Rr} + k\,\tilde{K}_{Rr} .$$

According to (1.32), here one component depends on the three others ; let :

$$\tilde{p}_{Rr} = \sqrt{1 - \tilde{I}_{Rr}^2 - \tilde{J}_{Rr}^2 - \tilde{K}_{Rr}^2} + i\,\tilde{I}_{Rr} + j\,\tilde{J}_{Rr} + k\,\tilde{K}_{Rr} . \tag{5.61}$$

$$\tilde{\lambda}_{Rr} , \tilde{I}_{Rr} , \tilde{J}_{Rr} , \tilde{K}_{Rr} \quad : \quad 4 \text{ unknowns (means)}$$

The length factors $\underline{\lambda}_{ik}$ are quotients of the instrumental length units, see (2.13) :

$$\underline{\lambda}_{ik} = \frac{\underline{\lambda}_k}{\underline{\lambda}_i} = \frac{\text{length unit in } P_k}{\text{length unit in } P_i} \tag{5.62}$$

$\tilde{\bar{\lambda}}_{ik}$: n-1 unknowns.(means)

According to (5.59), (5.60), (5.61) and (5.62) there is therefore a total number of :

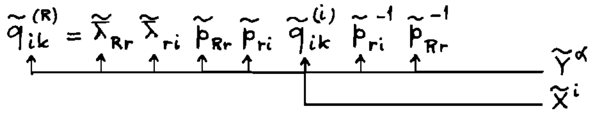
(5.5)	3n - 7
(5.60)	3n - 1
(5.61)	4
(5.62)	n - 1
	7n - 5 unknowns

5.6.2
Observation equations.

Distance measures, directions and zenith angles occur jointly per measured side of a network in the vector quaternion :

$$\tilde{q}^{(i)}_{ik} = o + i\,\tilde{s}_{ik}\cos\tilde{r}_{ik}\sin\tilde{\mathfrak{z}}_{ik} + j\,\tilde{s}_{ik}\sin\tilde{r}_{ik}\sin\tilde{\mathfrak{z}}_{ik} - k\,\tilde{s}_{ik}\cos\tilde{\mathfrak{z}}_{ik}\,. \tag{5.63}$$

For each network side we can now establish an equation in which the observation variates $\underline{X}^i$ and the unknowns $\underline{Y}^\alpha$ occur together :



$$\tilde{q}^{(R)}_{ik} = \tilde{\bar{\lambda}}_{Rr}\,\tilde{\bar{\lambda}}_{ri}\,\tilde{p}_{Rr}\,\tilde{p}_{ri}\,\tilde{q}^{(i)}_{ik}\,\tilde{p}^{-1}_{ri}\,\tilde{p}^{-1}_{Rr}$$

or, bringing all factors with $\tilde{Y}^\alpha$-quantities in the right-hand member :

$$\boxed{\tilde{q}^{(i)}_{ik} = \frac{1}{\tilde{\bar{\lambda}}_{Rr}\tilde{\bar{\lambda}}_{ri}}\,\tilde{p}^{-1}_{ri}\,\tilde{p}^{-1}_{Rr}\,\tilde{q}^{(R)}_{ik}\,\tilde{p}_{Rr}\,\tilde{p}_{ri}} \tag{5.64}$$

The i, j and k-components of the left-hand member are functions of the three observation variates s_{ik} , r_{ik} , $\mathfrak{z}_{ik}$; see (5.63). In order to obtain observation equations for these observation variates, we solve these as follows :

$$\begin{aligned} \ln s_{ik} &= \ln\sqrt{V_i\,V_i + V_j\,V_j + V_k\,V_k} \\ r_{ik} &= \arctan\frac{V_j}{V_i} + n\,\pi \;; \quad n=o \mid V_i > o \;; \quad n=1 \mid V_i < o \\ \mathfrak{z}_{ik} &= \operatorname{arccot}\frac{-V_k}{\sqrt{V_i\,V_i + V_j\,V_j}}\,. \end{aligned} \tag{5.65}$$

According to (5.5) approximate values X^i_o can now be computed from approximate values Y^α_o via (5.64) and (5.65).
Observation equations (5.7) are obtained by differentiating (5.64) and (5.65).

(5.65) gives, after substitution of polar coordinates in the coefficients and whilst adding Sc $\{\underline{\Delta q}_{ik}\} = 0$: (in the left hand member $\underline{s}_{ik} = \underline{s}_{ik} + \underline{\varepsilon}$, etc.)

$$(\underline{\Delta X}^{i})_{ik} = \begin{pmatrix} \Delta \underline{\ln s}_{ik} \\ \underline{\Delta r}_{ik} \\ \underline{\Delta \zeta}_{ik} \end{pmatrix} = \frac{1}{s_{ik}} \begin{pmatrix} 0 & \cos r \sin \zeta & \sin r \sin \zeta & -\cos \zeta \\ 0 & \frac{-\sin r}{\sin \zeta} & \frac{\cos r}{\sin \zeta} & 0 \\ 0 & \cos r \cos \zeta & \sin r \cos \zeta & \sin \zeta \end{pmatrix}_{ik} \cdot \begin{pmatrix} 0 \\ Vi\{\underline{\Delta q}^{(i)}_{ik}\} \\ Vj\{\underline{\Delta q}^{(i)}_{ik}\} \\ Vk\{\underline{\Delta q}^{(i)}_{ik}\} \end{pmatrix} \qquad (5.66)$$

The difference equation of (5.64) reads :

$$\underline{\Delta q}^{(i)}_{ik} = -q^{(i)}_{ik}\left[\Delta \underline{\ln \bar{\lambda}}_{Rr} + \Delta \underline{\ln \bar{\lambda}}_{ri}\right] - \left[\overset{-1}{p}_{ri}\,(\overset{-1}{p}\underline{\Delta} p)_{Rr}\, p_{ri} + (\overset{-1}{p}\underline{\Delta} p)^{(i)}_{ri}\right] q^{(i)}_{ik} +$$

$$+ q^{(i)}_{ik}\left[\qquad \text{"} \qquad \right] +$$

$$+ \frac{1}{\lambda_{Rr}\lambda_{ri}}\, \overset{-1}{p}_{ri}\, \overset{-1}{p}_{Rr}\, \underline{\Delta q}^{(R)}_{ik}\, p_{Rr}\, p_{ri} \cdot$$

In isomorphic matrices this reads :

$$(\overrightarrow{\underline{\Delta q}}^{(i)}_{ik}) = (-\overrightarrow{q}^{(i)}_{ik})\left[\Delta \underline{\ln \bar{\lambda}}_{Rr} + \Delta \underline{\ln \bar{\lambda}}_{ri}\right] + \left[(q^{(i)}_{ik}) - (\overline{q^{(i)}_{ik}})\right]\left[(P_{ri})^{-1}(\overrightarrow{\overset{-1}{p}\underline{\Delta} p})_{Rr} + (\overrightarrow{\overset{-1}{p}\underline{\Delta} p})^{(i)}_{ri}\right] +$$

$$+ \frac{1}{\bar{\lambda}_{Rr}\bar{\lambda}_{ri}}\,(P_{ri})^{-1}(P_{Rr})^{-1}(\overrightarrow{\underline{\Delta q}}^{(R)}_{ik}) \cdot$$

Here :

$$(-\overrightarrow{q}^{(i)}_{ik}) = \begin{pmatrix} 0 \\ -s_{ik}\cos r_{ik} \sin \zeta_{ik} \\ -s_{ik} \sin r_{ik} \sin \zeta_{ik} \\ s_{ik} \cos \zeta_{ik} \end{pmatrix}$$

$$(q^{(i)}_{ik}) - (\overline{q^{(i)}_{ik}}) = s_{ik} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\cos \zeta_{ik} & 2 \sin r_{ik} \sin \zeta_{ik} \\ 0 & -2 \cos \zeta_{ik} & 0 & -2 \cos r_{ik} \sin \zeta_{ik} \\ 0 & -2 \sin r_{ik} \sin \zeta_{ik} & 2 \cos r_{ik} \sin \zeta_{ik} & 0 \end{pmatrix} \qquad (5.67)$$

(5.61) furnishes :

$$(\overrightarrow{\overset{-1}{p}\underline{\Delta} p})_{Rr} = \underbrace{\frac{1}{S^2_{Rr}} \begin{pmatrix} 0 & 0 & 0 \\ S^2_{Rr} + I^2_{Rr} & S_{Rr}K_{Rr} + I_{Rr}J_{Rr} & -S_{Rr}J_{Rr} + I_{Rr}K_{Rr} \\ -S_{Rr}K_{Rr} + I_{Rr}J_{Rr} & S^2_{Rr} + J^2_{Rr} & S_{Rr}I_{Rr} + J_{Rr}K_{Rr} \\ S_{Rr}J_{Rr} + I_{Rr}K_{Rr} & -S_{Rr}I_{Rr} + J_{Rr}K_{Rr} & S^2_{Rr} + K^2_{Rr} \end{pmatrix}}_{\text{Suppose : } = (T_{Rr}) \cdot} \begin{pmatrix} \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta K}_{Rr} \end{pmatrix} \qquad (5.68)$$

The substitution of (5.67), add (5.68) in the right-hand member of (5.66) now gives the observation equations :

$$\begin{pmatrix} \Delta \underline{\ln s}_{ik} \\ \underline{\Delta r}_{ik} \\ \underline{\Delta z}_{ik} \end{pmatrix} = \begin{pmatrix} \Delta \underline{\ln \bar\lambda}_{Rr} + \Delta \underline{\ln \bar\lambda}_{ri} \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-\cos r \cos z}{\sin z} & \frac{-\sin r \cos z}{\sin z} & -1 \\ 0 & -\sin r & \cos r & 0 \end{pmatrix}_{ik} \left[(P_{ri})^{-1} (T_{Rr}) \begin{pmatrix} \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta K}_{Rr} \end{pmatrix} + \overrightarrow{(\overset{-1}{p}\underline{\Delta} p)}^{(i)}_{ri} \right] +$$

$$+ \frac{1}{\bar\lambda_{Rr} \bar\lambda_{ri} s_{ik}} \begin{pmatrix} 0 & \cos r \sin z & \sin r \sin z & -\cos z \\ 0 & \frac{-\sin r}{\sin z} & \frac{\cos r}{\sin z} & 0 \\ 0 & \cos r \cos z & \sin r \cos z & \sin z \end{pmatrix}_{ik} (P_{ri})^{-1} (P_{Rr})^{-1} \begin{pmatrix} 0 \\ \underline{\Delta X}^R_k - \underline{\Delta X}^R_i \\ \underline{\Delta Y}^R_k - \underline{\Delta Y}^R_i \\ \underline{\Delta Z}^R_k - \underline{\Delta Z}^R_i \end{pmatrix}$$

(5.69)

This is the most general form of observation equations for distance measures, directions and zenith angles.

Furthermore : $\underline{\bar\lambda}_{Rr} \underline{\bar\lambda}_{ri} \underline{s}_{ik}$ is the length of the side $P_i P_k$ in the R-system, to be designated by :

$$\ell_{ik} = \bar\lambda_{Rr} \bar\lambda_{ri} s_{ik} \tag{5.70}$$

We shall now work out (5.69) for the case in which the unit vectors of the (R)-system are parallel to those of the (r)-system, thus for approximate values :

$$\left. \begin{array}{l} S^{\circ}_{Rr} = 1 . \\ I^{\circ}_{Rr} = J^{\circ}_{Rr} = K^{\circ}_{Rr} = 0 . \end{array} \right\} \rightarrow \begin{array}{l} i^R // i^R \\ j^R // j^R \\ k^R // k^R \end{array}$$

$$\bar\lambda^{\circ}_{Rr} = 1 \quad \rightarrow \quad \text{length unit (R) = length unit (r) .}$$

Then (5.68) becomes :

$$\overrightarrow{(\overset{-1}{p}\underline{\Delta} p)}_{Rr} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta k}_{Rr} \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{\Delta I}_{Rr} \\ \underline{\Delta J}_{Rr} \\ \underline{\Delta k}_{Rr} \end{pmatrix} \tag{5.71}$$

and :

$$(P_{Rr}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.71}$$

Assume further :

$$(P_{ri})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & C \\ 0 & D & E & F \\ 0 & G & H & I \end{pmatrix} \tag{5.72}$$

Symbolic notation for (1.72) ;
A, B, C, D, E, F, G, H, I constitute
an orthogonal matrix

According to (3.7), the terms with $(p^{-1}\underline{\Delta}\ p)_{ri}$ can be expressed in differences $\underline{\Delta\theta}_r$, $\underline{\Delta\varphi}_r$, $\underline{\Delta \ln\lambda}_{r\ldots i}$, $\underline{\Delta\varphi}_i$ and $\underline{\Delta\theta}_i$.
Substitution of (5.70), (5.71), (5.72) and (3.7) in (5.69) now gives the observation equations :

$$\begin{aligned} \Delta \underline{\ln s}_{ik} = & - \Delta \underline{\ln \bar{\lambda}}_{Rr} - \Delta \underline{\ln \bar{\lambda}}_{ri} + \\ & + \frac{1}{l_{ik}} \left[\cos r_{ik} \sin z_{ik} A + \sin r_{ik} \sin z_{ik} D - \cos z_{ik} G \right] \underline{\Delta X}_{ik} + \\ & + \frac{1}{l_{ik}} \left[\quad \text{,,} \quad B \quad \text{,,} \quad E \quad \text{,,} \quad H \right] \underline{\Delta Y}_{ik} + \\ & + \frac{1}{l_{ik}} \left[\quad \text{,,} \quad C \quad \text{,,} \quad F \quad \text{,,} \quad I \right] \underline{\Delta Z}_{ik} . \end{aligned} \tag{5.73}$$

$$\begin{aligned} \underline{\Delta r}_{ik} = & \; 2\left[\frac{-\cos r \cos z}{\sin z} A - \frac{\sin r \cos z}{\sin z} D - G \right] \underline{\Delta I}_{Rr} + \\ & + 2\left[\quad \text{,,} \quad B - \quad \text{,,} \quad E - H \right] \underline{\Delta J}_{Rr} + \\ & + 2\left[\quad \text{,,} \quad C - \quad \text{,,} \quad F - I \right] \underline{\Delta K}_{Rr} + \\ & + \left[-\cos A_{ik} \frac{\cos z}{\sin z} (\sin\varphi_r \cos\varphi_i - \cos\lambda_{ri} \cos\varphi_r \sin\varphi_i) + \sin A_{ik} \frac{\cos z}{\sin z} \sin\lambda_{ri} \cos\varphi_r + \right. \\ & \qquad \left. + \cos\lambda_{ri} \cos\varphi_r \cos\varphi_i + \sin\varphi_r \sin\varphi_i \right] \underline{\Delta\theta}_r + \\ & + \left[-\sin A_{ik} \frac{\cos z}{\sin z} \cos\lambda_{ri} + \cos A_{ik} \frac{\cos z}{\sin z} \sin\lambda_{ri} \sin\varphi_i + \sin\lambda_{ri} \cos\varphi_i \right] \underline{\Delta\varphi}_r + \\ & + \left[-\cos A_{ik} \frac{\cos z}{\sin z} \cos\varphi_i + \sin\varphi_i \right] \underline{\Delta\lambda}_{r\ldots i} + \sin A_{ik} \frac{\cos z}{\sin z} \underline{\Delta\varphi}_i - \underline{\Delta\theta}_i + \\ & + \frac{1}{l_{ik}} \left[\frac{-\sin r}{\sin z} A + \frac{\cos r}{\sin z} D \right] \underline{\Delta X}_{ik} + \frac{1}{l_{ik}} \left[\frac{-\sin r}{\sin z} B + \frac{\cos r}{\sin z} E \right] \underline{\Delta Y}_{ik} + \\ & \qquad + \frac{1}{l_{ik}} \left[\frac{-\sin r}{\sin z} C + \frac{\cos r}{\sin z} F \right] \underline{\Delta Z}_{ik} . \end{aligned} \tag{5.74}$$

$$\begin{aligned}\underline{\Delta\zeta}_{ik} = {} & 2\left[-\sin r\,A+\cos r\,D\right]\underline{\Delta I}_{Rr}+2\left[-\sin r\,B+\cos r\,E\right]\underline{\Delta J}_{Rr}+2\left[-\sin r\,C+\cos r\,F\right]\underline{\Delta k}_{Rr}\\ & +\left[-\sin A_{ik}\left(\sin\varphi_r\cos\varphi_l-\cos\lambda_{ri}\cos\varphi_r\sin\varphi_i\right)-\cos A_{ik}\sin\lambda_{ri}\cos\varphi_r\right]\underline{\Delta\theta}_r +\\ & +\left[\cos A_{ik}\cos\lambda_{ri}+\sin A_{ik}\sin\lambda_{ri}\sin\varphi_i\right]\underline{\Delta\varphi}_r +\\ & -\sin A_{ik}\cos\varphi_i\,.\,\underline{\Delta\lambda}_{ri}-\cos A_{ik}\,\underline{\Delta\varphi}_i +\\ & +\frac{1}{l_{ik}}\left[\cos r\cos\zeta\,A+\sin r\cos\zeta\,D+\sin\zeta\,G\right]\underline{\Delta X}_{ik} +\\ & +\frac{1}{l_{ik}}\left[\quad ''\quad B+\quad ''\quad E\quad ''\quad H\right]\underline{\Delta Y}_{ik} +\\ & +\frac{1}{l_{ik}}\left[\quad ''\quad C\quad ''\quad F\quad ''\quad I\right]\underline{\Delta Z}_{ik}\,.\end{aligned}\tag{5.75}$$

Observation equations for astronomical observation variates

Astronomical latitudes occur both as observation variate and as unknown. The observation equations are :

$$\boxed{\underline{\Delta\varphi}_i+\underline{\varepsilon}=\underline{\Delta\varphi}_i^{\alpha}}\tag{5.76}$$

Of all astronomical longitudes (unknowns) one may be chosen, λ_0, because longitudes only occur as difference quantities. The observation equations are :

$$\boxed{\begin{array}{lll}\underline{\Delta\lambda}_{ik}+\underline{\varepsilon}=\underline{\Delta\lambda}_k-\underline{\Delta\lambda}_i & & \\ \text{or}: & =\underline{\Delta\lambda}_k & (\text{if } \lambda_0=\lambda_i)\,. \\ \text{or}: & =-\underline{\Delta\lambda}_i & (\text{if } \lambda_0=\lambda_k)\,.\end{array}}\tag{5.77}$$

An observation equation for the azimuth is obtained from (4.8) :

$$\underline{\Delta A}_{ab}=\underline{\Delta r}_{ab}+\underline{\Delta\theta}_a$$

We now substitute the observation equation for $\underline{\Delta r}_{ab}$, see (5.74) :

$$\underline{\Delta r}_{ab}=\cdots\cdots-\underline{\Delta\theta}_a+\cdots\cdots\cdots\cdots$$

hence :

$$\boxed{\underline{\Delta A}_{ab}=\left[\text{the observation equation of } \underline{\Delta r}_{ab}\text{, the term } -\underline{\Delta\theta}_a \text{ excluded}\right]}\tag{5.78}$$

5.6.3
Networks with parallel k-vectors.

Subsequehtly we work out (5.69) for a network with parallel k-unit vectors, as described in section 4.2.3. This means :

$$p_{ri} = \cos\tfrac{1}{2}[\theta_i-\theta_r] + k\sin\tfrac{1}{2}[\theta_i-\theta_r]\ .$$

then :

$$(P_{ri})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[\theta_i-\theta_r] & \sin[\theta_i-\theta_r] & 0 \\ 0 & -\sin[\theta_i-\theta_r] & \cos[\theta_i-\theta_r] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus in the formulae (5.72) etc.

$A = \cos[\theta_i-\theta_r]\ ;\quad E = A$

$B = \sin[\theta_i-\theta_r]\ ;\quad D = -B$

$C = F = G = H = 0$

$L = 1.$

Furthermore ; see fig 44 :

$$\cos r\ A - \sin r\ B = \cos[r_{ik}+\theta_i-\theta_r] =$$

$$= \cos[A_{ik}-\theta_r] =$$

$(5.71):\ \theta_R = \theta_r\ \cdots\cdots$

$$= \cos[A_{ik}-\theta_R] = \cos\psi_{ik}$$

and : $\cos r\ B - \sin r\ A = \sin\psi_{ik}\ .$

in which ψ_{ik} is the "argument" relative to the i-unit vector of the (R)-system.
With this, the observation equations become :

$$\Delta\underline{\ln s}_{ik} = -\Delta\underline{\ln\bar{\lambda}}_{Rr} - \Delta\underline{\ln\bar{\lambda}}_{ri} + \frac{\sin z_{ik}}{l_{ik}}\cos\psi_{ik}\,\underline{\Delta X}_{ik} + \frac{\sin z_{ik}}{l_{ik}}\sin\psi_{ik}\,\underline{\Delta Y}_{ik} + \\ -\frac{\cos z_{ik}}{l_{ik}}\,\underline{\Delta Z}_{ik}. \qquad (5.79)$$

$$\underline{\Delta r}_{ik} = -2\frac{\cos z_{ik}}{\sin z_{ik}}\cos\psi_{ik}\,\underline{\Delta I}_{Rr} - 2\frac{\cos z_{ik}}{\sin z_{ik}}\sin\psi_{ik}\,\underline{\Delta J}_{Rr} - 2\,\underline{\Delta k}_{Rr} + \\ + \underline{\Delta\theta}_r - \underline{\Delta\theta}_i + \sin A_{ik}\frac{\cos z_{ik}}{\sin z_{ik}}[\underline{\Delta\varphi}_i - \underline{\Delta\varphi}_r] + \\ + \left[-\cos A_{ik}\frac{\cos z_{ik}}{\sin z_{ik}}\cos\varphi_i + \sin\varphi_i\right]\underline{\Delta\lambda}_{r\ldots i} + \\ -\frac{\sin\psi_{ik}}{l_{ik}\sin z_{ik}}\,\underline{\Delta X}_{ik} + \frac{\cos\psi_{ik}}{l_{ik}\sin z_{ik}}\,\underline{\Delta Y}_{ik}\ . \qquad (5.80)$$

$$\begin{aligned} \underline{\Delta J}_{ik} = & -2 \sin \psi_{ik} \; \underline{\Delta I}_{Rr} + 2 \cos \psi_{ik} \; \underline{\Delta J}_{Rr} + \\ & - \cos A_{ik} \left[\underline{\Delta \varphi}_i - \underline{\Delta \varphi}_r \right] - \sin A_{ik} \cos \varphi_i \; \underline{\Delta \lambda}_{ri} + \\ & + \frac{1}{l_{ik}} \cos J_{ik} \cos \psi_{ik} \; \underline{\Delta X}_{ik} + \frac{1}{l_{ik}} \cos J_{ik} \sin \psi_{ik} \; \underline{\Delta Y}_{ik} + \frac{1}{l_{ik}} \sin J_{ik} \; \underline{\Delta Z}_{ik} \,. \end{aligned} \qquad (5.81)$$

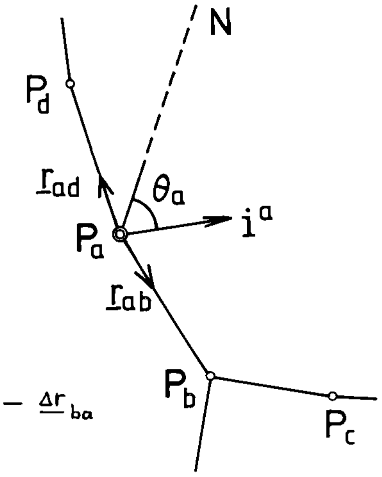
Remark :
In these observation equations all k-vectors are parallel !

Appendix

1. Coefficients of $\underline{\Delta\theta}_a$ and $\underline{\Delta r}_{a..}$ in orientations $\underline{\Delta\theta}_i$.

The first orientation is :

$$\underline{\theta}^{o} = \underline{\theta}_{\alpha} .$$



Other orientations are computed via the sides of the network, as a function of observation variates ; see (3.30) :

$$\underline{\theta}_b = \arctan \frac{Vj\{\underline{p}_{b'a}\, \underline{q}_{ab}^{(b')}\, \underline{p}_{b'a}^{-1}\}}{Vi\{\underline{p}_{b'a}\, \underline{q}_{ab}^{(b')}\, \underline{p}_{b'a}^{-1}\}} - \underline{r} \qquad (+\pi) .$$

See (3.40) :

$$\underline{\Delta\theta}_b = \left[\frac{-\cos r}{\sin\zeta\cos\zeta}\right]_{ab}^{b'} Vi\left\{(q^{-1}\underline{\Delta}q)_{ab}^{(b')}\right\} - \left[\frac{\sin r}{\sin\zeta\cos\zeta}\right]_{ab}^{b'} Vj\left\{(q^{-1}\underline{\Delta}q)_{ab}^{(b')}\right\} - \underline{\Delta r}_{ba}$$

in which, see (3.11) :

$$V\{(q^{-1}\underline{\Delta}q)_{ab}^{(b')}\} = q_{ab}^{-1\,(b')}\, (p^{-1}\underline{\Delta}p)_{b'a}^{(b')}\, q_{ab}^{(b')} - (p^{-1}\underline{\Delta}p)_{b'a}^{(b')} + p_{b'a}\, (q^{-1}\underline{\Delta}q)_{ab}^{(a)}\, p_{b'a}^{-1}$$

$$\underline{\Delta\theta}_a \;\cdots \text{equal coefficients} \cdots\cdots\; \underline{\Delta r}_{ab}$$

In (3.15) it is shown, that in this formula the coefficients of $\underline{\Delta\theta}_a$ and $\underline{\Delta r}_{ab}$ are equal. This means, that also in $\underline{\Delta\theta}_b$ these difference quantities have equal coefficients ; suppose :

$$\underline{\Delta\theta}_b = t_{ba}\left[\underline{\Delta\theta}_a + \underline{\Delta r}_{ab}\right] + t'_{ba}\, \underline{\Delta\varphi}_a + t''_{ba}\, \underline{\Delta\lambda}_{ab} + t'''_{ba}\, \underline{\Delta\varphi}_b + t''''_{ba}\, \underline{\Delta\zeta}_{ab} - \underline{\Delta r}_{ba}$$

Via the next networkside, P_bP_c, we obtain in a similar way :

$$\underline{\Delta\theta}_c = t_{cb}\left[\underline{\Delta\theta}_b + \underline{\Delta r}_{bc}\right] + t'_{cb}\, \underline{\Delta\varphi}_b + \cdots\cdots\text{other terms}\cdots\cdots$$

$$= t_{cb}\, t_{ba}\left[\underline{\Delta\theta}_a + \underline{\Delta r}_{ab}\right] + t_{cb}\, \underline{\Delta r}_{bc} + \cdots\text{other terms}\cdots$$

and via P_aP_d :

$$\underline{\Delta\theta}_d = t_{da}\left[\underline{\Delta\theta}_a + \underline{\Delta r}_{ad}\right] + \cdots\text{other terms}\cdots$$

2. Coefficients of $\underline{\Delta\theta}_a$ and $\underline{\Delta r}_{a.}$ in zero-mean variate $\underline{V}_{ik}^{(r)}$.

We consider the zero-mean variate :

$$\underline{V}_{ik}^{(r)} = Vc\{(q^{-1}\underline{\Delta}q)_{ik}^{(r)} - (q^{-1}\underline{\Delta}q)_{ki}^{(r)}\}$$

There will occur three orientations in $\underline{V}_{ik}^{(r)}$:

$$\underline{\Delta\theta}_r \;,\; \underline{\Delta\theta}_i \;,\; \underline{\Delta\theta}_k \;.$$

Suppose : (symbolic notation)

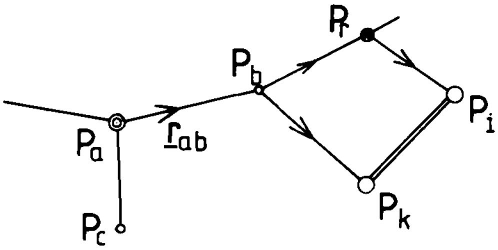
$$(q^{-1}\underline{\Delta} q)^{(r)}_{ik} = \Delta \underline{\ln \bar{\lambda}}_{ri} + \Delta \underline{\ln s}_{ik} + v^{1}_{ik}\, \underline{\Delta\theta}_{r} + v^{2}_{ik}\, \underline{\Delta\varphi}_{r} + v^{3}_{ik}\, \underline{\Delta\lambda}_{r\ldots i} +$$

$$+ v^{4}_{ik}\, \underline{\Delta\varphi}_{i} + v^{5}_{ik}\left[\underline{\Delta\theta}_{i} + \underline{\Delta r}_{ik}\right] + v^{6}_{ik}\, \underline{\Delta \mathfrak{J}}_{ik} \cdot$$

There has to be made a distinction between several different situations :

I P_i is not P_a and P_k is not P_a ;

θ_r , θ_i and θ_k via r_{ab}



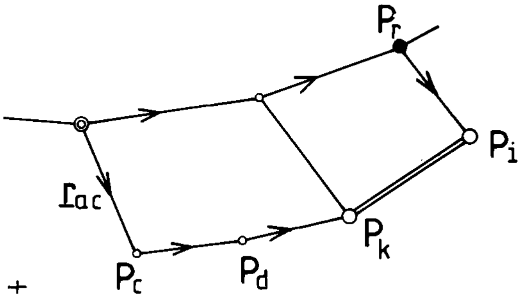
Now $\underline{V}^{(r)}_{ik}$ becomes :

$$\underline{V}^{(r)}_{ik} = v^{1}_{ik}\, \underline{\Delta\theta}_{r} + v^{5}_{ik}\, \underline{\Delta\theta}_{i} + v^{1}_{ki}\, \underline{\Delta\theta}_{r} + v^{5}_{ki}\, \underline{\Delta\theta}_{k} + \ldots \text{other terms} \ldots\ldots =$$

$$= \left[v^{1}_{ik} + v^{1}_{ki}\right] t_{rb}\, t_{ba}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] + v^{5}_{ik}\, t_{ir}\, t_{rb}\, t_{ba}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] +$$

$$+ v^{5}_{ki}\, t_{kb}\, t_{ba}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] + \ldots\ldots$$

II Idem, but : θ_k via P_c, P_d .

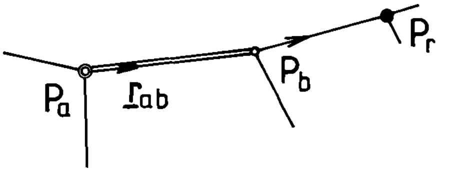


Now $\underline{V}^{(r)}_{ik}$ becomes :

$$\underline{V}^{(r)}_{ik} = \left[(v^{1}_{ik} + v^{1}_{ki})\, t_{rb}\, t_{ba} + v^{5}_{ik}\, t_{ir}\, t_{rb}\, t_{ba}\right]\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] +$$

$$+ v^{5}_{ki}\, t_{kd}\, t_{dc}\, t_{ca}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ac}\right] + \ldots \text{other terms} \ldots$$

III $P_i \equiv P_a$; $P_k \equiv P_b$

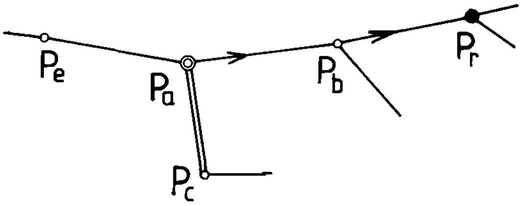


Now $\underline{V}^{(r)}_{ik} \equiv \underline{V}^{(r)}_{ab}$ becomes :

$$\underline{V}^{(r)}_{ab} = v^{1}_{ab}\, \underline{\Delta\theta}_{r} + v^{5}_{ab}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] + v^{1}_{ba}\, \underline{\Delta\theta}_{r} + v^{5}_{ba}\, \underline{\Delta\theta}_{b} + \ldots\ldots =$$

$$= \left[(v^{1}_{ab} + v^{1}_{ba})\, t_{rb}\, t_{ba} + v^{5}_{ab} + v^{5}_{ba}\, t_{ba}\right]\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] + \ldots \text{other terms} \ldots$$

IV $P_i \equiv P_a$; $P_k \equiv P_c$



Now $\underline{V}^{(r)}_{ik} \equiv \underline{V}^{(r)}_{ac}$ becomes :

$$\underline{V}^{(r)}_{ac} = v^{1}_{ac}\, \underline{\Delta\theta}_{r} + v^{5}_{ac}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ac}\right] + v^{1}_{ca}\, \underline{\Delta\theta}_{r} + v^{5}_{ca}\, \underline{\Delta\theta}_{c} + \ldots\ldots =$$

$$= \left[v^{1}_{ac} + v^{1}_{ca}\right] t_{rb}\, t_{ba}\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ab}\right] + \left[v^{5}_{ac} + v^{5}_{ca}\, t_{ca}\right]\left[\underline{\Delta\theta}_{a} + \underline{\Delta r}_{ac}\right] + \ldots\ldots$$

Conclusion :

In all different positions of the zero-mean variate $\underline{V}_{ik}^{(r)}$ with respect to P_a and P_r, the coefficient of $\underline{\Delta\theta}_a$ is equal to the sum of the coefficients of all directions in P_a, occurring in $\underline{V}_{ik}^{(r)}$:

$$\underline{V}_{ik}^{(r)} = u_b\,\underline{\Delta r}_{ab} + 0\,\underline{\Delta r}_{ac} + 0\,\underline{\Delta r}_{ae} + u_b\,\underline{\Delta\theta}_a + \ldots\ldots \text{ other terms } \ldots\ldots$$

or :

$$= u_b\,\underline{\Delta r}_{ab} + u_c\,\underline{\Delta r}_{ac} + 0\,\underline{\Delta r}_{ae} + \left[u_b+u_c\right]\underline{\Delta\theta}_a + \ldots\ldots \text{ other terms } \ldots\ldots$$

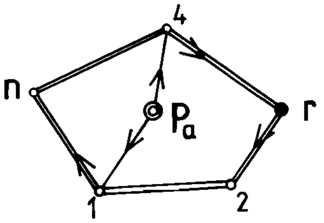
or :

$$= u_b\,\underline{\Delta r}_{ab} + u_c\,\underline{\Delta r}_{ac} + u_e\,\underline{\Delta r}_{ae} + \left[u_b+u_c+u_e\right]\underline{\Delta\theta}_a + \ldots \text{ other terms } \ldots$$

3. The coefficients of $\underline{\Delta\theta}_a$ and $\underline{\Delta r}_{a..}$ in a closed polygon.

We now consider the closed polygon $P_1\ P_2\ P_3\ P_4\ P_n$, with the zero-mean variate :

$$\underline{W}_{1\ldots n}^{(r)} = \underline{\Delta q}_{12}^{(r)} + \underline{\Delta q}_{23}^{(r)} + \underline{\Delta q}_{34}^{(r)} + \underline{\Delta q}_{4n}^{(r)} + \underline{\Delta q}_{n1}^{(r)} .$$



Introduction of a symbolic notation for $\underline{\Delta q}_{ik}^{(r)}$:

$$\underline{\Delta q}_{ik}^{(r)} = q_{ik}^{(r)}\left[\underline{\Delta \ln \bar{\lambda}}_{ri} + \underline{\Delta \ln s}_{ik}\right] + w_{ik}^{1}\,\underline{\Delta\theta}_r + w_{ik}^{2}\,\underline{\Delta\varphi}_r + w_{ik}^{3}\,\underline{\Delta\lambda}_{r\ldots i} +$$

$$+ w_{ik}^{4}\,\underline{\Delta\varphi}_i + w_{ik}^{5}\left[\underline{\Delta\theta}_i + \underline{\Delta r}_{ik}\right] + w_{ik}^{6}\,\underline{\Delta J}_{ik} .$$

In the network shown in figure, with $P_r \equiv P_3$, the coefficients of $\underline{\Delta\theta}_a$ and $\underline{\Delta r}_{a1}$, $\underline{\Delta r}_{a4}$ become :

$$\underline{W}_{1\ldots n}^{(r)} = \left[\begin{array}{l} w_{12}^{1}\,t_{34}\,t_{4a} + \\ + w_{23}^{1}\,t_{34}\,t_{4a} + w_{23}^{5}\,t_{23}\,t_{34}\,t_{4a} + \\ + w_{34}^{1}\,t_{34}\,t_{4a} + w_{34}^{5}\,t_{34}\,t_{4a} + \\ + w_{4n}^{1}\,t_{34}\,t_{4a} + w_{4n}^{5}\,t_{4a} + \\ + w_{n1}^{1}\,t_{34}\,t_{4a} \end{array}\right]\left[\underline{\Delta\theta}_a + \underline{\Delta r}_{a4}\right] + \left[\begin{array}{l} w_{12}^{5}\,t_{1a} + \\ \\ \\ \\ + w_{n1}^{5}\,t_{n1}\,t_{1a} \end{array}\right]\left[\underline{\Delta\theta}_a + \underline{\Delta r}_{a1}\right] + \ldots\ldots$$

For this zero-mean variate the same conclusion may be drawn :

$$\underline{W}_{1\ldots n}^{(r)} = u_1\,\underline{\Delta r}_{a1} + u_4\,\underline{\Delta r}_{a4} + \left[u_1+u_4\right]\underline{\Delta\theta}_a + \ldots \text{ other terms } \ldots\ldots$$

4. The condition equations.

We will now prove, that all condition equations mentioned in chapter 5 are composed from zero-mean variates $\underline{V}_{ik}^{(r)}$ and $\underline{W}_{1\ldots n}^{(r)}$:

Polygon condition :

From (5.25) it becomes apparent that, by splitting up the $\underline{\Delta\Pi}$-quantities, the polygon condition reads :

$$\underline{\Delta V}^{(r)}_{1\cdots n} = q_{n1} \sum \left[(q^{-1}\underline{\Delta q})^{(r)}_{ik} - (q^{-1}\underline{\Delta q})^{(r)}_{ki} \right] =$$

$$= q_{n1} \sum \underline{V}^{(r)}_{ik} \, .$$

Network condition :

From (5.36) it becomes apparent that this also applies to the so-called "b-terms" of the coordinate condition ; in the "a-terms", however, the terms $\underline{\Delta q}_{n-1,n}$ must be replaced by $-\underline{\Delta q}_{n,n-1}$ in order to reach similarity between the a-terms and $\underline{W}_{1\ldots n}$;
the a-terms of (5.36) :

$$- \underline{\Delta q}_{1n} - \underline{\Delta q}_{21} - \cdots - \underline{\Delta q}_{n-1,n-2} + \underline{\Delta q}_{n-1,n} =$$

$$= - \sum_{\text{polygon}} \underline{\Delta q}_{ik} + \underline{\Delta q}_{n-1,n} + \underline{\Delta q}_{n,n-1} =$$

$$= - \underline{W}^{(r)}_{1\cdots n} + q_{n-1,n} \left[(q^{-1}\underline{\Delta q})_{n-1,n} - (q^{-1}\underline{\Delta q})_{n,n-1} \right] =$$

$$= - \underline{W}^{(r)}_{1\cdots n} + q_{n-1,n} \, \underline{V}^{(r)}_{n-1,n}$$

therefore also the a-terms of the coordinate condition are composed of zero quantities of the types $\underline{V}$ and $\underline{W}$.

Z-condition :

(5.26) :

$$\underline{\Delta Z}^{(r)}_{ik} = Vk \left\{ \underline{\Delta q}^{(r)}_{ik} + \underline{\Delta q}^{(r)}_{ki} \right\} =$$

$$= Vk \left\{ q_{ik} \left[(q^{-1}\underline{\Delta q})^{(r)}_{ik} - (q^{-1}\underline{\Delta q})^{(r)}_{ki} \right] \right\} =$$

$$= Vk \left\{ q_{ik} \, \underline{V}^{(r)}_{ik} \right\} .$$

A-condition :

From (5.39) it becomes apparent that the differences $\underline{\Delta\theta}_a$, $\underline{\Delta r}_{ai}$ do occur in the A-condition via :

1- the $\underline{\Delta Z}_{ik}$ — quantities ; these are of the type $\underline{V}^{r}_{ik}$.

2- the orientation $\underline{\Delta\theta}_i$; the latter is :

$$\underline{\Delta\theta}_i = t_{i\ldots a} \left[\underline{\Delta\theta}_a + \underline{\Delta r}_{a.} \right] + \ldots\ldots \text{other terms} \ldots\ldots$$

R-condition :

In the R-condition neither directions in P_a, nor $\underline{\Delta\theta}_a$ occur, because (5.27') fulfills 3.1.2.1.

Conclusion :

All types of condition equations are composed of zero-mean variates of the types $\underline{V}_{ik}$ and $\underline{W}_{1.....n}$; therefore the coefficient of $\underline{\Delta\theta}_a$ equals the sum of the coefficients of all directions in P_a , occurring in that condition equation :

$$\boxed{\begin{aligned} \underline{\Delta y}^p &= u_1^p \, \underline{\Delta r}_{a1} + u_2^p \, \underline{\Delta r}_{a2} + \cdots + u_n^p \, \underline{\Delta r}_{an} + \left[u_1^p + u_2^p + \dots + u_n^p \right] \underline{\Delta\theta}_a + \\ &\qquad + \dots \text{all other terms.} \end{aligned}}$$

Remark : $u_1^p \dots u_n^p$ can be $= 0$, but at least one of them $\neq 0$ in every condition equation N, V, Z and A .

1	Alberda, J.E.	Vertical angles; Deviations of the vertical and adjustment. Netherlands Geodetic Commission Vol.1, nr. 1. 1961
2	Baarda, W.	Polygontheory in the complex plane. Lecture notes of the geodetic institute TH-Delft 1966-1969
3	Baarda, W.	S-transformations and criterionmatrices. Netherlands Geodetic Commission Vol. 5, nr. 1. 1973
4	Baarda, W.	Vereffeningstheorie
5	Baarda, W.	Statistical concepts in geodesy. Netherlands Geodetic Commission Vol. 2, nr. 4. 1967
6	Baarda, W.	A testing procedure for use in geodetic networks. Netherlands Geodetic Commission Vol. 2, nr. 5. 1968
7	Baarda, W.	A personal report on activities in special study group No. 1.14 - Travaux de l'AIG, 24 Paris
8	Baarda, W.	Manuscripts 1960 - 1962
9	Baarda, W.	De levende aarde (diesrede THD 9/1/'76) NGT 1976/2 Mathematical Models (OEEPE publ. off. No. 1) Mathematical Geodesy in relation to the Netherlands Geodetic Commission (Delft, 1979)
10	Bahnert, G.	Refraktionseinflüsse auf trigonometrisches Nivellement (Vermessungstechnik 29/3)
11	Boerjan, R.R.	Voordracht N.G.L.-Congres - NGT/Geodesia 1981/12
12	Elmiger, A. and Wunderlin, N.	Dreidimensionale Berechnung von geodätischen Netzen (FIG-Congres 1981)
13	Glissmann, T. and Williams, D.	Ein koinzidenzverfahren zur Messung von refraktionsfreien Richtungen; ZfV 103/5
14	Brand, L.	Vector and Tensor Analysis. New York 1947
15	Hotine, M.	Mathematical Geodesy. Essa Monograph 2, Washington D.C. 1969
16	Meerdink, E.F.	Het mathematische model van triangulaties op bol en ellipsoide. T.H. Delft 1961
17	Molenaar, M.	A further inquiry into the theory of S-transformations and criterion matrices. Delft 1981
18	Quee, H.	Afstudeerscriptie 1971

19 Quee, H. On the stochasticity of alignment functions for the guidance of automated track maintenance machines. Proceedings of the FIG-Commission 5 symposium, Aalborg 1982

20 Quee, H. Een kriteriumtheorie voor boogkorrekties. NGT 1978/4/5

21 Ramsayer, K. Several publications on the testnet Stuttgart ZfV 104/1 (1979); ZfV 105/10 (1980)

22 Torge, W. and Wenzel, H.G. Dreidimensionale Ausgleichung des Testnetzes Westharz. DGK Reihe B Heft 32, 1978

23 Vermaat, E. Driehoek en viervlak; afstudeerscriptie 1970

24 Grafarend, E.W. and Schaffrin, B. Vectors, quaternions and spinors - a discussion of Algebras underlying threedimensional Geodesy. Forty years of thought I, Delft 1982