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S-TRANSFORMATIONS AND CRITERION MATRICES

by

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1 AN OUTLINE OF THE IDEAS AND THEIR ORIGIN

An answer to problematical points stated earlier

In the publication "A Testing Procedure for Use in Geodetic Networks" [9], a sketch was given for a possible programme of subjects to be studied in Special Study Group No. 1.14*) of the International Geodetic Association. This was an extension of first ideas expressed in 1962 [4a], with a supplement in [4]. Three main problems were formulated:

(1.1) the construction of an artifical covariance matrix, to serve as a mathematical translation of the lower limit of precision required by the purposes of geodetic networks
(1.2) the use of statistical tests in connection with the adjustment of networks in order to assure the reliability of geodetic networks in relation to their purpose in society
(1.3) the consequences for geodetic networks of a relativation of the concepts north and time.

In [9] {see also [10] for basic theory}, a first solution for (1.2) was given, which since then has been applied to the computation of many densification networks in The Netherlands. The results were very satisfactory, but the method has so far not led to reactions from the members of S.S.G. No. 1.14, and has received little attention from geodesists outside The Netherlands.

The present paper presents a solution for (1.1), a consistent theory which replaces the first draft discussed in November 1964 at Stockholm with the then President of S.S.G. No. 1.14, Professor L. ASPLUND. The theory has been developed since 1962 by the author and J. E. ALBERDA, along seemingly very different lines. Many discussions have first led to an improvement of individual ideas, and then, when in 1969 the present paper was drafted, to the insight that the two theories must be identical in essence. A separate publication by ALBERDA is to be expected; perhaps the differences in argumentation will lead to a clarification of the rather difficult line of thought.

In both investigations, use is made of the theory of the so-called S-transformations, {Dutch: Schrankingstransformaties; English: transformations related to Similarity transformations}, formulated by BAARDA around 1944, and applied in the HTW-1956 [1], published in [7], [3] and [5] in the form of "Similarity covariance transformations". The present theory was, however, only made possible by a more consistent approach of the S-transformations based on the theory of difference equations for the linearization of functional relationships in adjustment theory. This theory was developed in [1]; a first treatment can be found in [4] section 3.4.

Both researchers also used the criteria for the form and the size of standard ellipses of

*) S.S.G. No. 1.14: Specifications for Fundamental Networks in Geometric Geodesy.
[1] See the notes at the end of this section.

coordinates of points in a network, which were more or less sketchily formulated by BAARDA in the HTW-1956. ALBERDA chose the way of a further consistent elaboration of this formulation, whereas BAARDA in the first instance rejected the approach of the HTW-1956, because it was in conflict with studies on model theory, such as given in [4], section 4, and in the "Polygon Theory in the Complex Plane" [2].

On the origin of the theory of criterion matrices

It will be clear that the roots of this development lie in the past; in particular it is connected to the work of J. M. TIENSTRA. This concerns not only his calculus of observations,[2] from which eventually [1] was developed; more relevant is his introduction of complex numbers [3] in plane surveying, including already the introduction of distance ratios, along with angles, as computational quantities. He also gave a first formulation of the maximum admissible size of standard ellipses of coordinates in the HTW-1938.[4] A standard ellipse was required to lie within a circle of radius *d* cm, *d* being chosen from three possible values according to the economic value of parcels of land. In the norm of circular standard ellipses, one recognizes ideas from the surveying literature published in the German language during the Thirties, and no doubt TIENSTRA was influenced by these papers.

The formulae for the reconnaissance of a geodetic network, as given in the HTW-1938, were tested by BAARDA for practical usefulness around 1940; a certain degree of inconsistency was then apparent.

Around 1944, this led to an analysis of the basic line of thought, and to the idea from which eventually the model theory in section 4 of [4] was developed. The starting point was the introduction of distance ratios and angles as fundamental quantities, whereas scale and orientation were considered as artificial and non-essential elements of the theory. This implied that coordinates, too, had to be considered as artificial and non-essential elements. Consequently the concept of coordinate had to receive its proper place: coordinates were only to be defined in an operational sense, as functions of distance ratios and angles. Thus, in 1944, a first intuitive version of the theory of S-transformations was established, which in a later algebraic formulation made possible the definition of coordinates in a sharply indicated system. It could hardly be realized at the time that only after twenty-five years a first well-rounded version of the theory could be established!

Around 1950, TIENSTRA was requested to take part in a revision of the HTW-1938. His all too early death in 1951 implied that this task had to be taken over by BAARDA, and in a premature stage of the theory, a formulation had to be given which would meet the requirements of surveying practice. Once more, many ideas could be adopted from publications in the German language, of which only the greatly appreciated work of Dr. E. PINKWART is mentioned here. Hence, the HTW-1956 was a mixture of Tienstra's ideas with the theory of S-transformations. The construction of criterion circles – limiting the size of standard ellipses – was now based on a much more complicated theory, and their radius *d* was made dependent on the mutual distance of points. Although TIENSTRA was partially followed in the introduction of different functions for the determination of *d* in areas of different land value, the further adaption of precision criteria was to be found in the introduction of standard deviations Δd, characterizing the process of the description of a terrain "point" by a mathematical point.

Although this offered satisfactory possibilities for the analysis of many surveying problems, the system of reconnaissance formulae for networks in the HTW-1956 proved not to be entirely consistent.

Around 1958, the questions not answered by the designed theory, led to an entirely modified geometrical theory based on complex numbers, and aimed, among other things, at the explanation of a number of differences between the computing models of classical triangulation networks and the trilateration networks which were more and more applied after 1945. This theory had its roots in the ideas developed in 1944, and within some years it grew into the "Polygon Theory in the Complex Plane", see [7], [2], [4].

Looking back over the twenty-five years needed to establish the theory, one is surprised at the long and winding path that had to be followed before an essentially very simple line of thought could be formulated, that forms a suitable start for the solution of virtually all plane surveying problems. It is even more surprising that on the basis of these views a much better insight into the essence of distance- and direction measurement is obtained, even so much better that better directives for the measuring process could be given. As had been expected, a much better model was obtained for the foundation of computer programmes, although as a consequence the adjustment theory had to be built up in a more consistent way. Curiously, this also opened the possibility to give the theory of S-transformations its present form, with as a forerunner one of the examples treated in section 17 of [1].

In the present paper, a new form of the precision criteria for geodetic networks is developed from the theory of S-transformations. This replaces the criteria chosen in the HTW-1956, like they in their turn replaced the criteria of the HTW-1938. Now, however, the criteria are not formulated for a single standard ellipse, but for the total covariance matrix pertaining to the coordinate variates of the points of the network. Therefore a criterion matrix is introduced, enabling the formulation of a criterion for the whole consistent with criteria for partial problems. It is curious that the construction of the criterion matrix in many respects bears resemblance to the construction of an artificial positive definite matrix in the studies in potential theory by KRARUP;[5] both matrices must obey the fundamental properties of the model theory to which they belong.

In this paper, the construction of a criterion matrix for two-dimensional coordinate variates in the plane is preceded by the construction of a criterion matrix for one-dimensional coordinate variates. As an example of the latter, height variates from levelling have been chosen, presupposing that a transformation has been executed so that a strictly additive computing model is valid for "height differences" made dimensionless. One can then also develop S-transformations, an idea stemming from 1959.

Now one may wonder what is the essence of the one- and two-dimensional theories considered. In the former, the essential variates are differences, and they are therefore invariant with respect to a translation. In the latter, the essential variates are Π-variates or functions of Π-variates, and therefore they are invariant with respect to a similarity transformation. This formulation is based on Klein's[6] statement that a geometrical model theory can be characterized by its invariance with respect to certain transformations. It is true that this view originally led to the generalization of the two-dimensional theory {complex numbers} to the three-dimensional theory using quaternions, but on closer investigation, this characterization of a model theory does not give sufficient insight into the essence of such a theory.

Therefore it will be tried to give a more satisfactory characterization of the whole of the geodetic theories designed; some unpublished parts will be taken into consideration as well. Clemency is asked for the primitive formulation by a geodesist who is not a mathematician.

Some algebraic considerations[7]

The above mentioned dimensionless additive "height differences" actually offer a less attractive starting point as an example of one-dimensional quantities. Firstly, because the definition of such a quantity follows from a more comprehensive theory {the rewriting of Green's theorems of potential theory, leading to a transformed presentation of Molodenski's integral equations} and, secondly, because the additive computing model makes only limited use of the properties of the system of real numbers, to be denoted by R.

The algebraic structure of R is indicated as a field, a field has very strong algebraic properties such as the distributive property and the commutative and associative properties of addition and multiplication. For further considerations, it is particularly important that R with every element a contains an element a^{-1} for which $a^{-1} \cdot a = 1$; because of this, R is a commutative division algebra. This implies that $ab = 0$ only if $a = 0$, or $b = 0$, or $a = b = 0$; hence R does not contain divisors of zero. Finally, it is remarked that R is an ordered field.

The field R of real numbers is often chosen as the field over which an n-dimensional vector space with n basis elements or -vectors is constructed.

If also multiplication of the elements of the vector space is defined such that the distributive and associative properties hold, one obtains associative linear algebras over R, also called systems of hypercomplex numbers. The algebras may be division algebras, but often this will *not* be the case.

Now two-dimensional quantities are considered, in which real coordinate quantities x and y are compounded into the complex coordinate quantity z. If now one wishes to explore the essential properties of the functional computing model to be used, the measuring process must be the starting point, because the model of functional relationship is linked to this process. This implies that the model theory of section 4 in [4] points the way here.

The essential quantity in the linking-up procedure is then the Π-quantity defined as a function of coordinate quantities:

$$e^{\Pi_{jik}} = \frac{z_{ik}}{z_{ij}} \quad \ldots \quad (1.4)$$

In the further elaboration of the theory as "Polygon theory in the complex plane", and in the theory of S-transformations, all properties of the system of complex numbers {to be denoted by C} turn out to be necessary, except the commutativity of multiplication. For in C, there is no difference between post- and premultiplication; e.g. the right hand member of (1.4) may be written as:

$$z_{ik} \cdot z_{ij}^{-1} = z_{ij}^{-1} \cdot z_{ik} \qquad (1.5)$$

The commutativity of multiplication simplifies the study of the complex number theory as it is now, but it must be noted that the construction of the theory was actually made more difficult by it. It was the development of the quaternion-theory, for which (1.5) is not valid, that pointed the way out of the tangle of formula-combinations in the complex number theory.

The algebraic structure of C is again a field, but not an ordered field. A field is a division algebra, which in view of (1.4) is a necessary condition. Hence, C has no divisors of zero, this also proves to be a necessary condition for the development of the theory. Finally, C can also be considered as a two-dimensional vector space over R, with basis elements 1, i.

It is an interesting question how the designed "polygon theory" must be generalized to obtain a "spatial polygon theory" in three dimensions. In considering this question it must be noted that the measuring process in three dimensions, although it is more complete, does not in essence differ from the two-dimensional case. The model theory to be linked to the situation will therefore not be different in essence from the two-dimensional theory, although it will be more extensive. There are now three real coordinate quantities x, y and z, which can be compounded into the vector q. But what next?[8]

Already WEIERSTRASS around 1860, and later HILBERT, showed that an extension of C is not possible without giving up one of the properties. From what has been remarked with respect to (1.5) it seems most plausible to give up the commutativity of multiplication. The algebraic structure of such a system of numbers is a skew field; the numbers are Hamilton's quaternions and we shall denote the system by H. H can be constructed as a two-dimensional {complex} vector space over C with basis elements 1, j and thence as a four-dimensional vector space over R with basis elements 1, i, j, k {the basis element i in H should not be confused with the basis element i in C}.

H is an example of an associative linear algebra over R, but according to FROBENIUS {1878} it is also the most general associative linear algebra over R without divisors of zero, hence H is the only non-commutative division algebra over R. Further, H can be interpreted as a system of hypercomplex numbers.

The vector q is written:

$$q = x \cdot \mathrm{i} + y \cdot \mathrm{j} + z \cdot \mathrm{k} \qquad (1.6)$$

and instead of (1.4), one of the following quaternions is formed:

$$Q_{jik} = q_{ik} \cdot q_{ij}^{-1} \qquad (1.7')$$

$$Q_{jik}^{T} = q_{ij}^{-1} \cdot q_{ik} \qquad (1.7'')$$

The non-commutativity of multiplication consequently implies the possibility of choice between left- and right-division {VAN DER WAERDEN[7]}. The connection with the geodetic measuring process led to the choice of the left-division (1.7′) in the "Spatial polygon theory".

The place of Q_{jik} in the three-dimensional theory is somewhat less central than the place of Π_{jik} in the two-dimensional theory, for Q_{jik} is not completely invariant with respect to a similarity transformation. But it is important that, also in the quaternion theory, S-transformations can be developed as a generalization of the S-transformations in the complex number theory, so that application of criterion matrices in the three-dimensional theory is possible.

It is important to note that R is embedded in C and C is embedded in H, so that H comprises both R and C. If now (1.6) is written as:

$$q = 0 \cdot 1 + x \cdot \mathrm{i} + y \cdot \mathrm{j} + z \cdot \mathrm{k} \quad \ldots \ldots \quad (1.8)$$

then H comprises also the vectors $(0, x, y, z)$ in three-dimensional space. But (1.7) shows that in general the product or quotient of two vectors is not a vector, but a quaternion. Vector multiplication thus shows that vector algebra with all properties of H is therefore not possible, and, in particular, the division of vectors is not uniquely defined.

A still further possibility of extension is considered. The algebra of Cayley numbers, or algebra of octonions, can be constructed as a two-dimensional vector space over H, and, consequently as an eight-dimensional vector space over R. But then both the commutative and the associative property of multiplication are lost. It seems that only recently J. F. ADAMS [8] has conclusively proved that the absence of divisors of zero is limited to vector spaces of the dimensions 1, 2, 4, 8 over R, so that an extension further than octonions is certainly impossible. But to work without the associativity of multiplication seems impossible in the constructed "polygon theory", because combinations of successive factors in chains of multiplication can usually be interpreted as recognizable quantities in this theory. Therefore the octonions will not be usable for a geodetic model theory, and the quaternions will be the most general number system for geodetic purposes.

Everyone is familiar with the calculus based on the real numbers R; the extension of R to C led to the comprehensive theory of functions of complex variables. The extension of C to H, or, more generally, to Grassmann's hypercomplex numbers, has not led to analogous function theories. As a non-mathematician one is then groping in the dark. Prof. H. MORITZ traced an American publication [9] in which an algebra is developed that is commutative and non-associative {with respect to multiplication}; it is constructed as a three-dimensional vector space over R, with a complicated multiplication table for the three basis elements. An interesting generalized function theory can thus be built up, but this algebra comprises divisors of zero, and is therefore not usable for our purposes.

In the spring of 1970, Dr. G. KIRSCHMER drew attention to an interesting publication by W. EICHHORN,[10] which answered the question if generalization of the theory of functions of a complex variable is at all possible for the number systems or algebras considered here. EICHHORN comes to the conclusion that in general there will be a difference between the possibilities of differentiability and integrability of functions on these number systems, except for the system of complex numbers.

For the time being, there seems therefore to be no hope that the "Spatial polygon theory" – which is already in an advanced stage of development, and which is capable of describing and logically connecting all geometric methods in geodesy – can in an equally logic manner be connected with potential-theoretic methods in geodesy.

Curiously, vector algebra, with its weak algebraic properties, has grown into a vector analysis, so that GIBBS seems to have won completely over HAMILTON in this respect.[11] This is still stronger emphasized by the enormous flight taken by tensor analysis. But it must be realized that every algebra in n-dimensional spaces has weaker properties as $n > 2$ increases,[8] so that this development certainly implies losses. One of them is the loss of the possibility of division, which is admitted neither for vectors nor for tensors. The introduction of dimensionless functions of quantities, which proves to be essential in the process of linking measurement to computation, can then only be artificially done later on, by a transformation of relationships obtained by means of an available mathematical theory. It then turns out to be very difficult to see through the essential questions of the model theory, as the example of the problems of potential theory in geodesy has taught us.

Further speculative considerations

The approach to geodesy from an algebraic point of view finds its origin in an evening's discussion between HOTINE and the author in a Munich Bierkeller, during an I.A.G. Symposium in 1956, when HOTINE presented his first study of the computation of spatial networks.[12] Hotine's problem was the development of a reliable and systematic method for finding and establishing the most desirable form {from the numerical point of view} of condition equations in spatial networks, and the statistical implications of this problem. The answer to his question could not then be given, and HOTINE has {seemingly} bypassed this problem when in a next stage of his theory, following the advice of American geodesists, he treated the adjustment problem in the form of observation equations {parametric condition equations} instead of in the form of condition equations. Indeed, Hotine's choice of a tensor-analytic form of model relations leads almost automatically to the solution by observation equations, although this form has in general the disadvantage that essential model-theoretical questions are masked.

Ever since the discussion mentioned above, the author has given his attention to these problems. The question arose whether the introduction of curvilinear spatial coordinates was essential for the solution. It became clear that the vector methods in {linear} Euclidean space which have been and are being developed by geodesists, do not make a very satisfactory impression; they often bear an artificial character and are not very transparent. After what has been said in connection with (1.8) this is explicable. Who came nearest to a more algebraic form of solution was perhaps FRIEDRICH, in a very interesting little book[13] on vector calculus, in which he actually replaced two-dimensional vectors by complex numbers but did not draw the full consequence of the algebra of C. His solution for three-dimensional vectors has much in common with certain aspects of the quaternion theory, but he certainly has not drawn the consequence of the algebra of H.

The first more or less intuitive, and later systematic, application of the linear algebra of complex numbers for geodetic problems in the plane, led via KLEIN,[6] around 1962 to the linear algebra of quaternions as an aid to the solution of the problems of three-dimensional geometric geodesy. Gradually, it became clear that the answer to the problem of 1956 could be given by this approach to spatial problems. Curiously, the next stimulus to the development of a quaternion theory was given by Bjerhammar's study in potential theory,[14] presented to the second symposium on "Tridimensional Geodesy" in Cortina d'Ampezzo, 1962.

BJERHAMMAR proved in principle that via a transformation of Green's theorems a connection could be made between geometric and gravimetric problems in geodesy. This led the author to a further exploration in this direction, and to an extension of the transformation of Green's theorems to a point where quantities from the quaternion theory were recognized in the formulas. Although not all theoretical difficulties have been solved, it is virtually certain that the integral equations thus obtained determine distance ratios between terrestrial points. This means that the gravimetric model theory can contribute geometric elements to the quaternion theory, which according to (1.7) pertains essentially to geometric geodesy.

With this result, the dilemma raised by Eichhorn's work[10] can also be solved; i.e. it is not necessary to search for a function theory based on the algebra H; it is sufficient to transform the formulation of the potential-theoretical problems in geodesy according to an available mathematical theory. Of course, this is an emergency solution, but there is only a slight hope that it will be a temporary one.

This logical discord between geometric and gravimetric problems in geodesy has for years been a clearly recognized fact, and the question arises if it is not noticable in Hotine's work as well. Only a few years ago, HOTINE told the author that he doubted whether he would ever be able to design a single elegant connecting theory for the whole of geodesy. And even now, the study of his masterwork[15] on mathematical geodesy leaves open questions in this respect.

Whereas the quaternion theory gives a reasonably good answer to the problem posed in 1956, its connection with the problems of gravimetric geodesy and with the possibility of the three-dimensional generalization of the theory of S-transformations seems capable of giving a solution of the problem (1.3). This would result in a complete parallelism between geodesy in two-dimensional Euclidean space and geodesy in three-dimensional Euclidean space, with parallel relationships also for one-dimensional problems, as far as these can be considered as realistic. The interrelated algebraic properties of R, C and H are an aid in understanding the corresponding geodetic model theories in their progressive degree of generalization and complexity.

These properties also help in the search for the essential concepts and relationships in geodetic model theories. This increases in importance when the theory of stationary random processes in {gravimetric} geodesy is applied as a least-squares estimation. This necessitates the establishment of covariance functions, partly based on functional relationships between geodetic quantities. Least-squares computations always give a result if the covariance matrices have relatively simple matrix properties. If, however, the functional relationships have not been derived from a correctly established geodetic model theory, then a fictitious result is obtained, and the very object[5] for which the apparatus of stationary random processes is introduced, is not attained.

The search for a system of interconnected criteria for reliability and precision

In [9], a general method for testing model hypotheses was developed, the "B-method" of testing, whose application to geodetic networks led to the introduction of the concepts "internal reliability" and "external reliability" of a network. Both concepts pertain to a measure for the control of mistakes or gross errors in the measurement or in the computing model by redundant observations; the effectiveness of the checks depends heavily on the configuration of the net.

It has long been known that the configuration of a net is also in a high degree decisive for the precision of the coordinate variates of its points. A measure for the "precision" of networks is, in this connection, the covariance matrix of these variates in a suitable S-system.

From the mathematical formulation of these measures, a certain connection between them is apparent, and this connection will have to find expression in the criteria to be formulated, if the system of criteria is to be consistent.

The present paper offers the possibility to formulate "precision" criteria for networks by choosing values for the parameters (15.a.26) of the criterion matrix. This also implies the exertion of an influence on the form of the "confidence regions" [8] for coordinates of points of the network. In [9], section 10, the size of these regions had already been made dependent on the parameters α_0 and β_0 which determine the B-method of testing; the proper place and the significance of these "confidence regions" in the argumentation of [9] were, however, not quite clear yet. The theory of "confidence regions" seems therefore to be the point where criteria for "reliability" and for "precision" of networks come together, and can be adapted to each other. But the connecting theory is still in a tentative stage.

Finally, the main problem remains: the choice of the values of criterion parameters so that they are in agreement with the social purpose of geodetic networks,[16] as was already indicated in the problem points (1.1) and (1.2). Among other things, this will necessitate an adaption to values of parameters from the descriptive model of the objects measured – such as (15.a.27). And, of course, only an economically justified compromise[16] will actually make measurement and computation possible. The total connecting theory may then lead to the establishment of a consistent system of rules for the reconnaissance of networks.

Naturally, this connecting theory will have to be supplemented with contributions based on other points of view, before we can speak of a genuine decision theory. Prof. H. Wolf has often pointed out the desirability of a many-sided approach, lastly in a beautiful survey of ideas for possible criteria in A.V.N. 1970,[17] in which, however, the aspect of error checks in networks was left out of consideration. This survey shows how far geodetic practice still has to go before a consistent system of criteria can be established. Perhaps this image can be simplified by finding the target functions {"Zielfunktionen"}[17] that are invariant with respect to an S-transformation. A provisional analysis suggests that a reformulation is desirable.

Introduction to the next sections

a. *Difference equations. Elimination of parameters*

In the next section of this paper, only relationships between mean values of variates are considered. The advantage is that no difference in treatment is necessary between derived variates {being functions of observed variates} and the functions of observed variates which, if put equal to zero, express "laws of nature" or "condition model" in an adjustment problem. This will first be elucidated, in connection with the line of thought of [4], especially section 3.4. The fundamental idea is that to a measuring process is linked a computing model, whose functional relationships hold for the mean values of variates. Let there be given the vector of observed variates $(\underset{\sim}{x}^i)$, and the vector $(\underset{\sim}{x}^r)$ of either observed or derived variates. From the computing model, algebraically independent relationships can be established; let $(\tilde{y}^\alpha)$ be a vector of mean values of non-measured variates which figure as parameters in the relationships:

$$\left.\begin{array}{l}(\tilde{x}^r) = (X^r\{\ldots, \tilde{x}^i, \ldots, \tilde{y}^\alpha, \ldots\}); \quad r = \ldots, \quad i = \ldots, \quad \alpha = \ldots \\ \{\text{range } r\} \geqslant \{\text{range } \alpha\}\end{array}\right\} \quad \ldots\ldots\ldots \quad (1.9)$$

It is also essential that approximate values (X^r_0), (X^i_0) and (Y^α_0) respectively, satisfy (1.9):

$$(X^r_0) = (X^r\{\ldots, X^i_0, \ldots, Y^\alpha_0, \ldots\}) \quad \ldots\ldots\ldots \quad (1.10)$$

With (1.10), (1.9) can be expanded in Taylor's series:

$$(\tilde{x}^r - X^r_0) = (A^r_i)(\tilde{x}^i - X^i_0) + (A^r_\alpha)(\tilde{y}^\alpha - Y^\alpha_0) + \ldots \quad \ldots\ldots\ldots \quad (1.11')$$

Let:

$$\left.\begin{array}{l}\tilde{x}^r - X^r_0 = \widetilde{\Delta x}^r \\ \tilde{x}^i - X^i_0 = \widetilde{\Delta x}^i \\ \tilde{y}^\alpha - Y^\alpha_0 = \widetilde{\Delta y}^\alpha\end{array}\right\} \quad \ldots\ldots\ldots \quad (1.11'')$$

Then, omitting terms of higher order {provided this is admissible}, from (1.11) follow the difference equations:

$$(\widetilde{\Delta x}^r) = (A^r_i)(\widetilde{\Delta x}^i) + (A^r_\alpha)(\widetilde{\Delta y}^\alpha) \quad \ldots\ldots\ldots \quad (1.12)$$

Using relationships of the computing model, the vector $(\tilde{y}^\alpha)$ can often be eliminated from (1.9). If the computational solution can be restricted to the use of difference equations – i.e. neglecting higher-order terms in the series expansions – then the following relationships X^v as transformations of (1.9) need not be known:

$$\left.\begin{array}{l}(\tilde{x}^v) \ = (X^v\{\ldots, \tilde{x}^r, \ldots\}) \\ (X^v_0) = (X^v\{\ldots, X^r_0, \ldots\}) \\ \widetilde{\Delta x}^v \ = \tilde{x}^v - X^v_0\end{array}\right\} \quad \ldots\ldots\ldots \quad (1.13)$$

Expansion of (1.13) into series gives the difference equations:

$$(\widetilde{\Delta x}^v) = (A^v_r)(\widetilde{\Delta x}^r) \quad \ldots\ldots\ldots \quad (1.14)$$

(1.12) substituted into (1.14) gives:

$$(\widetilde{\Delta x}^v) = (A^v_r)(A^r_i)(\widetilde{\Delta x}^i) + (A^v_r)(A^r_\alpha)(\widetilde{\Delta y}^\alpha) \quad \ldots\ldots\ldots \quad (1.15)$$

The homogeneous linear equations, with unknowns A_r:

$$\left.\begin{array}{l} (A_\alpha^r)^*(A_r)^* = (0) \\ \text{give the solution:} \\ \quad \text{number of linearly independent vectors } (A_r)^* = \\ \quad = \{\text{range } r\} - \{\text{range } \alpha\} \underset{\substack{\text{denote}\\ \text{by}}}{=} \{\text{range } v\} \\ \text{together forming the matrix } (A_r^v)^*. \end{array}\right\} \quad (1.16)$$

According to (1.16), a matrix (A_r^v) can therefore always be found, which eliminates $(\widetilde{\Delta y}^\alpha)$ in (1.15). Then (1.15):

$$\left.\begin{array}{ll} (A_r^v)(A_\alpha^r) = (0) \rightarrow (A_r^v) \\ (A_r^v)(A_i^r) = (A_i^v) \\ (\widetilde{\Delta x}^v) \quad = (A_i^v)(\widetilde{\Delta x}^i) \end{array}\right\} \quad (1.17)$$

In the case of S-transformations one finds:

$$(A_i^r) = (\delta_i^r),\ (\delta_i^r) \text{the unit matrix} \quad (1.18)$$

Hence: (1.17):

$$(\widetilde{\Delta x}^v) = (A_r^v)\cdot[(\delta_i^r)(\widetilde{\Delta x}^i)]$$

or with (1.14):

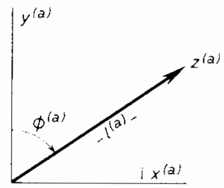
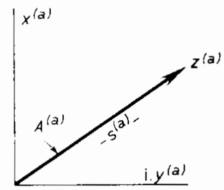
$$(A_r^v)(\widetilde{\Delta x}^r) = (A_r^v)\cdot[(\delta_i^r)(\widetilde{\Delta x}^i)] \quad (1.19)$$

(1.12) and (1.14)–(1.19) indicate the line of thought followed in (2.1)–(2.7); in connection with (1.13) reference can be made to the intuitive interpretation of S-transformations in the note of section 4. Finally, (1.19) may serve as a condition model, like in (5.3), provided reduction to zero is made.

IT IS REPEATED ONCE MORE, THAT THIS METHOD IS ONLY APPLICABLE WITHIN THE REGION OF VALIDITY OF THE DIFFERENCE EQUATIONS.

b. *x, y-coordinate systems used*

The x,y-system used in The Netherlands differs from the one adopted in most other countries. The latter system is used in [4], [5] and [9], but the Dutch systems was used in the development of a complex of interconnected theories. In order to avoid manuscript errors, the Dutch system is therefore used in this paper; a change from one system to the other means only changing x and y in the formulas. Furthermore, the notation for polar coordinates follows Dutch practice. The following table gives the connections for an arbitrary (a)-system:

	this paper {Dutch}	[4], [5], [9] {international}
coordinate system		
z bearing distance	$z^{(a)} = y^{(a)} + \mathrm{i} \cdot x^{(a)}$ $\varphi^{(a)}$ $l^{(a)}$	$z^{(a)} = x^{(a)} + \mathrm{i} \cdot y^{(a)}$ $A^{(a)}$ $s^{(a)}$
direction $\{P_j \rightarrow P_k\}$ distance-measure	$r_{jk} \equiv \varphi_{jk}^{(j)}$ $s_{jk} \equiv l_{jk}^{(j)}$	$r_{jk} \equiv A_{jk}^{(j)}$ $d_{jk} \equiv s_{jk}^{(j)}$

(1.20)

c. *Outline of the next sections*

The paper consists of four parts. The first part {sections 2–10} treats aspects of the construction, and application of S-transformations on coordinate variates from geodetic networks in the plane. The second part {sections 11–13} gives the parallelism with networks of one-dimensional coordinate variates, having in mind levelling networks, and continues by treating the construction of a criterion matrix. The third part {sections 14–17} treats the construction of a criterion matrix for two-dimensional coordinate variates in the plane, with a further elaboration of S-transformations. Finally, the fourth part contains some examples.

The *first part* is devoted to a very extensive treatment of a subject matter that is very simple in essence. Test computations, based on a number of memorandums [6], indicated in the spring of 1969 the necessity of a sharper definition of S-transformations, in order to establish unambiguously the rank of the transformation matrix. Among other things, this is a requirement for the application of the law of propagation of variances, because many covariance matrices are numerically near-singular. The elaboration of this is found in sections 2 and 3. In section 4, a connection with earlier studies is made, and the place of the similarity transformation in the theory is considered. In a note, a possible intuitive approach to S-transformations is pointed out; this approach may be clarifying when applying the theory. The sections 5–7 contain studies on the interaction between S-transformations and the adjustment procedure for densification networks; here one meets the complication that there are points in the network whose coordinates have been fixed in a previous stage, and are now introduced as "given" quantities. The sections 8–10 treat the *main problem* of this paper, viz. the testing of the computed covariance matrix of coordinates with respect to a pre-established criterion matrix, or alternatively, the description {in the sense of replacement} of the computed covariance matrix by a criterion matrix that resembles it as well as possible {in the latter case it will usually be termed "substitute matrix"}. Testing or description turns out to be possible only within one S-system; this implies that the theory of S-transformations is indispensable. In section 17 it is demonstrated, by means of a practical method of network computation, why these transformations could escape notice, with the consequences for the interpretation of the results obtained.

Having in mind the example of levelling networks in section 11, it is tried in the *second part* to explain the assumption, in section 12, of the optional function d_{ij}^2. In the construction of the criterion matrix, it is introduced as a positive "polynomial" {better: posynomial} of the distance l_{ij} between the points P_i and P_j of the network. It is sufficient to compute l_{ij} from approximate coordinates in a limited number of significant digits. In section 13, the variance Δd_i^2 is introduced, describing the uncertainty of definition of a terrain point P_i as a mathematical point. In the note to section 13, the requirements to be fulfilled by the optional function d_{ij}^2 are investigated.

The *third part* presents the generalization to two-dimensional coordinate variates; like in the second part, the construction of criterion matrices is grafted upon S-transformations. Because one is dealing with a genuine complex number theory, some small parts of the calculus of observations had to be treated by complex numbers in section 14. Among them are treated the consequences of the requirement of circular point- and relative standard ellipses in the construction of the criterion matrix. The construction itself is developed in section 15, supplemented by a different approach to S-transformations in section 16, and a later development in section 15a. After ample consideration, both courses of development have been maintained and presented together, in order to show the reader different aspects of the theory. The course followed in section 15 dates from 1962–1963; eventually it led via the G-functions to an optional function d_{ij}^2 of the same type as the one mentioned above. Formulas for the computation of the criterion matrix proved, however, to be subject to some inconvenient restrictions. In the summer of 1970, the theory of section 15.a was therefore developed, which was more directly aimed at the computations, and included the above mentioned variances Δd_i^2. Here, the significance of the parameters of the criterion matrix is indicated, and a way is sketched towards the determination of numerical values to be assigned to these parameters. Therefore, section 15.a can be considered as the ***nucleus*** of this paper as far as the construction of the criterion matrix is concerned, the more so because this section – contrary to sections 15 and 16 – is practically unconnected to the only partially published "Polygon theory in the complex plane" [2]. Finally, in section 17 an adjustment model is developed, as an application of S-transformations to the connection of several coordinate systems belonging to {stochastically} independent groups of coordinate variates in different networks. Another aspect of section 17 was already mentioned in the summary of the first part of this paper.

A generalization of the theory to networks of three-dimensional coordinate variates proves to be possible, but it has not yet been elaborated sufficiently.

The question remains if the choice of the optional function d_{ij}^2 has been made sufficiently general, although it is very satisfactory that for one-, two- and three-dimensional situations the same optional function can be used. This question is open to further research.

Finally, it should be noted that the text of this paper was written between the summer of 1969 and the spring of 1971, because of which small differences in terminology were almost unavoidable. But an effort has been made to elucidate as clearly as possible the many aspects of the theory designed.

An alternative summary of the theory with some additions was given in [11] and [12].

Notes to Section 1

1. W. Baarda, D. de Groot, F. Harkink – Handleiding voor de Technische Werkzaamheden van het Kadaster {Manual for the Technical Activities of the Cadastral Service in The Netherlands} – The Hague, 1956. In this paper, the manual is referred to as "HTW-1956".
For a sketch of the theoretical ideas and the constructions of a system of rules for reconnaissance, see:
W. Baarda, D. de Groot – Opzet en techniek van kadastrale metingen {Plan and Technique of Cadastral Surveys} – Reports 12th Congress of the Netherlands Federation of Land Surveyors, June 1952, pages 2–30. Also reprinted by the Geodetic Institute, Delft.
A summary and a concrete elaboration is given in:
W. A. Claessen, A. H. Kooimans – Principles of the Calculus of Observations and its Application to Cadastral and Engineering Surveys – Compte Rendu du Neuvième Congrès International des Géomètres 1958, Waltman, Delft, 1959, pages 300–319.

2. J. M. Tienstra – Theory of the adjustment of normally distributed observations – Argus, Amsterdam, 1956.

3. J. M. Tienstra – Nets of Triangles Consisting of Points with circular Error-Curves – Kon. Akad. van Wetenschappen, Proceedings Vol. XXXVI, No. 6, Amsterdam, 1933.

4. J. M. Tienstra, Th. L. Kwisthout, P. J. Hamelberg – Handleiding voor de Technische werkzaamheden van het Kadaster {Manual for the Technical Activities of the Cadastral Service in The Netherlands} – The Hague, 1938. In this paper, the manual is referred to as "HTW-1938".

5. T. Krarup – A contribution to the mathematical foundation of physical geodesy – Publ. No. 44. Danish Geodetic Institute, Copenhagen, 1969.
H. Moritz – Least-squares estimation in physical geodesy – Report No. 130, Department of Geodetic Science, Ohio State University, 1970.
H. Moritz – A generalized least-squares model – Studia geoph. et geod. 14, 1970, pages 353–362.

6. F. Klein – Elementary mathematics from an advanced standpoint. Arithmetic, algebra, analysis – Dover Publ., New York, 1939.

7. Some works on algebraic concepts {also quaternions} ranged according to increasing complexity:
Jagjit Singh – Mathematical Ideas. Their Nature and Use – Hutchinson, London, 1959.
J. C. H. Gerretsen – Wiskunde – In: Eerste Ned. Syst. Ingerichte Encyclopaedie. Deel IV, Amsterdam, 1949.
E. T. Bell – Mathematics. Queen and Servant of Science – McGraw-Hill, New York, 1965.
G. Birkhoff, S. Maclane – A Survey of Modern Algebra – 3rd ed., MacMillan, New York, 1965.
L. Brand – Vector and Tensor Analysis – Wiley, New York, 1947.
M. Lagally, W. Franz – Vorlesungen über Vektorrechnung – 7. Auflage, Akad. VerlagsGesellschaft, Leipzig 1964.
B. L. van der Waerden – Algebra – Springer-Verlag, Heidelberg, 1966, 1967.

8. The algebraic considerations presented in this paper mainly follow the excellent book:
H. B. Griffiths, P. J. Hilton – A Comprehensive Textbook of Classical Mathematics. A Contemporary Interpretation – Van Nostrand Reinhold, London, 1970.
For a deeper analysis the work of Pickert is important, see the following article and its summary in Euclides:
G. Pickert, H. G. Steiner – Komplexe Zahlen und Quaternionen – In: Grundzüge der Mathematik. Band I, 2 Auflage, V. d. Hoeck & Ruprecht, Göttingen, 1962.
P. G. J. Vredenduin – Uitbreiding van getalsystemen {Extension of number systems} – Euclides, maandblad voor de didactiek van de wiskunde, September 1965.
The question whether the designed geodetic theories make complete use of R, C or H, is still open.

Thanks are due to H. C. VAN DER HOEK, who raised this point, and was willing to study possible algebraic consequences.

9. T. E. PHIPPS, JR. – Function Theory in Three Dimensions – McLean, Virginia, 1966.
   T. E. PHIPPS, JR. – Solution of the Two-Dimensional Poisson Equation using Three-Dimensional Harmonic Polynomials – McLean, Virginia, 1966.
10. W. EICHHORN – Eine aus Fragen der Verallgemeinerung der Funktionentheorie erwachsende Charakterisierung der Algebra der komplexen Zahlen. – Jahresbericht der Deutschen Mathematiker Vereinigung, 71. Band (1969), Heft 3, S 123–137.
11. See the historical introduction in:
    A. P. WILLS – Vector Analysis with an Introduction to Tensor Analysis – Dover Publ., New York, 1958.
    and the discussion papers in:
    The Scientific Papers of J. W. GIBBS. Vol. 2 – Dover Publ., New York, 1961.
12. M. HOTINE – Adjustment of Triangulation in Space – Tolworth, 1956.
13. K. FRIEDRICH – Neue Grundlagen und Anwendungen der Vektorrechnung – Oldenbourg, München, 1921.
14. A. BJERHAMMAR – A General World Geodetic Problem – Stockholm, 1962.
15. M. HOTINE – Mathematical Geodesy – U.S. Department of Commerce, ESSA Monograph 2, Washington, 1969.
16. The embedding of a geodetic model in the surrounding social structure is elaborated for some examples in:
    W. BAARDA – The Future of the Géomètre – Archives 12th Int. Congress of Surveyors, Vol. A, paper 2A, London, 1968.
    For the compromise between "consumer" and "producer" – as a simplified situation characterizing the relation between society and the geodesist – in which the decision process results in the compromise, recorded in the contract between consumer and producer, see:
    W. BAARDA – Statistics: A Compass for the Land Surveyor – Archives 12th Int. Congress of Surveyors, Vol. B, paper 507 C, London, 1968.
17. H. WOLF – Polarität und Optimierung bei freien und eingeschalteten geodätischen Netzen – Allgemeine Vermessungs-Nachrichten 1970, Heft 8, pages 291–198.

2 THE MODEL OF THE SIMILARITY TRANSFORMATION

The transition from a z-system to a $z^{(a)}$-system is effected according to [4] appendix, p. 63:

$$\tilde{z}_i^{(a)} = \tilde{\gamma}^{(a)} \cdot \tilde{z}_i + \tilde{\delta}^{(a)}; \qquad i, j = \ldots \tag{2.1}$$

Difference equations are deduced by introducing approximate values fulfilling the same relations (2.1) see [4] section 3.4:

$$z_i^{(a)^0} = \gamma^{(a)^0} \cdot z_i^0 + \delta^{(a)^0} \tag{2.2}$$

Ignoring terms of higher order, one obtains from (2.1) and (2.2):

$$(\tilde{z}_i^{(a)} - z_i^{(a)^0}) = \gamma^{(a)^0} \cdot (\tilde{z}_i - z_i^0) + z_i^0 \cdot (\tilde{\gamma}^{(a)} - \gamma^{(a)^0}) + (\tilde{\delta}^{(a)} - \delta^{(a)^0}) \tag{2.3'}$$

or:

$$\boxed{\widetilde{\Delta z}_i^{(a)} = \gamma^{(a)^0} \cdot \widetilde{\Delta z}_i + z_i^0 \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)}} \tag{2.3''}$$

Now add two "base points" P_r, P_s to the points P_i. From (2.3) follows then:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(a)} \\ \widetilde{\Delta z}_r^{(a)} \\ \widetilde{\Delta z}_s^{(a)} \end{pmatrix} = \begin{pmatrix} \gamma^{(a)^0} \cdot \delta_i^i & 0 & 0 & z_i^0 & \delta_0^i \\ 0 & \gamma^{(a)^0} & 0 & z_r^0 & 1 \\ 0 & 0 & \gamma^{(a)^0} & z_s^0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{\Delta z}_i \\ \widetilde{\Delta z}_r \\ \widetilde{\Delta z}_s \\ \widetilde{\Delta \gamma}^{(a)} \\ \widetilde{\Delta \delta}^{(a)} \end{pmatrix} \tag{2.4}$$

in which (δ_i^i) is a unit matrix, and
(δ_0^i) a column vector all of whose elements are 1.

From the model relations (2.4), $\widetilde{\Delta \gamma}^{(a)}$ and $\widetilde{\Delta \delta}^{(a)}$ can be eliminated by a so-called S-transformation, introducing quantities in the $\underset{r,s}{\text{S}}$-system:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(a)(rs)} \\ \widetilde{\Delta z}_r^{(a)(rs)} \\ \widetilde{\Delta z}_s^{(a)(rs)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & -\dfrac{z_{si}^0}{z_{sr}^0} & -\dfrac{z_{ri}^0}{z_{rs}^0} \\ 0 & 1 - \dfrac{z_{sr}^0}{z_{sr}^0} = 0 & -\dfrac{z_{rr}^0}{z_{rs}^0} = 0 \\ 0 & -\dfrac{z_{ss}^0}{z_{sr}^0} = 0 & 1 - \dfrac{z_{rs}^0}{z_{rs}^0} = 0 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(a)} \\ \widetilde{\Delta z}_r^{(a)} \\ \widetilde{\Delta z}_s^{(a)} \end{pmatrix} \tag{2.5'}$$

and analogous quantities without the index (a) . . . . (2.5'')

The application of (2.5) to (2.4), and use of relations like:

$$\frac{z_{si}^0}{z_{sr}^0} + \frac{z_{ri}^0}{z_{rs}^0} = 1 \quad \text{for each } i \qquad (2.6)$$

results in:

$$\left.\begin{aligned} \widetilde{\Delta z}_i^{(a)(rs)} &= \gamma^{(a)0} \cdot \widetilde{\Delta z}_i^{(rs)}; \quad i, j = \ldots \\ \widetilde{\Delta z}_r^{(a)(rs)} &= \widetilde{\Delta z}_r^{(rs)} = 0 \\ \widetilde{\Delta z}_s^{(a)(rs)} &= \widetilde{\Delta z}_s^{(rs)} = 0 \end{aligned}\right\} \qquad (2.7)$$

Now (2.5), and hence (2.7), is established in accordance with the approach to S-transformations given earlier in [3], leading to a singular transformation matrix. Therefore a better approach is found if one works with a non-singular transformation matrix, in a way that does not cause the loss of information. In fact, $\underline{\Delta\gamma}^{(a)}$ and $\underline{\Delta\delta}^{(a)}$ are then considered as "implicitly given derived variates", see section 17 of [1]. One obtains:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(a)(rs)} \\ \\ \widetilde{\Delta z}_r^{(a)} \\ \\ \widetilde{\Delta z}_s^{(a)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & -\frac{z_{si}^0}{z_{sr}^0} & -\frac{z_{ri}^0}{z_{rs}^0} \\ \\ 0 & 1 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(a)} \\ \\ \widetilde{\Delta z}_r^{(a)} \\ \\ \widetilde{\Delta z}_s^{(a)} \end{pmatrix} \qquad (2.8')$$

and analogous quantities without the index (a) . . . . (2.8'')

Applying (2.8) to (2.4) results in:

$$\left.\begin{aligned} \widetilde{\Delta z}_i^{(a)(rs)} &= \gamma^{(a)0} \cdot \widetilde{\Delta z}_i^{(rs)}; \qquad i, j = \ldots \\ \widetilde{\Delta z}_r^{(a)} &= \gamma^{(a)0} \cdot \widetilde{\Delta z}_r + z_r^0 \cdot \widetilde{\Delta\gamma}^{(a)} + \widetilde{\Delta\delta}^{(a)} \\ \widetilde{\Delta z}_s^{(a)} &= \gamma^{(a)0} \cdot \widetilde{\Delta z}_s + z_s^0 \cdot \widetilde{\Delta\gamma}^{(a)} + \widetilde{\Delta\delta}^{(a)} \end{aligned}\right\} \qquad (2.9)$$

The case: $\tilde{\gamma}^{(a)} - 1 \approx \tilde{\delta}^{(a)} \approx 0$

This case, most frequently met in practice, implies for (2.2):

$\gamma^{(a)0} = 1$	$\delta^{(a)0} = 0$
$z_i^{(a)0} = z_i^0$	
$z_r^{(a)0} = z_r^0$; $z_s^{(a)0} = z_s^0$	

. . . . (2.10)

From (2.7) and (2.9) follows, with (2.10):

$$\widetilde{\Delta z}_i^{(a)(rs)} = \widetilde{\Delta z}_i^{(rs)}; \quad i,j = \dots \qquad (2.11')$$

$$\begin{aligned} \widetilde{\Delta z}_i^{(rs)} &= \widetilde{\Delta z}_i^{(sr)} \\ \widetilde{\Delta z}_r^{(rs)} &= \widetilde{\Delta z}_s^{(rs)} = 0 \end{aligned} \qquad (2.11'')$$

$$\begin{aligned} \widetilde{\Delta z}_r^{(a)} &= \widetilde{\Delta z}_r + z_r^0 \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)} \\ \widetilde{\Delta z}_s^{(a)} &= \widetilde{\Delta z}_s + z_s^0 \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)} \end{aligned} \qquad (2.11''')$$

Meaning of $\widetilde{\Delta z}_i^{(rs)}$

From (2.5) or (2.8) follows, with (2.6) ,see also [4] section 4.2 and appendix:

$$\begin{aligned} \widetilde{\Delta z}_i^{(rs)} &= \widetilde{\Delta z}_i - \frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_r - \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_s = -\frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_{ir} - \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_{is} = \\ &= -\frac{z_{ri}^0 \cdot z_{is}^0}{z_{rs}^0}\left(\frac{\widetilde{\Delta z}_{is}}{z_{is}^0} - \frac{\widetilde{\Delta z}_{ir}}{z_{ir}^0}\right) = \\ &= -\frac{z_{ri}^0 \cdot z_{is}^0}{z_{rs}^0} \cdot \widetilde{\Delta \Pi}_{ris} \end{aligned} \qquad (2.12)$$

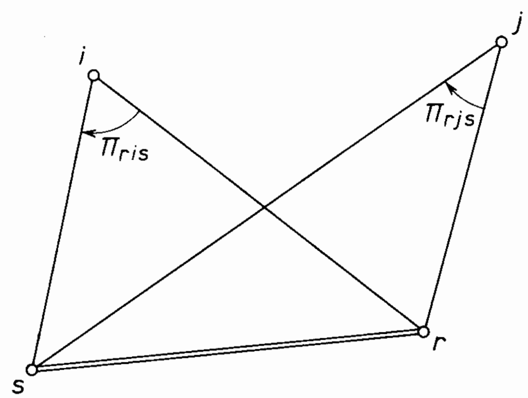


Fig. 2-1

(2.12) is very important for a good understanding of the following sections; also very helpful for a better understanding of (2.11).

Just as important is (2.12) for the interpretation of standard ellipses, as an example of which [5] may serve, in particular the additional figures 1a–4a, see note at the end of section 2. The interesting thing is that the dashed circular standard ellipses do *not* depend on the shape of the net: they follow from the choice of the criterion matrix. The standard ellipses drawn in full lines are dependent on the shape of the net; they are found by applying the law of propagation of variances to the functions expressing the coordinates in terms of measured quantities.

Figure 1a: In the choice of the $S_{1,3}$-system, the net to the left of 11–13 appears to be relatively good, to the right of 11–13 it is relatively bad as far as precision is concerned.

Figure 2a: The same applies in the choice of the $S_{11,13}$-system.

Figure 3a: If the $S_{17,21}$-system is chosen, the full-drawn standard ellipses on the left become large too, because the Π-quantities concerned are now also functions of the poor part of the net on the right. But the partial net 17-19-21 now appears to be relatively good too.

Figure 4a: In the choice of the $S_{3,21}$-system, almost all Π-quantities are affected by the poor part of the net. These standard ellipses therefore give the least reliable impression of the qualities of different parts of the net.

Remark

A more direct way to Π-quantities is the following, see figure 2-1 and [4] section 4.2 and appendix:

Consider the difference quantities:

$$\tilde{z}_{ri}^{(a)} = \tilde{z}_{rs}^{(a)} \cdot e^{\tilde{\Pi}_{sri}} \quad \Big| \quad \tilde{z}_{si}^{(a)} = \tilde{z}_{sr}^{(a)} e^{-\tilde{\Pi}_{isr}}$$

with transformation of system:

$$\begin{array}{r|l} (a) \rightarrow (h) & (a) \rightarrow (h') \\ \tilde{z}_{ri}^{(h)} = \tilde{\gamma}_{(a)}^{(h)} \cdot \tilde{z}_{ri}^{(a)} & \tilde{z}_{si}^{(h')} = \tilde{\gamma}_{(a)}^{(h')} \cdot \tilde{z}_{si}^{(a)} \end{array}$$

or, with $\ln z_{ij} = \Lambda_{ij}$:

$$\tilde{\Lambda}_{ri}^{(h)} = \widetilde{\ln \gamma}_{(a)}^{(h)} + \tilde{\Lambda}_{rs}^{(a)} + \tilde{\Pi}_{sri} \quad \Big| \quad \tilde{\Lambda}_{si}^{(h')} = \widetilde{\ln \gamma}_{(a)}^{(h')} + \tilde{\Lambda}_{sr}^{(a)} - \tilde{\Pi}_{isr}$$

hence:

$$\begin{array}{r|l} \widetilde{\Delta\Lambda}_{ri}^{(h)} = \widetilde{\Delta \ln \gamma_{(a)}^{(h)}} + \widetilde{\Delta\Lambda}_{rs}^{(a)} + \widetilde{\Delta\Pi}_{sri} & \widetilde{\Delta\Lambda}_{si}^{(h')} = \widetilde{\Delta \ln \gamma_{(a)}^{(h')}} + \widetilde{\Delta\Lambda}_{sr}^{(a)} - \widetilde{\Delta\Pi}_{isr} \\ \text{with } \gamma_{(a)}^{(h)0} = 1 & \text{with } \gamma_{(a)}^{(h')0} = 1 \end{array}$$

Now put:

$$\widetilde{\Delta \ln \gamma_{(a)}^{(h)}} = -\widetilde{\Delta\Lambda}_{rs}^{(a)} \quad \Big| \quad \widetilde{\Delta \ln \gamma_{(a)}^{(h')}} = -\widetilde{\Delta\Lambda}_{sr}^{(a)}$$

But $\widetilde{\Delta\Lambda}_{rs}^{(a)} = \widetilde{\Delta\Lambda}_{sr}^{(a)}$, hence (h)-system $\equiv$ (h')-system, call this the (rs)-system. Then:

$$\begin{array}{r|l} \widetilde{\Delta\Lambda}_{ri}^{(rs)} = \widetilde{\Delta\Pi}_{sri} & \widetilde{\Delta\Lambda}_{si}^{(rs)} = -\widetilde{\Delta\Pi}_{isr} \\ \widetilde{\Delta z}_{ri}^{(rs)} = z_{ri}^{0} \cdot \widetilde{\Delta\Pi}_{sri} & \widetilde{\Delta z}_{si}^{(rs)} = -z_{si}^{0} \cdot \widetilde{\Delta\Pi}_{isr} \end{array}$$

But in a mathematically consistent triangle we have:

$$\widetilde{\Delta\eta}_{(i),s,r} \equiv 0 = z^0_{si} \cdot \widetilde{\Delta\Pi}_{isr} + z^0_{ri} \cdot \widetilde{\Delta\Pi}_{sri}$$

or:

$$\widetilde{\Delta z}^{(rs)}_{ri} = \widetilde{\Delta z}^{(rs)}_{si} \underset{\text{denote by}}{=} \widetilde{\Delta z}^{(rs)}_{i}$$

In such a triangle we also have:

$$\widetilde{\Delta\eta}_{(s),r,i} \equiv 0 = z^0_{rs} \cdot \widetilde{\Delta\Pi}_{sri} + z^0_{is} \cdot \widetilde{\Delta\Pi}_{ris}$$

hence:

$$\boxed{\widetilde{\Delta z}^{(rs)}_{i} = z^0_{ri} \cdot \widetilde{\Delta\Pi}_{sri} = z^0_{si} \cdot \widetilde{\Delta\Pi}_{rsi} = -\frac{z^0_{ri} \cdot z^0_{is}}{z^0_{rs}} \cdot \widetilde{\Delta\Pi}_{ris}} \quad \text{. . . . . cf. (2.12)}$$

Consequently:

$$\widetilde{\Delta A}^{(a)}_{ri} = \widetilde{\Delta A}^{(a)}_{rs} + \widetilde{\Delta\Pi}_{sri}, \quad \text{with} \quad z^0_{ri} \cdot \widetilde{\Delta\Pi}_{sri} = \widetilde{\Delta z}^{(rs)}_{i},$$

hence:

$$\begin{aligned} \widetilde{\Delta z}^{(rs)}_{i} &= \widetilde{\Delta z}^{(a)}_{ri} - z^0_{ri} \cdot \widetilde{\Delta A}^{(a)}_{rs} = \\ &= \widetilde{\Delta z}^{(a)}_{i} - \widetilde{\Delta z}^{(a)}_{r} - \frac{z^0_{ri}}{z^0_{rs}} (\widetilde{\Delta z}^{(a)}_{s} - \widetilde{\Delta z}^{(a)}_{r}) = \\ &= \widetilde{\Delta z}^{(a)}_{i} - \frac{z^0_{rs} + z^0_{ir}}{z^0_{rs}} \widetilde{\Delta z}^{(a)}_{r} - \frac{z^0_{ri}}{z^0_{rs}} \widetilde{\Delta z}^{(a)}_{s} \end{aligned}$$

or:

$$\boxed{\widetilde{\Delta z}^{(rs)}_{i} = \widetilde{\Delta z}^{(a)}_{i} - \frac{z^0_{si}}{z^0_{sr}} \cdot \widetilde{\Delta z}^{(a)}_{r} - \frac{z^0_{ri}}{z^0_{rs}} \cdot \widetilde{\Delta z}^{(a)}_{s}} \quad \text{. . . . . cf. (2.8)}$$

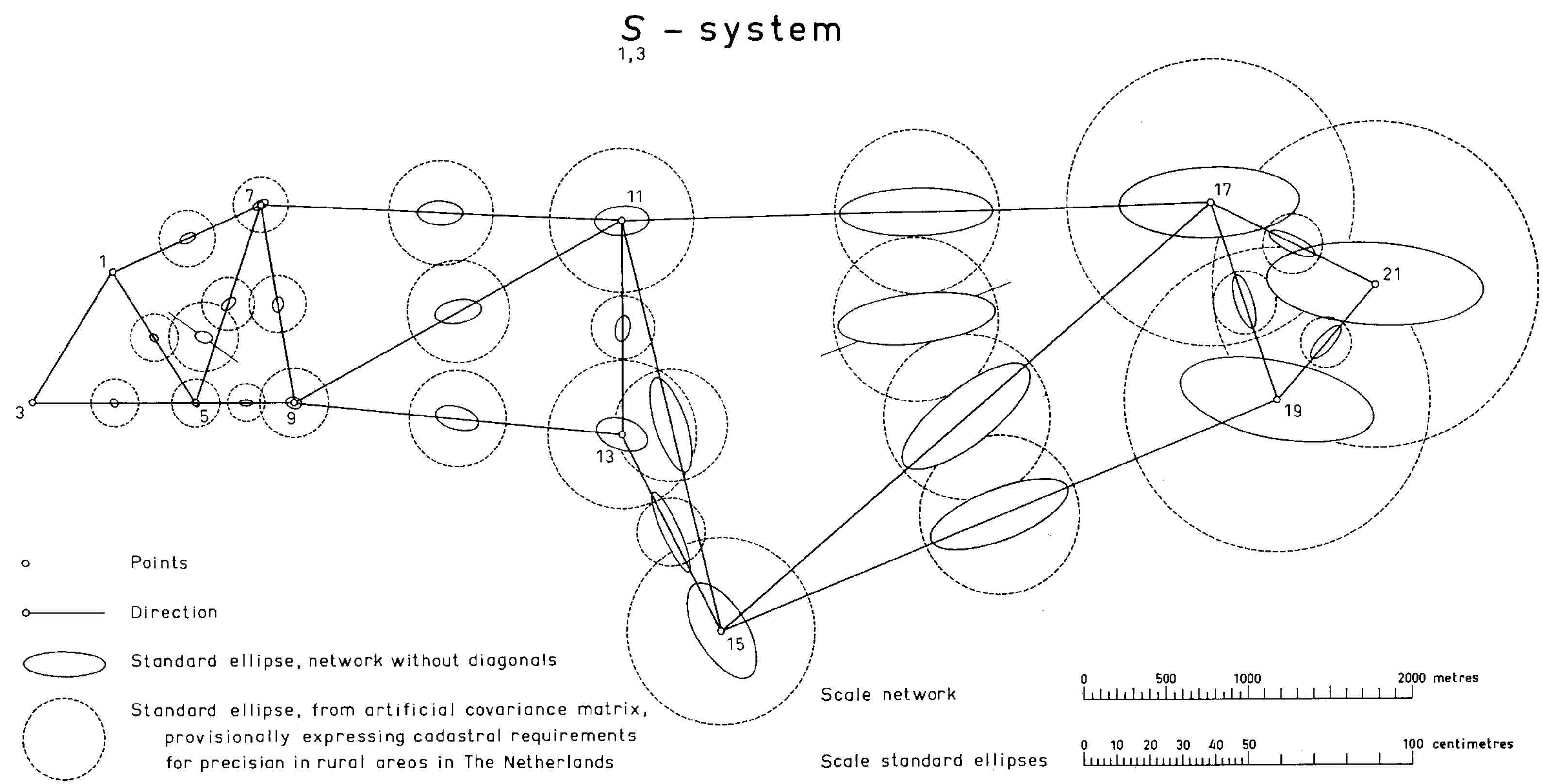
Note to section 2. Remarks to figures in [5]

The previously mentioned figures 1a–4a of [5] are included here. The full-drawn ellipses follow from the adjustment of the network under the assumption of a standard deviation for the uncorrelated direction variates $\underline{r}$:

$$\sigma_r = \text{constant} = 7 \text{ dmgr (centesimal seconds)}$$

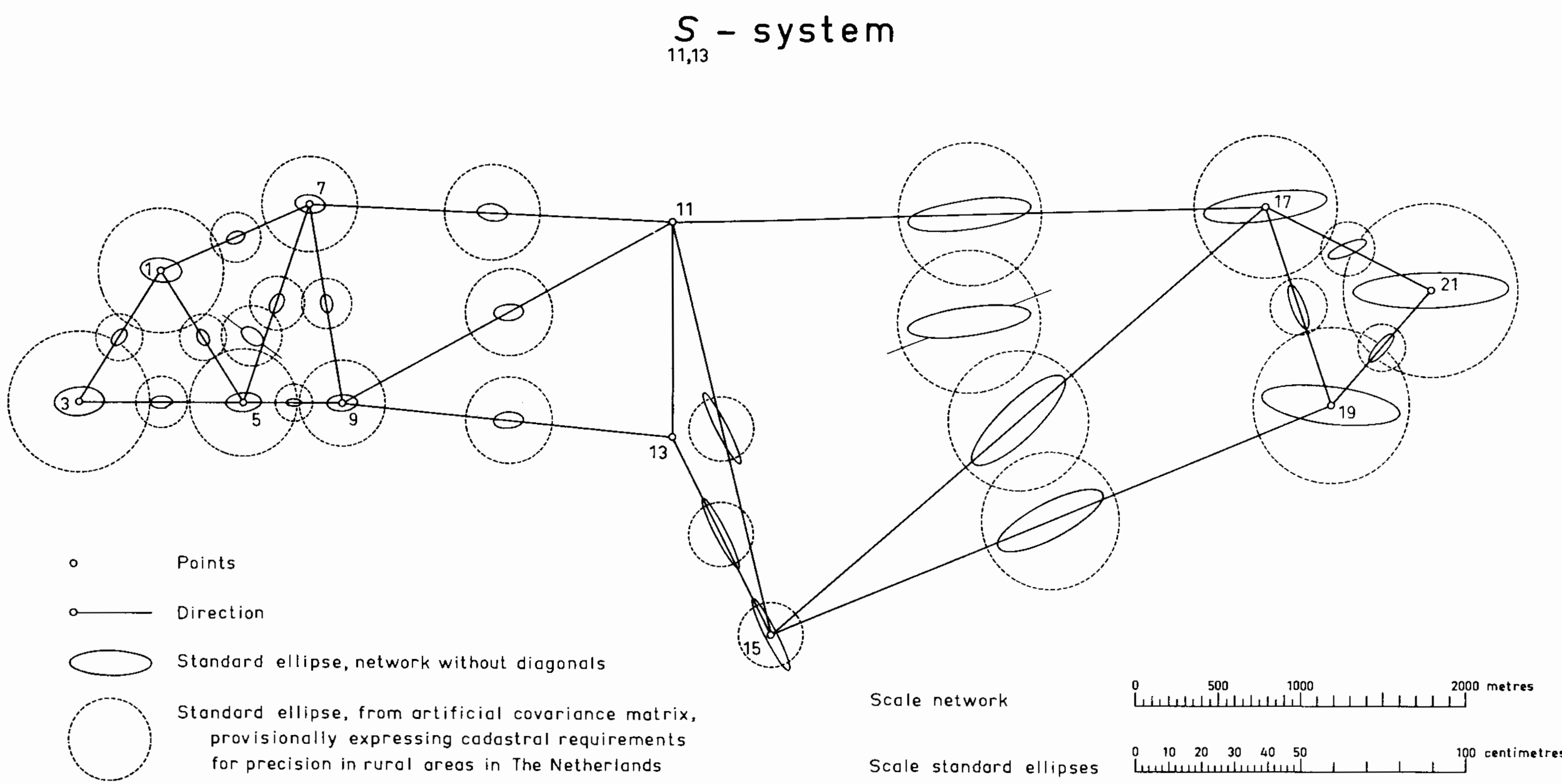
The criterion of precision is established in relation with the criterion for rural areas {areas type 2} used in the HTW-1956. This implies the construction of an artificial covariance matrix, the criterion matrix, with only one parameter, c_1 {see (15.a.26)}. The corresponding circular standard ellipses are drawn in dashed lines, in such a way that in the $S_{3,21}$-system the radius of the circle in P_{13} is chosen according to:

$$\tfrac{1}{2} d_A \cdot \sqrt{2} = \sqrt{\frac{36}{2}\{4 \text{ km} + 0.005\}} \approx 8.5 \text{ cm}$$



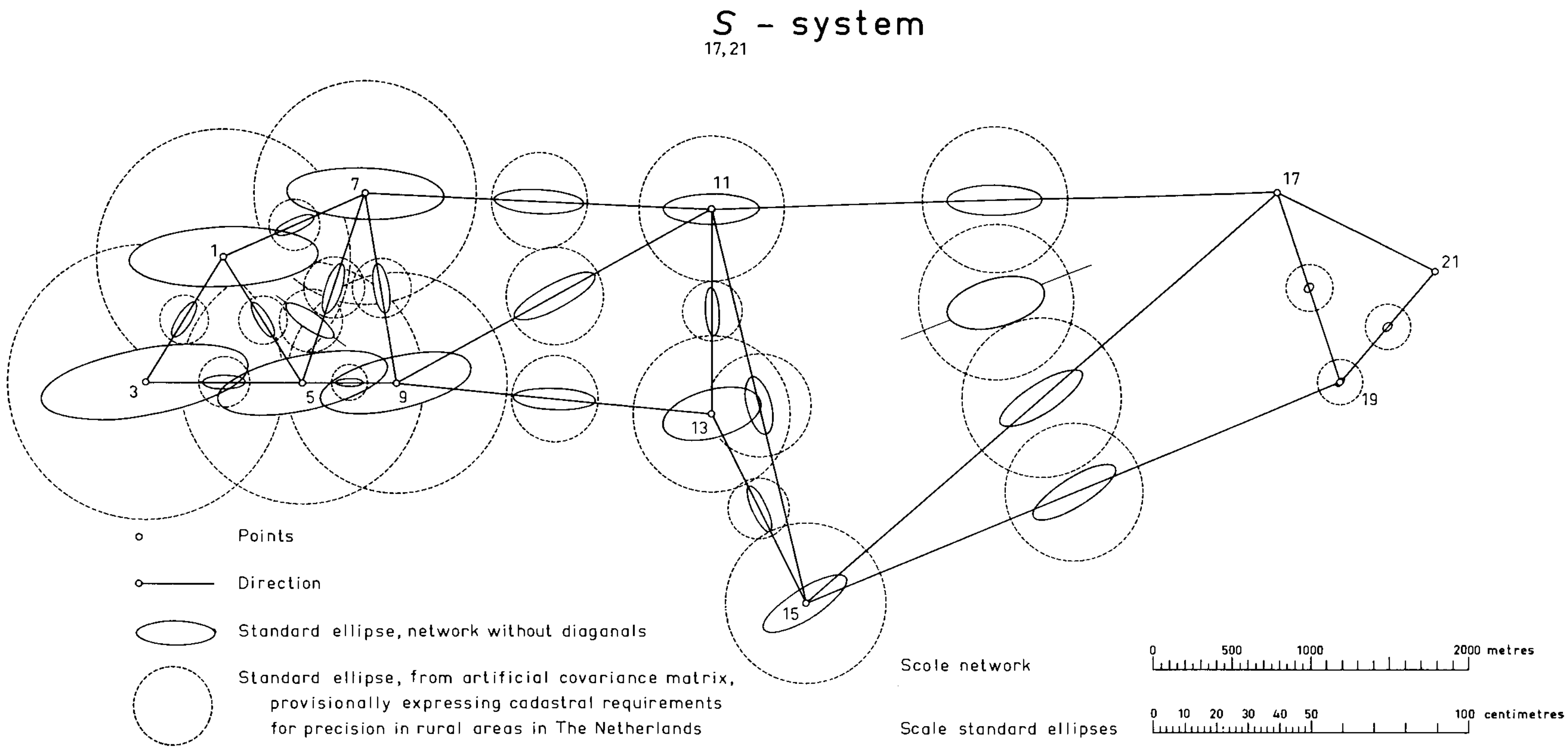
S - system
1,3
1
3
5
7
9
11
13
15
17
19
21
Points
Direction
Standard ellipse, network without diagonals
Standard ellipse, from artificial covariance matrix,
provisionally expressing cadastral requirements
for precision in rural areas in The Netherlands
Scale network
0 500 1000 2000 metres
Scale standard ellipses
0 10 20 30 40 50 100 centimetres

Fig. 1a



S – system
11,13
7
1
3
5
9
11
13
15
17
19
21
Points
Direction
Standard ellipse, network without diagonals
Standard ellipse, from artificial covariance matrix,
provisionally expressing cadastral requirements
for precision in rural areas in The Netherlands
Scale network
0
500
1000
2000 metres
Scale standard ellipses
0
10
20
30
40
50
100 centimetres

Fig. 2a



$S_{17,21}$ - system
1
3
5
7
9
11
13
15
17
19
21
Points
Direction
Standard ellipse, network without diaganals
Standard ellipse, from artificial covariance matrix, provisionally expressing cadastral requirements for precision in rural areas in The Netherlands
Scale network
0 500 1000 2000 metres
Scale standard ellipses
0 10 20 30 40 50 100 centimetres

Fig. 3a

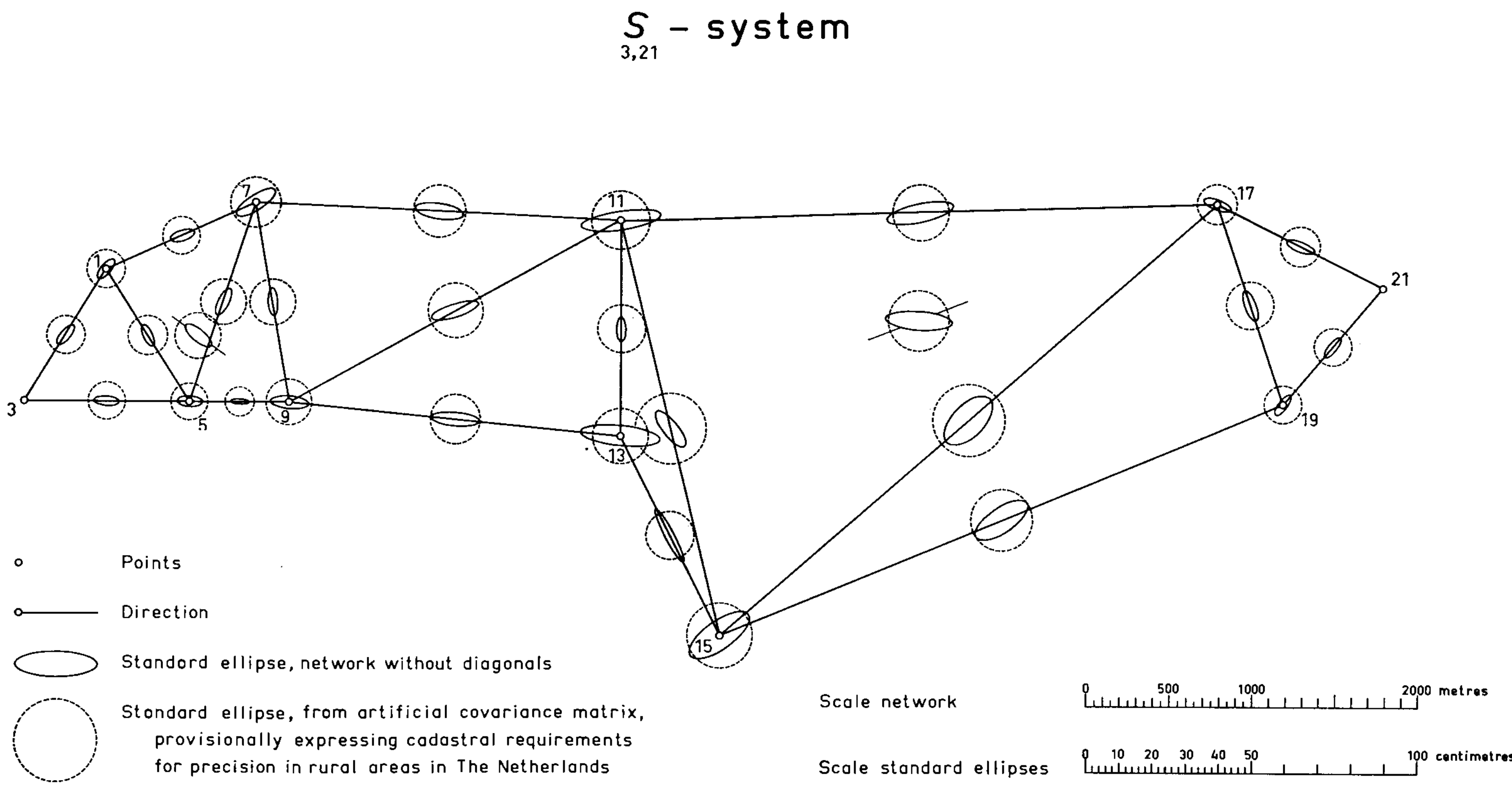

S 3,21 - system
3
1
5
7
9
11
13
15
17
19
21
Points
Direction
Standard ellipse, network without diagonals
Standard ellipse, from artificial covariance matrix, provisionally expressing cadastral requirements for precision in rural areas in The Netherlands
Scale network
0 500 1000 2000 metres
Scale standard ellipses
0 10 20 30 40 50 100 centimetres

Fig. 4a

implying an assumed "distance of densification" {HTW-1956}:

$$A = \tfrac{1}{2}\{l_{3,21} \approx 8 \text{ km}\} = 4 \text{ km}$$

This results in the parameter value:

$$c_1 \approx 17 \text{ cm}^2/\text{km}$$

From the four figures it is evident that the degree in which the dashed standard ellipses enclose or intersect the full-drawn ones, is not invariant with respect to an S-transformation. This is explained by applying the general eigenvalue problem of section 8, or with (8.21):

$$\lambda_{\max} \approx 36 \quad \gg \{c_1 \approx 17\}$$

Nevertheless the influence of the poorly conditioned part of the net between the pairs of points P_{11}, P_{13} and P_{17}, P_{19} is evident. The question whether the correct parameter for the criterion matrix has been chosen will not be discussed here.

3 A NON-SINGULAR S-TRANSFORMATION

The transformation (2.5) is singular; (2.8) is non-singular. Which of the two should be considered as more directive?

It should be borne in mind that in a network coordinates can only be computed starting from a computational base. The criterion matrix according to [6] is also constructed with respect to a computational base. This means that (2.5) is more directive than (2.8).

For a further study alternative base points P_v, P_w are added to the points P_i, P_r and P_s. Then we have, assuming (2.10):

$$\begin{array}{ll|l}
\widetilde{\Delta z}_i^{(rs)} & = \widetilde{\Delta z}_i - \dfrac{z^0_{si}}{z^0_{sr}}\widetilde{\Delta z}_r - \dfrac{z^0_{ri}}{z^0_{rs}}\widetilde{\Delta z}_s & \leftrightarrow \widetilde{\Delta \Pi}_{ris} \\
\widetilde{\Delta z}_v^{(rs)} & = \widetilde{\Delta z}_v - \dfrac{z^0_{sv}}{z^0_{sr}}\widetilde{\Delta z}_r - \dfrac{z^0_{rv}}{z^0_{rs}}\widetilde{\Delta z}_s & \leftrightarrow \widetilde{\Delta \Pi}_{rvs} \\
\widetilde{\Delta z}_w^{(rs)} & = \widetilde{\Delta z}_w - \dfrac{z^0_{sw}}{z^0_{sr}}\widetilde{\Delta z}_r - \dfrac{z^0_{rw}}{z^0_{rs}}\widetilde{\Delta z}_s & \leftrightarrow \widetilde{\Delta \Pi}_{rws}
\end{array} \qquad (3.1)$$

$$\begin{array}{ll}
\widetilde{\Delta z}_i^{(a)(rs)} & = \widetilde{\Delta z}_i^{(rs)} \\
\widetilde{\Delta z}_v^{(a)(rs)} & = \widetilde{\Delta z}_v^{(rs)} \\
\widetilde{\Delta z}_w^{(a)(rs)} & = \widetilde{\Delta z}_w^{(rs)} \\
\widetilde{\Delta z}_r^{(a)} & = \widetilde{\Delta z}_r + z^0_r \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)} \\
\widetilde{\Delta z}_s^{(a)} & = \widetilde{\Delta z}_s + z^0_s \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)}
\end{array} \qquad (3.2)$$

Now (2.5) and (2.8) can also be applied using P_v, P_w as base points. This gives:

$$\begin{array}{ll|l}
\widetilde{\Delta z}_i^{(vw)} & = \widetilde{\Delta z}_i - \dfrac{z^0_{wi}}{z^0_{wv}}\widetilde{\Delta z}_v - \dfrac{z^0_{vi}}{z^0_{vw}}\widetilde{\Delta z}_w & \leftrightarrow \widetilde{\Delta \Pi}_{viw} \\
\widetilde{\Delta z}_r^{(vw)} & = \widetilde{\Delta z}_r - \dfrac{z^0_{wr}}{z^0_{wv}}\widetilde{\Delta z}_v - \dfrac{z^0_{vr}}{z^0_{vw}}\widetilde{\Delta z}_w & \leftrightarrow \widetilde{\Delta \Pi}_{vrw} \\
\widetilde{\Delta z}_s^{(vw)} & = \widetilde{\Delta z}_s - \dfrac{z^0_{ws}}{z^0_{wv}}\widetilde{\Delta z}_v - \dfrac{z^0_{vs}}{z^0_{vw}}\widetilde{\Delta z}_w & \leftrightarrow \widetilde{\Delta \Pi}_{vsw}
\end{array} \qquad (3.3)$$

$$\begin{array}{ll}
\widetilde{\Delta z}_i^{(a)(vw)} & = \widetilde{\Delta z}_i^{(vw)} \\
\widetilde{\Delta z}_r^{(a)(vw)} & = \widetilde{\Delta z}_r^{(vw)} \\
\widetilde{\Delta z}_s^{(a)(vw)} & = \widetilde{\Delta z}_s^{(vw)} \\
\widetilde{\Delta z}_v^{(a)} & = \widetilde{\Delta z}_v + z^0_v \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)} \\
\widetilde{\Delta z}_w^{(a)} & = \widetilde{\Delta z}_w + z^0_w \cdot \widetilde{\Delta \gamma}^{(a)} + \widetilde{\Delta \delta}^{(a)}
\end{array} \qquad (3.4)$$

The most conspicuous feature of the formulas (3.3) and (3.4) is that r and s have changed places with v and w.

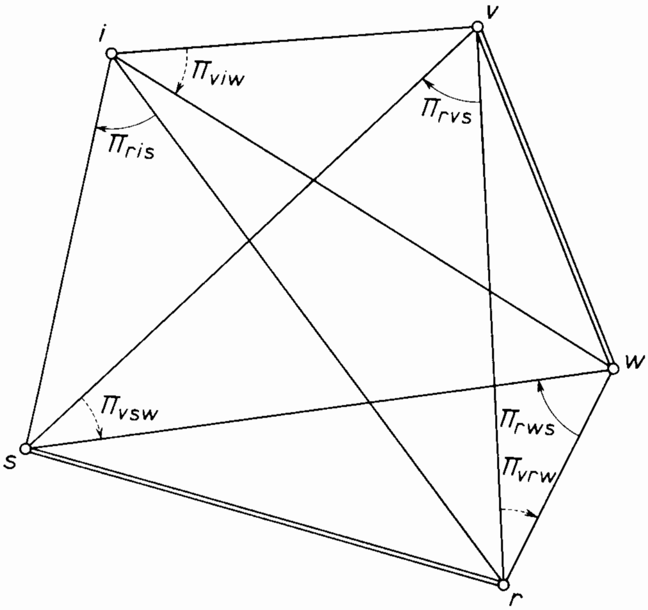


Fig. 3-1

This change of places is illustrated in figure 3-1. The determinant quantities are:

in the $\underset{r,s}{\text{S}}$-system: $\widetilde{\Delta\Pi}_{ris}$, $\widetilde{\Delta\Pi}_{rvs}$, $\widetilde{\Delta\Pi}_{rws}$

in the $\underset{v,w}{\text{S}}$-system: $\widetilde{\Delta\Pi}_{viw}$, $\widetilde{\Delta\Pi}_{vrw}$, $\widetilde{\Delta\Pi}_{vsw}$

According to the introduction of this section, (3.1) and (3.3) are essential. Therefore, there must be a transformation from one to the other, which is in fact a transformation of Π-quantities.

Now one can arrive at the formulas (3.3) by applying this S-transformation to the quantities in the $\underset{r,s}{\text{S}}$-system. By substitution of (3.1) one finds:

$$\left.\begin{aligned}
\widetilde{\Delta z}_i^{(rs)(vw)} &= \widetilde{\Delta z}_i^{(rs)} - \frac{z^0_{wi}}{z^0_{wv}}\,\widetilde{\Delta z}_v^{(rs)} - \frac{z^0_{vi}}{z^0_{vw}}\,\Delta z_w^{(rs)} = \widetilde{\Delta z}_i^{(vw)} \\
\widetilde{\Delta z}_r^{(rs)(vw)} &= \widetilde{\Delta z}_r^{(rs)} - \frac{z^0_{wr}}{z^0_{wv}}\,\widetilde{\Delta z}_v^{(rs)} - \frac{z^0_{vr}}{z^0_{vw}}\,\Delta z_w^{(rs)} = \widetilde{\Delta z}_r^{(vw)} \\
\widetilde{\Delta z}_s^{(rs)(vw)} &= \Delta z_s^{(rs)} - \frac{z^0_{ws}}{z^0_{wv}}\,\widetilde{\Delta z}_v^{(rs)} - \frac{z^0_{vs}}{z^0_{vw}}\,\widetilde{\Delta z}_w^{(rs)} = \widetilde{\Delta z}_s^{(vw)} \\
&\text{in which: } \widetilde{\Delta z}_r^{(rs)} = \widetilde{\Delta z}_s^{(rs)} = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3.5)$$

The transformation (3.5) of $\underset{r,s}{S}$- to $\underset{v,w}{S}$-system can therefore be written as:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_r^{(vw)} \\ \widetilde{\Delta z}_s^{(vw)} \end{pmatrix} = \underbrace{\begin{pmatrix} \delta_i^i & -\dfrac{z_{wi}^0}{z_{wv}^0} & -\dfrac{z_{vi}^0}{z_{vw}^0} \\ 0 & -\dfrac{z_{wr}^0}{z_{wv}^0} & -\dfrac{z_{vr}^0}{z_{vw}^0} \\ 0 & -\dfrac{z_{ws}^0}{z_{wv}^0} & -\dfrac{z_{vs}^0}{z_{vw}^0} \end{pmatrix}}_{\text{denote by: } (S_{(rs)}^{(vw)})} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(rs)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (3.6)$$

In analogy, the transformation of $\underset{v,w}{S}$-system to $\underset{r,s}{S}$-system can be written:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(rs)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} = \underbrace{\begin{pmatrix} \delta_i^i & -\dfrac{z_{si}^0}{z_{sr}^0} & -\dfrac{z_{ri}^0}{z_{rs}^0} \\ 0 & -\dfrac{z_{sv}^0}{z_{sr}^0} & -\dfrac{z_{rv}^0}{z_{rs}^0} \\ 0 & -\dfrac{z_{sw}^0}{z_{sr}^0} & -\dfrac{z_{rw}^0}{z_{rs}^0} \end{pmatrix}}_{\text{denote by: } (S_{(vw)}^{(rs)})} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_r^{(vw)} \\ \widetilde{\Delta z}_s^{(vw)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (3.7)$$

Substitution of (3.7) in (3.6) with an application of relations between coefficients like (2.6) then gives:

$$(S_{(rs)}^{(vw)}) \cdot (S_{(vw)}^{(rs)}) = \begin{pmatrix} \delta_i^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.8)$$

(3.8) shows that the S-transformations of the type (3.6) and hence (3.7) are non-singular, in addition the inverse of the transformation matrix has also been found. In this form of the transformation, the change of places between r, s and v, w is particularly clear.

This type of S-transformation is exceptionally important, because the covariance matrix of the coordinates in a newly measured network as well as the criterion matrix are constructed in an S-system, whereas a given $z^{(a)}$-system can always be reduced to an S-system by means of S-transformations as indicated in section 2.

Within a group of S-transformations the choice of the computational base is unimportant in that a change of the computational base results in a *non-singular* transformation, so that no information is lost. In particular, the comparison between two covariance matrices of coordinates of the same group of points in the same S-system {e.g. one obtained by applying the law of propagation of variances to measured quantities, the other a criterion matrix} must therefore lead to conclusions that are independent of the choice of the S-system.

Note to section 3. An alternative form of the non-singular S-transformation treated

Take from (3.6) the non-singular transformation:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_r^{(vw)} \\ \widetilde{\Delta z}_s^{(vw)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & 0 & 0 \\ 0 & -\frac{z_{wr}^0}{z_{wv}^0} & -\frac{z_{vr}^0}{z_{vw}^0} \\ 0 & -\frac{z_{ws}^0}{z_{wv}^0} & -\frac{z_{vs}^0}{z_{vw}^0} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} \quad (3.9)$$

and substitute this into the right-hand member of (3.7):

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(rs)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} = (S_{(vw)}^{(rs)}) \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_r^{(vw)} \\ \widetilde{\Delta z}_s^{(vw)} \end{pmatrix} \quad (3.10)$$

Executing the matrix multiplication concerned then results in:

$$\boxed{\begin{pmatrix} \widetilde{\Delta z}_i^{(rs)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & +\frac{z_{wi}^0}{z_{wv}^0} & +\frac{z_{vi}^0}{z_{vw}^0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix}} \quad (3.11)$$

The inversion of (3.11) is then:

$$\begin{pmatrix} \widetilde{\Delta z}_i^{(vw)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & -\frac{z_{wi}^0}{z_{wv}^0} & -\frac{z_{vi}^0}{z_{vw}^0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta z}_i^{(rs)} \\ \widetilde{\Delta z}_v^{(rs)} \\ \widetilde{\Delta z}_w^{(rs)} \end{pmatrix} \quad (3.12)$$

which formula is the equivalent of (2.8), only with the restriction that the transformation (3.12) gives the transition of quantities in the $\underset{r,s}{\text{S}}$-system to quantities in the $\underset{v,w}{\text{S}}$-system, thereby including:

$$\underline{\Delta z}_v^{(rs)} \quad \text{and} \quad \underline{\Delta z}_w^{(rs)} \quad \text{as "free" variates*)} \quad (3.13)$$

*) "Free" observation variates do not occur in condition equations of an adjustment problem, but they are correlated with {"tied"} variates occurring in these equations; see [1], section 12.

If the transformation (3.9) is made to follow (3.12), then (3.6) is obtained once more:

$$\begin{pmatrix} \Delta\tilde{z}_i^{(vw)} \\ \Delta\tilde{z}_r^{(vw)} \\ \Delta\tilde{z}_s^{(vw)} \end{pmatrix} = (S_{(rs)}^{(vw)}) \cdot \begin{pmatrix} \Delta\tilde{z}_i^{(rs)} \\ \Delta\tilde{z}_v^{(rs)} \\ \Delta\tilde{z}_w^{(rs)} \end{pmatrix} \quad \ldots\ldots \quad (3.14)$$

In this way a completely closed system of transformation formulas is obtained, always making possible the return from the $\underset{v,w}{\mathrm{S}}$-system to the $\underset{r,s}{\mathrm{S}}$-system:

a. either by using (3.11) if the computation of (3.13) is followed {compare (8.2) in [3]} (3.15)

b. or by using (3.10), and hence (3.7), if the base points P_r and P_s are included in the $\underset{v,w}{\mathrm{S}}$-system {compare (8.3) in [3]} (3.16)

Densification networks can often be computed in one or more local $\underset{v,w}{\mathrm{S}}$-systems. The subsequent computation of the covariance matrix of coordinates in the original $\underset{r,s}{\mathrm{S}}$-system was studied in [3].

In this situation, there is a practical difference between the two methods of computation (3.15) and (3.16). If the computation is done according to (3.15), the position of the base points P_r and P_s need not be known, whereas it must be known if the computation is done according to (3.16). This is of special importance if the original system has the characteristics of an (*a*)-system, as in (2.8′), rather than an $\underset{r,s}{\mathrm{S}}$-system. This situation is sometimes found in national networks, where the coordinates of the "datum point" as well as the length and the azimuth of an initial side are given a probability distribution. Such an (*a*)-system might be considered as an $\underset{r,s}{\mathrm{S}}$-system in which the base points P_r and P_s are not specified. The computation according to (3.15) in such a case prevents loss of information when an S-transformation is executed.

The computation of densification measurements in a local $\underset{v,w}{\mathrm{S}}$-system has a curious consequence for the computation of corrections to coordinates resulting from the adjustment procedure.

In the $\underset{v,w}{\mathrm{S}}$-system we have, according to (2.7):

$$\underline{\Delta z}_v^{(vw)} = \underline{\Delta z}_w^{(vw)} = 0$$

hence for correction variates $\underline{\varepsilon}$:

$$\left.\begin{aligned} &\underline{\varepsilon}_{z_v^{(vw)}} = \underline{\varepsilon}_{z_w^{(vw)}} = 0 \\ &\underline{\varepsilon}_{z_r^{(vw)}} \neq 0, \quad \underline{\varepsilon}_{z_s^{(vw)}} \neq 0 \end{aligned}\right\} \quad \ldots\ldots \quad (3.17)$$

In the $\underset{r,s}{S}$-system we have:

$$\underline{\Delta z}_r^{(rs)} = \underline{\Delta z}_s^{(rs)} = 0$$

hence:

$$\left.\begin{array}{l} \underline{\varepsilon}_{z_r^{(rs)}} = \underline{\varepsilon}_{z_r^{(rs)}} = 0 \\ \underline{\varepsilon}_{z_v^{(rs)}} \neq 0, \quad \underline{\varepsilon}_{z_w^{(rs)}} \neq 0 \end{array}\right\} \qquad (3.18)$$

This apparent contradiction is caused by the fact that in S-transformations the correction-variates of coordinates are transformed in the same way as the corresponding coordinate-variates. For example, from (3.10) follows:

$$\begin{pmatrix} \underline{\varepsilon}_{z_i^{(rs)}} \\ \underline{\varepsilon}_{z_v^{(rs)}} \\ \underline{\varepsilon}_{z_w^{(rs)}} \end{pmatrix} = (S^{(rs)}_{(vw)}) \cdot \begin{pmatrix} \underline{\varepsilon}_{z_i^{(vw)}} \\ \underline{\varepsilon}_{z_r^{(vw)}} \\ \underline{\varepsilon}_{z_s^{(vw)}} \end{pmatrix} \qquad (3.19)$$

In practical cases of densification measurements the effect (3.19) can be eliminated by assigning a pseudo covariance matrix to the given coordinates of higher order, such that the corrections to these given coordinates become zero. One of the possible methods has been elaborated in section 7.

If P_r, P_s, P_v and P_w are points of the higher-order network and P_i $\{i = \ldots\}$ are new points in the densification network then follows in this case for the computed set of pseudo least squares corrections e, see (3.19):

$$\left.\begin{array}{l} e_{z_r^{(vw)}} = e_{z_s^{(vw)}} = e_{z_v^{(rs)}} = e_{z_w^{(rs)}} = 0 \\ e_{z_i^{(rs)}} = e_{z_i^{(vw)}} \end{array}\right\} \qquad (3.20)$$

It is not strictly necessary to act according to (3.20). On the one hand, this follows from the interpretation of $\Delta z^{(rs)}$-variates as $\Delta\Pi$-variates in (2.12), so that the interpretation of "coordinates" should be used with care, and on the other hand this follows from the considerations in section 5, where, however, the assignment of corrections to given coordinates is clearly coupled to an (a)-system.

4 THE CONNECTION WITH EARLIER IDEAS

The broadening of the line of thought is best shown by referring to earlier publications, such as [3] and [5], in particular [5], because of the deceptively short and simple picture it presents.

Lagrange's interpolation formula

The execution of a similarity transformation in the form of the "connection to the points P_r, P_s by a similarity transformation" will be elaborated first.

In the case (2.10), it follows from (2.11) that:

$$\widetilde{\Delta z}_i^{(a)(rs)} - \widetilde{\Delta z}_i^{(rs)} = 0 \quad \ldots \quad (4.1)$$

or, worked out with (2.5) or (2.8):

$$(\widetilde{\Delta z}_i^{(a)} - \widetilde{\Delta z}_i) - \frac{z_{si}^0}{z_{sr}^0}(\widetilde{\Delta z}_r^{(a)} - \widetilde{\Delta z}_r) - \frac{z_{ri}^0}{z_{rs}^0}(\widetilde{\Delta z}_s^{(a)} - \widetilde{\Delta z}_s) = 0$$

or, again with (2.10):

$$\boxed{(\tilde{z}_i^{(a)} - \tilde{z}_i) = \frac{z_{si}^0}{z_{sr}^0}(\tilde{z}_r^{(a)} - \tilde{z}_r) + \frac{z_{ri}^0}{z_{rs}^0}(\tilde{z}_s^{(a)} - \tilde{z}_s)} \quad \ldots \quad (4.2)$$

i.e. Lagrange's interpolation formula.

If so desired, one can also determine $\widetilde{\Delta\gamma}^{(a)}$ and $\widetilde{\Delta\delta}^{(a)}$ from (2.11). Again assuming (2.10), one obtains:

$$\left.\begin{aligned} \tilde{z}_r^{(a)} - \tilde{z}_r &= z_r^0 \cdot \widetilde{\Delta\gamma}^{(a)} + \widetilde{\Delta\delta}^{(a)} \\ \tilde{z}_s^{(a)} - \tilde{z}_s &= z_s^0 \cdot \widetilde{\Delta\gamma}^{(a)} + \widetilde{\Delta\delta}^{(a)} \end{aligned}\right\} \quad \ldots \quad (4.3)$$

From (4.3) follows:

$$\boxed{\begin{aligned} \widetilde{\Delta\gamma}^{(a)} &= \frac{\tilde{z}_{rs}^{(a)} - \tilde{z}_{rs}}{z_{rs}^0} \\ \widetilde{\Delta\delta}^{(a)} &= \frac{z_s^0(\tilde{z}_r^{(a)} - \tilde{z}_r) - z_r^0(\tilde{z}_s^{(a)} - \tilde{z}_s)}{z_{rs}^0} \end{aligned}} \quad \ldots \quad (4.4)$$

Transition of $\underset{r,s}{\text{S}}$-system to (a)-system

From (4.1) follows:

$$\widetilde{\Delta z}_i^{(a)} - \frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_r^{(a)} - \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_s^{(a)} - \widetilde{\Delta z}_i^{(rs)} = 0$$

or:

$$\boxed{\widetilde{\Delta z}_i^{(a)} = \widetilde{\Delta z}_i^{(rs)} + \frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_r^{(a)} + \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_s^{(a)}} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.5)$$

(4.5) can be used, among other things, to return from an S-system to a system of given coordinates.

The connection with earlier definitions of S-transformations

The difference between the present development and the approach given in [3] or [5] concerns the assumption of approximate values.

Consider, in case (2.10), with (4.1):

$$\left.\begin{aligned}
\widetilde{\Delta z}_i^{(a)(rs)} &= \widetilde{\Delta z}_i^{(rs)} \\
\widetilde{\Delta z}_i^{(rs)} &= \widetilde{\Delta z}_i - \frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_r - \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_s \\
\widetilde{\Delta z}_i &= \tilde{z}_i - z_i^0 \\
\widetilde{\Delta z}_r &= \tilde{z}_r - z_r^0 \\
\widetilde{\Delta z}_s &= \tilde{z}_s - z_s^0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.6)$$

In the present theory, z_i^0, z_r^0, z_s^0 can be chosen rather arbitrarily.

In [5], and other publications, one chooses the set of observed values as approximate values, hence:

$$\left.\begin{aligned}
z_i^0 &= z_i \\
z_r^0 &= z_r \\
z_s^0 &= z_s
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.7)$$

From (4.6) and (4.7) one then obtains for the set of derived values:

$$\left.\begin{aligned}
&\Delta z_i^{(a)(rs)} = \Delta z_i^{(rs)} = 0 \\
&\Delta z_i = \Delta z_r = \Delta z_s = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.8)$$

so that a particular case is found, in which only the covariance matrix is transformed when an S-transformation is executed.

Because usually

$$\left.\begin{aligned} z_i &\neq z_i^{(a)} \\ z_r &\neq z_r^{(a)} \\ z_s &\neq z_s^{(a)} \end{aligned}\right\} \qquad (4.9)$$

the formulas relating to the actual similarity transformation, such as (4.2)–(4.5) will in general result in outcomes for the observed values *different* from zero.

The position of the similarity transformation

The difficulty indicated in the previous paragraph can be avoided by applying a similarity transformation (4.2) to the set of observed values, before working with relations in the $\underset{r,s}{S}$-system.

Instead of (4.7) one chooses then:

$$\left.\begin{aligned} z_i^0 &= z_i^{(a)} \\ z_r^0 &= z_r^{(a)} \\ z_s^0 &= z_s^{(a)} \end{aligned}\right\} \qquad (4.10)$$

With (4.10) we have for the derived observations:

$$\left.\begin{aligned} &\Delta z_i^{(a)} = \Delta z_r^{(a)} = \Delta z_s^{(a)} = 0 \\ &\Delta z_i^{(a)(rs)} = \Delta z_i^{(rs)} = 0 \end{aligned}\right\} \qquad (4.11)$$

In this case, (4.5) implies only a transformation of the covariance matrix.

This explains why the connection of the observed values by a similarity transformation to coordinates $z_r^{(a)}$ and $z_s^{(a)}$ almost always precedes the considerations about variances in an S-system. But if more than two points P_r, P_s are given in the (a)-system, one must be more careful in treating this problem of adjustment.

Note to section 4. An intuitive interpretation of S-transformations

A more intuitive line of thought, connected to the original ideas of about 1944, can serve as an aid to see through the preceding formulas describing aspects of S-transformations.

We start with:

$$\text{variates } \begin{pmatrix} \underline{z}_i^{(a)} \\ \underline{z}_r^{(a)} \\ \underline{z}_s^{(a)} \end{pmatrix} \text{ and approximate values } \begin{pmatrix} z_i^0 \\ z_r^0 \\ z_s^0 \end{pmatrix} ;\ i = \ldots \qquad (4.12)$$

Execute the S-transformation to the $\underset{r,s}{S}$-system according to (2.5) or (2.8), again written in mean values:

$$\widetilde{\Delta z}_i^{(rs)} = \widetilde{\Delta z}_i^{(a)} - \frac{z_{si}^0}{z_{sr}^0}\widetilde{\Delta z}_r^{(a)} - \frac{z_{ri}^0}{z_{rs}^0}\widetilde{\Delta z}_s^{(a)} \quad \ldots\ldots\ldots\ldots (4.13)$$

$$(\tilde{z}_i^{(rs)} - z_i^0) = (\tilde{z}_i^{(a)} - z_i^0) - \frac{z_{si}^0}{z_{sr}^0}(\tilde{z}_r^{(a)} - z_r^0) - \frac{z_{ri}^0}{z_{rs}^0}(\tilde{z}_s^{(a)} - z_s^0)$$

$$(\tilde{z}_i^{(rs)} - \tilde{z}_i^{(a)}) = \frac{z_{si}^0}{z_{sr}^0}(z_r^0 - \tilde{z}_r^{(a)}) + \frac{z_{ri}^0}{z_{rs}^0}(z_s^0 - \tilde{z}_s^{(a)}) \quad \ldots\ldots\ldots\ldots (4.14)$$

But according to (2.7) we have:

$$\left.\begin{aligned} \widetilde{\Delta z}_r^{(rs)} &= \tilde{z}_r^{(rs)} - z_r^0 = 0 \quad \rightarrow \quad \tilde{z}_r^{(rs)} = z_r^0 \\ \widetilde{\Delta z}_s^{(rs)} &= \tilde{z}_s^{(rs)} - z_s^0 = 0 \quad \rightarrow \quad \tilde{z}_s^{(rs)} = z_s^0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.15)$$

Then (4.14) with (4.15) results in:

$$\boxed{(\tilde{z}_i^{(rs)} - \tilde{z}_i^{(a)}) = \frac{z_{si}^0}{z_{sr}^0}(\tilde{z}_r^{(rs)} - \tilde{z}_r^{(a)}) + \frac{z_{ri}^0}{z_{rs}^0}(\tilde{z}_s^{(rs)} - \tilde{z}_s^{(a)})} \quad \ldots\ldots\ldots\ldots (4.16)$$

If (4.16) is compared to (4.2), it is clear that Lagrange's interpolation formula has been found, so that the S-transformation (4.13) can be *interpreted* as the connection of $z^{(a)}$-quantities from (4.12) to the approximate values z_r^0, z_s^0 of the coordinates P_r, P_s of the S-system by a similarity transformation.

r,s

In analogy with (2.1) we would consequently have:

$$\tilde{z}_i^{(rs)} = \tilde{\gamma}_{(a)}^{(rs)} \cdot \tilde{z}_i^{(a)} + \tilde{\delta}_{(a)}^{(rs)} \quad \ldots\ldots\ldots\ldots (4.17')$$

and in analogy with (4.4), in view of (4.15) and (2.10):

$$\left.\begin{aligned} -1 + \tilde{\gamma}_{(a)}^{(rs)} &= \frac{z_{rs}^0 - \tilde{z}_{rs}^{(a)}}{z_{rs}^0} = -\frac{\widetilde{\Delta z}_{rs}^{(a)}}{z_{rs}^0} \\ \tilde{\delta}_{(a)}^{(rs)} &= \frac{z_s^0(z_r^0 - \tilde{z}_r^{(a)}) - z_r^0(z_s^0 - \tilde{z}_s^{(a)})}{z_{rs}^0} = \\ &= \frac{z_s^0}{z_{sr}^0} \cdot \widetilde{\Delta z}_r^{(a)} + \frac{z_r^0}{z_{rs}^0} \cdot \widetilde{\Delta z}_s^{(a)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.17'')$$

For all derivations in this note it is essential that (2.10) holds: $z_i^{(a)}$-values must only be slightly different from z_i^0-values. Therefore the interpretation of an S-transformation as a similarity transformation must be used judiciously, because this interpretation is only valid within the range of validity of the difference equations used.

5 ADJUSTMENT TO GIVEN COORDINATES

Suppose that from the measurement of a new network coordinates z have been obtained, whereas for a number of points one also has given coordinates $z'^{(a)}$. In view of the theory of section 3, these outcomes are arranged as follows:

measured (as step I of the adjustment)	*given*	range of indices
$\underline{z}_i$	—	$i,j = \ldots$
$\underline{z}_k$	$\underline{z}'^{(a)}_k$	$k,l = \ldots$
$\underline{z}_v$	$\underline{z}'^{(a)}_v$	
$\underline{z}_w$	$\underline{z}'^{(a)}_w$	
$\underline{z}_r$	$\underline{z}'^{(a)}_r$	
$\underline{z}_s$	$\underline{z}'^{(a)}_s$	

. . . . (5.1)

The choice of approximate values z_i^0 etc. is left open. For the situation is, that the choice $z_i^0 = z_i$ (4.7) is always possible, but the choice $z_i^0 = z_i^{(a)}$ (4.10) is not, because a preceding transformation connecting the network to the points P_r, P_s, gives for the other given points, e.g.:

$$\left.\begin{array}{l} z_k \rightarrow z_k^{(a)}, \quad z_k^{(a)} \neq z_k'^{(a)} \\ \text{in which: } z_r = z_r^{(a)} = z_r'^{(a)}, \quad z_s = z_s^{(a)} = z_s'^{(a)} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (5.2)$$

To begin with, we shall therefore leave the similarity transformation, e.g. in the form (4.5), out of consideration. Because the elimination of this part of the total problem resulted in (2.11′) as the base for the adjustment model, as far as (2.10) could be applied.

Changing to the $\underset{r,s}{S}$-system, and omitting the index (a) one obtains:

Adjustment model in step II	
"free" variates	$\underline{\Delta z}_i^{(rs)}$; $i,j = \ldots$
(r,s)-condition model	$\widetilde{\Delta z}_k^{(rs)} - \widetilde{\Delta z}_k'^{(rs)} = 0$; $k,l = \ldots$
	$\widetilde{\Delta z}_v^{(rs)} - \widetilde{\Delta z}_v'^{(rs)} = 0$
	$\widetilde{\Delta z}_w^{(rs)} - \widetilde{\Delta z}_w'^{(rs)} = 0$
"free" variates	$\underline{\Delta z}_r'^{(a)}$, $\underline{\Delta z}_s'^{(a)}$

. . . . . . (5.3)

The covariance matrix of observation variates, computed for $\Delta z'$-variates with (2.8′) becomes:

	$\underline{\Delta z}_j^{(rs)}$	$\underline{\Delta z}_l^{(rs)}$ $\underline{\Delta z}_v^{(rs)}$ $\underline{\Delta z}_w^{(rs)}$	$\underline{\Delta z}_l^{\prime(rs)}$ $\underline{\Delta z}_v^{\prime(rs)}$ $\underline{\Delta z}_w^{\prime(rs)}$	$\underline{\Delta z}_r^{\prime(a)}$ $\underline{\Delta z}_s^{\prime(a)}$
$\underline{\Delta z}_i^{(rs)}$	A	B	0	0
$\underline{\Delta z}_k^{(rs)}$ $\underline{\Delta z}_v^{(rs)}$ $\underline{\Delta z}_w^{(rs)}$	B^*	C	0	0
$\underline{\Delta z}_k^{\prime(rs)}$ $\underline{\Delta z}_v^{\prime(rs)}$ $\underline{\Delta z}_w^{\prime(rs)}$	0	0	D	E^*
$\underline{\Delta z}_r^{\prime(a)}$ $\underline{\Delta z}_s^{\prime(a)}$	0	0	E	F

(5.4)

Because in almost all cases there will be no correlation between z- and $z'^{(a)}$-variates, (5.4) seems to be sufficiently general.*) The matrices D, E and F will usually be derived from the theory of criterion matrices.

Then the results of step II will be in the most general case:

$$\underbrace{\begin{pmatrix} \underline{\Delta z}_{i.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{k.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{v.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{w.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{k.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{v.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{w.\mathrm{II}}^{(rs)} \\ \underline{\Delta z}_{r.\mathrm{II}}^{(a)} \\ \underline{\Delta z}_{s.\mathrm{II}}^{(a)} \end{pmatrix}}_{\text{vector 1}} = \underbrace{\begin{pmatrix} \underline{\Delta z}_i^{(rs)} \\ \underline{\Delta z}_k^{(rs)} \\ \underline{\Delta z}_v^{(rs)} \\ \underline{\Delta z}_w^{(rs)} \\ \underline{\Delta z}_k^{\prime(rs)} \\ \underline{\Delta z}_v^{\prime(rs)} \\ \underline{\Delta z}_w^{\prime(rs)} \\ \underline{\Delta z}_r^{\prime(a)} \\ \underline{\Delta z}_s^{\prime(a)} \end{pmatrix}}_{\text{vector 2}} + \begin{pmatrix} -B \\ -C \\ +D \\ +E \end{pmatrix} \cdot \begin{pmatrix} C+D \end{pmatrix}^{-1} \cdot \underbrace{\begin{pmatrix} \underline{\Delta z}_k^{(rs)} - \underline{\Delta z}_k^{\prime(rs)} \\ \underline{\Delta z}_v^{(rs)} - \underline{\Delta z}_v^{\prime(rs)} \\ \underline{\Delta z}_w^{(rs)} - \underline{\Delta z}_w^{\prime(rs)} \end{pmatrix}}_{\text{vector 3}} \quad \ldots\ldots \quad (5.5)$$

*) If certain measurements of directions and distances from the network which generated the "given" coordinates $z'^{(a)}$ are used again as oberservations in the network which generates the coordinates z, then correlation will occur. Therefore it is recommended to avoid this situation.

(5.5) is extremely useful for theoretical studies. But for the *practical execution* of an adjustment, the vector 3 will usually be written in a different form, for which (5.2) is followed because otherwise $z_r^{(a)}$ and $z_s^{(a)}$ cannot be defined.

(4.5) applied to vector 3 in (5.5) gives then:

$$
\begin{aligned}
\underline{\Delta z}_k^{(rs)} - \underline{\Delta z}_k^{\prime(rs)} &= \left\{ \underline{\Delta z}_k^{(a)} - \frac{z^0_{sk}}{z^0_{sr}} \underline{\Delta z}_r^{\prime(a)} - \frac{z^0_{rk}}{z^0_{rs}} \underline{\Delta z}_s^{\prime(a)} \right\} + \\
&\quad - \left\{ \underline{\Delta z}_k^{\prime(a)} - \frac{z^0_{sk}}{z^0_{sr}} \underline{\Delta z}_r^{\prime(a)} - \frac{z^0_{rk}}{z^0_{rs}} \underline{\Delta z}_s^{\prime(a)} \right\} = \\
&= \underline{\Delta z}_k^{(a)} - \underline{\Delta z}_k^{\prime(a)} = \\
&= \underline{z}_k^{(a)} - \underline{z}_k^{\prime(a)} \qquad \qquad (5.6)
\end{aligned}
$$

(5.6) means a transformation to connect the z-system in the points P_r and P_s, i.e. for the set of coordinates, because the stochastic addition of the effect of connection of $\underline{\Delta z}_r^{\prime(a)}$ and $\underline{\Delta z}_s^{\prime(a)}$ vanishes in the difference.

This is now the line of thought of (4.10), i.e. the z-system is computed {defined} *after* connecting to $\underline{z}_r^{\prime(a)}$ and $\underline{z}_s^{\prime(a)}$ omitting the stochastic effect of this. Approximate values are to be chosen as in (5.2):

$z_i \to z_i^{(a)}$	$z_i^0 = z_i^{(a)}$		
$z_k \to z_k^{(a)}$	$z_k^0 = z_k^{(a)}$	$\neq z_k^{\prime(a)}$	
$z_v \to z_v^{(a)}$	$z_v^0 = z_v^{(a)}$	$\neq z_v^{\prime(a)}$	
$z_w \to z_w^{(a)}$	$z_w^0 = z_w^{(a)}$	$\neq z_w^{\prime(a)}$	(5.7)
$z_r \to z_r^{\prime(a)}$	$z_r^0 = z_r^{\prime(a)}$		
$z_s \to z_s^{\prime(a)}$	$z_s^0 = z_s^{\prime(a)}$		

With (5.7) we have for the derived observations:

$\Delta z_i^{(a)} = 0$		$\Delta z_i^{(a)(rs)} = \Delta z_i^{(a)}$		
$\Delta z_k^{(a)} = 0$	$\Delta z_k^{\prime(a)} = z_k^{\prime(a)} - z_k^{(a)}$	$\Delta z_k^{(a)(rs)} = \Delta z_k^{(a)}$	$\Delta z_k^{\prime(a)(rs)} = \Delta z_k^{\prime(a)}$	
$\Delta z_v^{(a)} = 0$	$\Delta z_v^{\prime(a)} = z_v^{\prime(a)} - z_v^{(a)}$	$\Delta z_v^{(a)(rs)} = \Delta z_v^{(a)}$	$\Delta z_v^{\prime(a)(rs)} = \Delta z_v^{\prime(a)}$	(5.8)
$\Delta z_w^{(a)} = 0$	$\Delta z_w^{\prime(a)} = z_w^{\prime(a)} - z_w^{(a)}$	$\Delta z_w^{(a)(rs)} = \Delta z_w^{(a)}$	$\Delta z_w^{\prime(a)(rs)} = \Delta z_w^{\prime(a)}$	
$\Delta z_r^{\prime(a)} = 0$	$\Delta z_r^{\prime(a)} = 0$			
$\Delta z_s^{\prime(a)} = 0$	$\Delta z_s^{\prime(a)} = 0$			

Seen in this light, (5.8) has the same effect as (5.6) for the observation set.

This means that for the numerical computation – the stochastic situation remains the same as in (5.5) – one can introduce in (5.5) {approximate values in vector 1 are of course the same as in vector 2}:

Observation set in (5.5)		
vector 1 {see also (5.11)}	vector 2	vector 3
$z_{i.II}^{(a)(rs)}$	$z_i^{(a)}$	
$z_{k.II}^{(a)(rs)}$	$z_k^{(a)}$	
$z_{v.II}^{(a)(rs)}$	$z_v^{(a)}$	$z_k^{(a)} - z_k'^{(a)}$
$z_{w.II}^{(a)(rs)}$	$z_w^{(a)}$	
$z_{k.II}^{(a)(rs)}$	$z_k'^{(a)}$	$z_v^{(a)} - z_v'^{(a)}$
$z_{v.II}^{(a)(rs)}$	$z_v'^{(a)}$	
$z_{w.II}^{(a)(rs)}$	$z_w'^{(a)}$	$z_w^{(a)} - z_w'^{(a)}$
$z_{r.II}^{(a)}$	$z_r'^{(a)}$	
$z_{s.II}^{(a)}$	$z_s'^{(a)}$	

. . . . (5.9)

For the notation $z_{i.II}^{(a)(rs)}$ instead of $z_{i.II}^{(rs)}$, see the text after (5.10).

The transformation of $\underset{r,s}{\text{S}}$-system to (a)-system can be executed afterwards analogously to (4.5), hence as a continuation of (5.5):

$$\begin{pmatrix} \underline{\Delta z_{i.II}^{(a)}} \\ \underline{\Delta z_{k.II}^{(a)}} \\ \underline{\Delta z_{v.II}^{(a)}} \\ \underline{\Delta z_{w.II}^{(a)}} \\ \underline{\Delta z_{r.II}^{(a)}} \\ \underline{\Delta z_{s.II}^{(a)}} \end{pmatrix} = \begin{pmatrix} \delta_i^i & 0 & 0 & 0 & \dfrac{z_{si}^{(a)}}{z_{sr}^{(a)}} & \dfrac{z_{ri}^{(a)}}{z_{rs}^{(a)}} \\ 0 & \delta_k^k & 0 & 0 & \dfrac{z_{sk}^{(a)}}{z_{sr}^{(a)}} & \dfrac{z_{rk}^{(a)}}{z_{rs}^{(a)}} \\ 0 & 0 & \delta_v^v & 0 & \dfrac{z_{sv}^{(a)}}{z_{sr}^{(a)}} & \dfrac{z_{rv}^{(a)}}{z_{rs}^{(a)}} \\ 0 & 0 & 0 & \delta_w^w & \dfrac{z_{sw}^{(a)}}{z_{sr}^{(a)}} & \dfrac{z_{rw}^{(a)}}{z_{rs}^{(a)}} \\ 0 & 0 & 0 & 0 & \delta_r^r & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_s^s \end{pmatrix} \cdot \begin{pmatrix} \underline{\Delta z_{i.II}^{(rs)}} \\ \underline{\Delta z_{k.II}^{(rs)}} \\ \underline{\Delta z_{v.II}^{(rs)}} \\ \underline{\Delta z_{w.II}^{(rs)}} \\ \underline{\Delta z_{r.II}^{(a)}} \\ \underline{\Delta z_{s.II}^{(a)}} \end{pmatrix} \quad \ldots\ldots (5.10)$$

(5.10) gives, apart from a stochastic transformation, also in general a transformation of the observation set. This is the reason for the notation with vector 1 in (5.9).

In most cases in *practice*, given coordinates in densification networks will not be given corrections, see section 7. In this case (5.10) means only a stochastic transformation, and the observation set will *not* be altered. Then the notation of vector 1 in (5.9) can be simplified:

Given coordinates not corrected, see section 7	
(5.9)	$z^{(a)(rs)} \rightarrow z^{(a)}$

. . . . (5.11)

6 THE APPLICATION OF NON-SINGULAR S-TRANSFORMATIONS TO THE ADJUSTMENT MODEL

Although it is not essential for the following discussion, one can always add to the "free" variates in the last row of (5.3) the variates:

$$\left.\begin{array}{ll} \underline{\Delta z}_k'^{(a)} & ; \quad k, l = \ldots \\ \underline{\Delta z}_v'^{(a)}, \quad \underline{\Delta z}_w'^{(a)} & \end{array}\right\} \qquad (6.1)$$

Next, a non-singular S-transformation of the type (3.6) is applied, if necessary extended by a unit matrix on account of the "free" variates of the type (6.1):

$$\begin{pmatrix} \underline{\Delta z}_i^{(vw)} \\ \underline{\Delta z}_k^{(vw)} \\ \underline{\Delta z}_r^{(vw)} \\ \underline{\Delta z}_s^{(vw)} \\ \underline{\Delta z}_k'^{(vw)} \\ \underline{\Delta z}_r'^{(vw)} \\ \underline{\Delta z}_s'^{(vw)} \\ \underline{\Delta z}_v'^{(a)} \\ \underline{\Delta z}_w'^{(a)} \\ \underline{\Delta z}_r'^{(a)} \\ \underline{\Delta z}_s'^{(a)} \\ \underline{\Delta z}_k'^{(a)} \end{pmatrix} = \left(\begin{array}{c|c|c} S_{(rs)}^{(vw)} & 0 & 0 \\ \hline 0 & S_{(rs)}^{(vw)} & 0 \\ \hline 0 & 0 & \begin{array}{ccccc} \delta_v^v & 0 & 0 & 0 & 0 \\ 0 & \delta_w^w & 0 & 0 & 0 \\ 0 & 0 & \delta_r^r & 0 & 0 \\ 0 & 0 & 0 & \delta_s^s & 0 \\ 0 & 0 & 0 & 0 & \delta_k^k \end{array} \end{array}\right) \cdot \begin{pmatrix} \underline{\Delta z}_i^{(rs)} \\ \underline{\Delta z}_k^{(rs)} \\ \underline{\Delta z}_v^{(rs)} \\ \underline{\Delta z}_w^{(rs)} \\ \underline{\Delta z}_k'^{(rs)} \\ \underline{\Delta z}_v'^{(rs)} \\ \underline{\Delta z}_w'^{(rs)} \\ \underline{\Delta z}_v'^{(a)} \\ \underline{\Delta z}_w'^{(a)} \\ \underline{\Delta z}_r'^{(a)} \\ \underline{\Delta z}_s'^{(a)} \\ \underline{\Delta z}_k'^{(a)} \end{pmatrix} \qquad (6.2)$$

Then, (5.3) is transformed into:

Adjustment model in step II	
"free" variates	$\underline{\Delta z}_i^{(vw)}$; $i, j = \ldots$
(v, w)-condition model	$\widetilde{\Delta z}_k^{(vw)} - \widetilde{\Delta z}_k'^{(vw)} = 0$; $k, l = \ldots$ $\widetilde{\Delta z}_r^{(vw)} - \widetilde{\Delta z}_r'^{(vw)} = 0$ $\widetilde{\Delta z}_s^{(vw)} - \widetilde{\Delta z}_s'^{(vw)} = 0$
"free" variates	$\underline{\Delta z}_v'^{(a)}$, $\underline{\Delta z}_w'^{(a)}$ and, if desired: $\underline{\Delta z}_r'^{(a)}$, $\underline{\Delta z}_s'^{(a)}$, $\underline{\Delta z}_k'^{(a)}$

. . . . . . (6.3)

It is not necessary to write out the results of the transformation (6.2) applied to (5.4) and (5.5). For the results of an adjustment are invariant with respect to a non-singular transformation like (6.2). Of course the superscripts (rs) and (vw) are essential because they indicate different variates. Only *after* the transformation (5.10) and the analogous one with (6.3), these differences vanish, because then only superscript (a) occurs.

Presumably, the part of the adjustment problem above the bold line in (5.3) and (6.3) respectively, is the most essential, because one works here entirely in and $\overset{r,s}{\text{S}}$-, respectively an $\underset{v,w}{\text{S}}$-system; one system can be derived from the other by an $S_{(rs)}^{(vw)}$ and an $S_{(vw)}^{(rs)}$-transformation respectively, Here, the remarks given after (3.8) are also valid.

7 THE ADJUSTMENT TO GIVEN COORDINATES WITHOUT CORRECTING THEM

There is a great practical advantage in keeping the coordinates of "given" points fixed, i.e. in not giving them corrections. This has also the more or less theoretical advantage that the matrices (D), (E) and (F) in (5.4) remain unaltered. The latter advantage is decisive if these matrices are derived from the criterium matrix chosen!

The effect of not correcting the given coordinates should be evenly distributed over the net. Therefore, preference is given to the pseudo covariance matrix according to method **II** in (5.101) of [1].

This implies for (5.4) the "rule of thumb" (5.103) of [1]:

$$(D) \to (0); \quad (E) \to (0); \quad (F) \to (0) \qquad \text{(7.1)}$$

The matrices (A), (B) and (C) can, if so desired, be replaced by pseudo variance matrices, provided that essential properties are maintained. For example, one may use a criterion matrix in the $\underset{r,s}{\text{S}}$-system.

With (7.1), (5.5) becomes:

$$\underset{\text{vector 1}}{\begin{pmatrix} \underline{\Delta z}^{(rs)}_{i.\mathrm{II}} \\ \underline{\Delta z}'^{(rs)}_{k} \\ \underline{\Delta z}'^{(rs)}_{v} \\ \underline{\Delta z}'^{(rs)}_{w} \\ \underline{\Delta z}'^{(rs)}_{k} \\ \underline{\Delta z}'^{(rs)}_{v} \\ \underline{\Delta z}'^{(rs)}_{w} \\ \underline{\Delta z}'^{(a)}_{r} \\ \underline{\Delta z}'^{(a)}_{s} \end{pmatrix}} = \underset{\text{vector 2}}{\begin{pmatrix} \underline{\Delta z}^{(rs)}_{i} \\ \underline{\Delta z}^{(rs)}_{k} \\ \underline{\Delta z}^{(rs)}_{v} \\ \underline{\Delta z}^{(rs)}_{w} \\ \underline{\Delta z}'^{(rs)}_{k} \\ \underline{\Delta z}'^{(rs)}_{v} \\ \underline{\Delta z}'^{(rs)}_{w} \\ \underline{\Delta z}'^{(a)}_{r} \\ \underline{\Delta z}'^{(a)}_{s} \end{pmatrix}} + \begin{pmatrix} -B \\ \hline -C \\ \hline 0 \\ \hline 0 \end{pmatrix} \cdot \begin{pmatrix} C \end{pmatrix}^{-1} \cdot \underset{\text{vector 3}}{\begin{pmatrix} \underline{\Delta z}^{(rs)}_{k} - \underline{\Delta z}'^{(rs)}_{k} \\ \underline{\Delta z}^{(rs)}_{v} - \underline{\Delta z}'^{(rs)}_{v} \\ \underline{\Delta z}^{(rs)}_{w} - \underline{\Delta z}'^{(rs)}_{w} \end{pmatrix}} \qquad \text{(7.2)}$$

If now the line of thought (5.6)–(5.9) is applied, the following is valid for the observation set:

$$\left.\begin{aligned} \Delta z^{(a)}_{r.\mathrm{II}} &= \Delta z'^{(a)}_{r} = 0 \\ \Delta z^{(a)}_{s.\mathrm{II}} &= \Delta z'^{(a)}_{s} = 0 \end{aligned}\right\} \qquad \text{(7.3)}$$

so that the application of (5.10) then only means a transformation of the covariance matrix; compare (5.11).

(7.2) gives pseudo least-squares estimators. In order to obtain their covariance matrix, the law of propagation of variances must be applied, making use of the complete matrix (5.4), established in accordance with the most reliable information.

8 THE COMPARISON OF A COVARIANCE MATRIX {OF COORDINATES} WITH A CRITERION MATRIX

Compare two stochastic vectors $(\underset{\sim}{z})$ and $(\underset{\sim}{z}')$ with identical sample values but different covariance matrices G and λH. The difference in model parameters {now only a stochastic difference for the corresponding variates} $\widetilde{\Delta\gamma}$ and $\widetilde{\Delta\delta}$ is eliminated by transformation to a S-system {see section 2}.

r,s

Using a symbolic z-notation for coordinates, one obtains:

$$\overline{(\Delta z^{(rs)}),(\Delta z^{(rs)})^*} = G^{(rs)} \quad \text{(8.1)}$$

$$\overline{(\Delta z'^{(rs)}),(\Delta z'^{(rs)})^*} = \lambda H^{(rs)} \quad \text{(8.2)}$$

In (8.2), a {positive} scale factor λ has been introduced, because on the one hand the theory to be developed can serve to scale the matrix H with respect to G, and on the other hand the matrix G can be tested with respect to the matrix H when putting $\lambda = 1$.

Consider now an *arbitrary* linear function:

$$\underline{\Delta F} = \Lambda \cdot (\underline{\Delta z}^{(rs)}) \quad \text{(8.3)}$$

hence also:

$$\underline{\Delta F}' = \Lambda \cdot (\underline{\Delta z}'^{(rs)}) \quad \text{(8.4)}$$

and require for any such linear function:

$$\overline{\Delta F, \Delta F} \leqslant \overline{\Delta F', \Delta F'} \quad \text{(8.5)}$$

(8.1)–(8.5) then gives:

$$\Lambda \cdot (G^{(rs)} - \lambda \cdot H^{(rs)}) \cdot \Lambda^* \leqslant 0 \quad \text{(8.6')}$$

(8.6′) means that the matrix:

$$(G^{(rs)} - \lambda \cdot H^{(rs)}) \quad \text{(8.6'')}$$

must be *negative semi-definite*

For simplicity, we deviate slightly from the notation in section 13 of [1], and write for the Choleskey decomposition of $H^{(rs)}$:

$$H^{(rs)} = \Gamma^{-1}\cdot\Gamma^{*-1} \quad \ldots \quad (8.7)$$

Standardization of the $\Delta z'^{(rs)}$-sample space is obtained by the transformations:

$$\left.\begin{aligned} (\underline{\Delta \bar{z}^{(rs)}}) &= \Gamma\cdot(\underline{\Delta z^{(rs)}}) \\ (\underline{\Delta \bar{z}'^{(rs)}}) &= \Gamma\cdot(\underline{\Delta z'^{(rs)}}) \end{aligned}\right\} \quad \ldots \quad (8.8)$$

(8.6) then becomes, with the unit matrix I:

$$\Lambda\cdot\Gamma^{-1}\cdot(\Gamma\cdot G^{(rs)}\cdot\Gamma^{*}-\lambda\cdot I)\cdot\Gamma^{*-1}\cdot\Lambda^{*}\leqslant 0 \quad \ldots \quad (8.9)$$

Now apply an orthogonal transformation, such that the matrix $(\Gamma\cdot G^{(rs)}\cdot\Gamma^{*})$ is reduced to the diagonal matrix:

$$\mathrm{dg}[\lambda_1,\ldots,\lambda_n] \quad \ldots \quad (8.10)$$

hence by the orthogonal transformations:

$$\left.\begin{aligned} &C\cdot(\underline{\Delta \bar{z}^{(rs)}}) \\ &C\cdot(\underline{\Delta \bar{z}'^{(rs)}}), \quad \text{with} \quad C^{-1}=C^{*} \end{aligned}\right\} \quad \ldots \quad (8.11)$$

Then (8.9) becomes:

$$\Lambda\cdot\Gamma^{-1}C^{-1}(C\cdot\Gamma\cdot G^{(rs)}\cdot\Gamma^{*}\cdot C^{*}-\lambda\cdot I)\cdot C^{*-1}\cdot\Gamma^{*-1}\cdot\Lambda^{*}\leqslant 0$$

or, with (8.10)

$$(\Lambda\cdot\Gamma^{-1}\cdot C^{-1})\cdot\mathrm{dg}[\lambda_1-\lambda,\ldots,\lambda_n-\lambda]\cdot(\Lambda\cdot\Gamma^{-1}\cdot C^{-1})^{*}\leqslant 0 \quad \ldots \quad (8.12')$$

or:

$$\mathrm{dg}[\lambda_1-\lambda,\ldots,\lambda_n-\lambda] \quad \text{negative semi-definite} \quad \ldots \quad (8.12'')$$

(8.12) is fulfilled by:*)

$$\boxed{\lambda \geqslant \{\lambda_i\}_{\max}} \quad \ldots \quad (8.13)$$

$\lambda_1, \ldots, \lambda_i, \ldots, \lambda_n$ can be computed directly as the eigenvalues of the matrix (8.6'').

---

*) See, e.g., L. Mirsky – An Introduction to Linear Algebra – Oxford, 1955, {in particular chapter 13}.

Before continuing, it should be remarked that (8.12) and therefore also (8.13) are invariant with respect to a non-singular transformation, apart from a possible change in the ordening of the eigenvalues $\lambda_1, \ldots, \lambda_n$, see e.g. the book by MIRSKY, pages 119 and 199.

Now let us consider:

$$\left.\begin{array}{l} (\underline{\Delta z}^{(vw)}) = S\cdot(\underline{\Delta z}^{(rs)}) \\ (\underline{\Delta z}'^{(vw)}) = S\cdot(\underline{\Delta z}'^{(rs)}) \end{array}, \quad S \equiv (S_{(rs)}^{(vw)})\right\} \qquad (8.14)$$

Then (8.6) becomes:

$$A\cdot S^{-1}\cdot(S\cdot G^{(rs)}\cdot S^{*}-\lambda\cdot S\cdot H^{(rs)}\cdot S^{*})\cdot S^{*-1}\cdot A^{*} =$$

$$= (A\cdot S^{-1})\cdot(G^{(vw)}-\lambda\cdot H^{(vw)})\cdot(A\cdot S^{-1})^{*} \leqslant 0 \qquad (8.15)$$

Let:

$$H^{(vw)} = \bar{\Gamma}^{-1}\cdot\bar{\Gamma}^{*-1} \qquad (8.16)$$

then the result of the standardization of the $\Delta z'^{(vw)}$-sample space and the subsequent orthogonal transformation can be written as:

$$\bar{C}\cdot\bar{\Gamma}\cdot(\underline{\Delta z}^{(vw)}) \qquad \text{resp.} \qquad \bar{C}\cdot\bar{\Gamma}\cdot(\underline{\Delta z}'^{(vw)}) \qquad (8.17)$$

such, that for (8.15) we have:

$$(A\cdot S^{-1}\cdot\bar{\Gamma}^{-1}\cdot\bar{C}^{-1})\cdot \mathrm{dg}[\lambda_1'-\lambda, \ldots, \lambda_n'-\lambda]\cdot(A\cdot S^{-1}\cdot\bar{\Gamma}^{-1}\cdot\bar{C}^{-1})^{*} \leqslant 0 \qquad (8.18)$$

Up to, at most, an orthogonal transformation, the standardizations in (8.8) and (8.17) give identical results, so that:

$$(\underline{\Delta \bar{z}}^{(vw)}) = \bar{\bar{C}}\cdot(\underline{\Delta \bar{z}}^{(rs)})$$

or:

$$\bar{C}\cdot\bar{\Gamma}\cdot S\cdot(\underline{\Delta z}^{(rs)}) = \bar{\bar{C}}\cdot\bar{C}\cdot\Gamma\cdot(\underline{\Delta z}^{(rs)})$$

or with (8.11):

$$= C\cdot\Gamma\cdot(\underline{\Delta z}^{(rs)}) \qquad (8.19)$$

From (8.19) it follows indeed, with (8.12) and (8.18), that:

$$\boxed{\begin{array}{l} \mathrm{dg}[\lambda_1'-\lambda, \ldots, \lambda_n'-\lambda] = \mathrm{dg}[\lambda_1-\lambda, \ldots, \lambda_n-\lambda] \\ \text{up to the ordering of the diagonal terms} \end{array}} \qquad (8.20)$$

(8.13) can now be used for two purposes:

I: *Given* $G^{(rs)}$, $H^{(rs)}$. *Sought* λ
Compute maximum eigenvalue from (8.6) and
choose λ from $\lambda \geqslant \{\lambda_i\}_{\max}$ . . . . . . (8.21)
This is the "scaling" of the criterion matrix.

II: *Given* $G^{(rs)}$ {from reconnaissance of the network}
and $H^{(rs)}$, already "scaled", so that $\lambda = 1$.
Problem: does $G^{(rs)}$ fulfill (8.6)?
Compute maximum eigenvalue from (8.6) and . . . . . . (8.22)
check if $\{\lambda_i\}_{\max} \leqslant 1$
This is a test on the required precision of the network investigated.

In both cases one obtains the situation of Fig. 8-1.

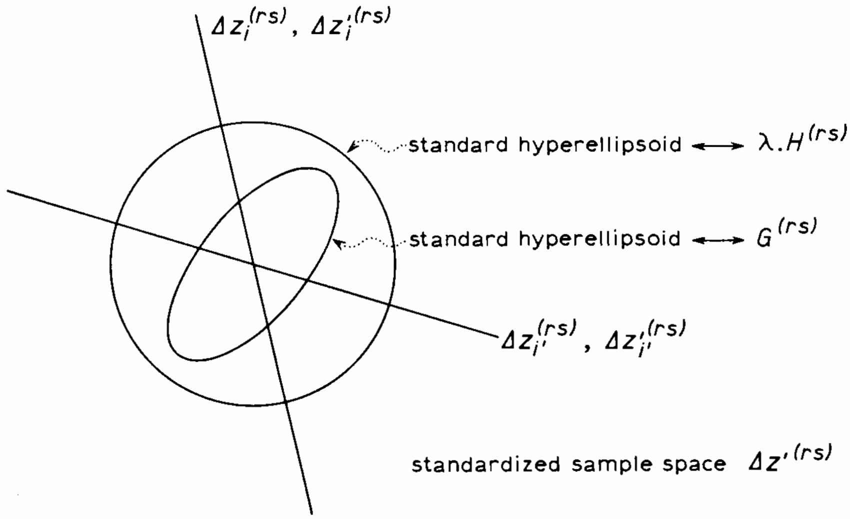


Fig. 8-1

Note to section 8. Change of the form of $H^{(rs)}$

In section 15 ff. it will be shown that the criterion matrix $H^{(rs)}$ is constructed out of formules containing several parameters. A change of the parameter values results in a change of the form of the standard hyperellipsoid belonging to the matrix $H^{(rs)}$, or, briefly, changes the form of $H^{(rs)}$.

This is now applied to the situation in Fig. 8-1, illustrating the course of things in (8.8)–(8.13). The inner standard hyperellipsoid in Fig. 8-1 refers to the covariance matrix:

$$(C \cdot \Gamma \cdot G^{(rs)} \cdot \Gamma^* \cdot C^*) = \mathrm{dg}\,[\lambda_1, \ldots, \lambda_n]$$

so that the eigenvalues $\lambda_1, \ldots, \lambda_n$ represent the squares of the semi-axes. The outer standard hyperellipsoid {here a hypersphere} in Fig. 8-1 refers to the covariance matrix:

$$\lambda \cdot (C \cdot \Gamma \cdot H^{(rs)} \cdot \Gamma^* \cdot C^*) = \lambda \cdot (I)$$

so that λ represents the square of the hypersphere radius.

The forms of the two hyperellipsoids will agree better with each other as the ratios of the eigenvalues $\lambda_1, \ldots, \lambda_n$ approach 1. The same applies to the two standard hyperellipsoids belonging to $H^{(rs)}$ and $G^{(rs)}$, or shorter, the form of $H^{(rs)}$ then corresponds closer to $G^{(rs)}$. A possible criterion of "fit" can be:

Form of $H^{(rs)}$ agrees better with form of $G^{(rs)}$
as: $\dfrac{\{\lambda_i\}_{\max}}{\{\lambda_i\}_{\min}} \rightarrow \text{smaller} \rightarrow 1$

. . . . (8.23)

9 THE COMPARISON OF A COVARIANCE MATRIX WITH A CRITERION MATRIX IN AN ADJUSTMENT TO GIVEN COORDINATES

As in section 8, primes will be used for z-variates to which the criterion matrix has been assigned to serve as their covariance matrix; this is, provisionally, contrary to the prime-notation in sections 5–7. Furthermore, the splitting up of the indices k, v, w can be abandoned, so that, with the addition of other given points P_m, P_v, P_w, (5.1) becomes:

measured (as step I of the adjustment problem	*given*	refers to criterion matrix	range of indices
$\underline{z}_i$		$\underline{\Delta z}_i'^{(rs)}$	$i, j\ = \dots$
$\underline{z}_k$	$\underline{z}_k^{(a)}$	$\underline{\Delta z}_k'^{(rs)}$	$k, l = \dots$
$\underline{z}_r$	$\underline{z}_r^{(a)}$		
$\underline{z}_s$	$\underline{z}_s^{(a)}$		
	$\underline{z}_m^{(a)}$	$\underline{\Delta z}_m'^{(rs)}$	$m, n = \dots$
	$\underline{z}_v^{(a)}$	$\underline{\Delta z}_v'^{(rs)}$	
	$\underline{z}_w^{(a)}$	$\underline{\Delta z}_w'^{(rs)}$	

. . . . (9.1)

In Fig. 9-1 the situation envisaged in (9.1) is illustrated. A new network {of arbitrary shape} contains new points P_i and given points P_k and P_r, P_s. Apart from these given points there are many more given points, the nearest of which, P_m, have been indicated, as well as the base points P_v, P_w, used previously.

Now assume that the coordinates $\underline{z}^{(a)}$ in the second column in the $\underset{v,w}{\mathrm{S}}$-system fulfill (8.22), or symbolically:

$$\overline{\begin{bmatrix} \Delta z_k^{(a)(vw)} \\ \Delta z_r^{(a)(vw)} \\ \Delta z_s^{(a)(vw)} \\ \Delta z_m^{(a)(vw)} \end{bmatrix}, \begin{bmatrix} \Delta z_l^{(a)(vw)} \\ \Delta z_r^{(a)(vw)} \\ \Delta z_s^{(a)(vw)} \\ \Delta z_n^{(a)(vw)} \end{bmatrix}^*} \leqslant \overline{\begin{bmatrix} \Delta z_k'^{(vw)} \\ \Delta z_r'^{(vw)} \\ \Delta z_s'^{(vw)} \\ \Delta z_m'^{(vw)} \end{bmatrix}, \begin{bmatrix} \Delta z_l'^{(vw)} \\ \Delta z_r'^{(vw)} \\ \Delta z_s'^{(vw)} \\ \Delta z_n'^{(vw)} \end{bmatrix}^*} \quad \dots\dots\dots\dots \quad (9.2)$$

then, according to section 8, we have in the $\underset{r,s}{\mathrm{S}}$-system:

$$\overline{\begin{pmatrix}\Delta z_k^{(a)(rs)}\\ \Delta z_m^{(a)(rs)}\\ \Delta z_v^{(a)(rs)}\\ \Delta z_w^{(a)(rs)}\end{pmatrix}, \begin{pmatrix}\Delta z_l^{(a)(rs)}\\ \Delta z_n^{(a)(rs)}\\ \Delta z_v^{(a)(rs)}\\ \Delta z_w^{(a)(rs)}\end{pmatrix}^*} \leqslant \overline{\begin{pmatrix}\Delta z_k'^{(rs)}\\ \Delta z_m'^{(rs)}\\ \Delta z_v'^{(rs)}\\ \Delta z_w'^{(rs)}\end{pmatrix}, \begin{pmatrix}\Delta z_l'^{(rs)}\\ \Delta z_n'^{(rs)}\\ \Delta z_v'^{(rs)}\\ \Delta z_w'^{(rs)}\end{pmatrix}^*} \quad \ldots\ldots\ldots\ldots \quad (9.3)$$

so that from the point of view of precision no corrections need be applied to the given coordinates in the new adjustment problem which serves the determination of the points P_k. Hence the method of section 7 can be applied, in which for the simplification of the computation no pseudo covariances in the matrices A, B and C in (5.4) and hence (7.2) are used.

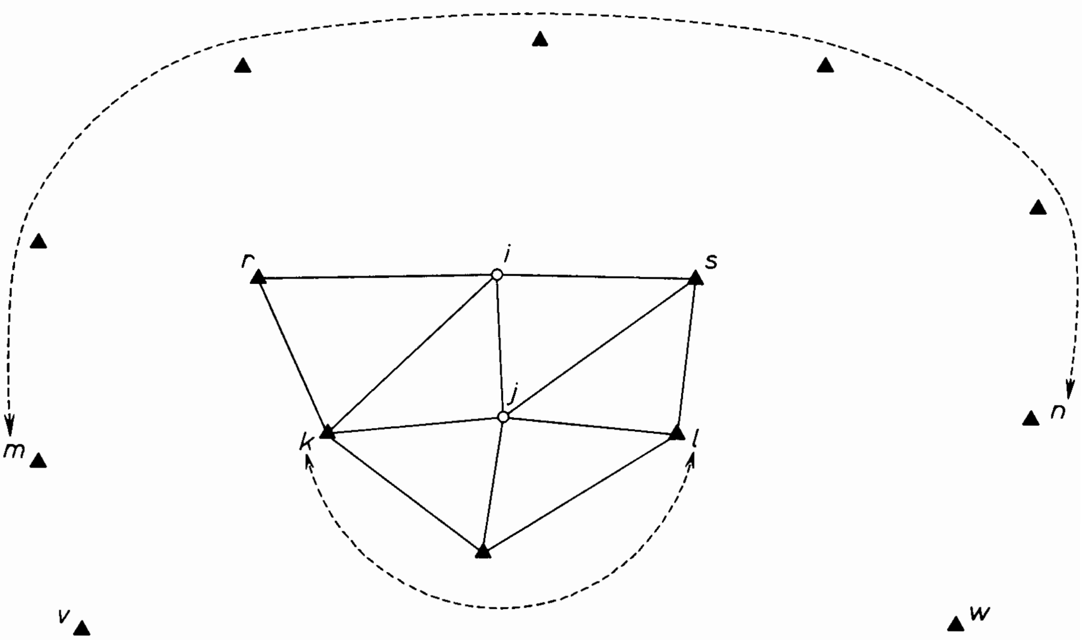


Fig. 9-1

If now the base points P_v, P_w are left out of consideration, the result (7.2) can be written in the form:

$$\begin{pmatrix}\underline{\Delta z}_{i.\mathrm{II}}^{(rs)}\\ \underline{\Delta z}_k^{(a)(rs)}\\ \underline{\Delta z}_m^{(a)(rs)}\end{pmatrix} = \begin{pmatrix}\delta_i^i & -\Lambda_k^i & +\Lambda_k^i & 0\\ 0 & 0 & \delta_k^k & 0\\ 0 & 0 & 0 & \delta_m^m\end{pmatrix} \cdot \begin{pmatrix}\underline{\Delta z}_i^{(rs)}\\ \underline{\Delta z}_k^{(rs)}\\ \underline{\Delta z}_k^{(a)(rs)}\\ \underline{\Delta z}_m^{(a)(rs)}\end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (9.4)$$

Usually one will not have the matrix D {or E or F} in (5.4) at his disposal. Instead of the left-hand member of (9.3) one takes as safe estimate the right-hand member of (9.3). In this case, (5.4), as far as it refers to (9.4) can be written in symbolic form:

	$\underline{\Delta z}_j^{(rs)}$	$\underline{\Delta z}_l^{(rs)}$	$\underline{\Delta z}_l^{(a)(rs)}$	$\underline{\Delta z}_n^{(a)(rs)}$
$\underline{\Delta z}_i^{(rs)}$	g^{ij}	g^{il}	0	0
$\underline{\Delta z}_k^{(rs)}$	g^{kj}	g^{kl}	0	0
$\underline{\Delta z}_k^{(a)(rs)}$	0	0	h^{kl}	h^{kn}
$\underline{\Delta z}_m^{(a)(rs)}$	0	0	h^{ml}	h^{mn}

. . . . (9.5)

$$\left.\begin{array}{l}(A^i_k) = (g^{il})(\bar{g}_{lk}), \quad (\bar{g}_{lk}) = (g^{kl})^{-1} \\ (g^{ij.\mathrm{II}}) = (g^{ij}) - (g^{il})(\bar{g}_{lk})(g^{kj})\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.6)$$

(9.5) and (9.6) then give with (9.4) the covariance matrix:

	$\underline{\Delta z}_{j.\mathrm{II}}^{(rs)}$	$\underline{\Delta z}_l^{(a)(rs)}$	$\underline{\Delta z}_n^{(a)(rs)}$
$\underline{\Delta z}_{i.\mathrm{II}}^{(rs)}$	$[g^{ij.\mathrm{II}} + A^i_k h^{kl} A^j_l]$	$[A^i_k h^{kl}]$	$[A^i_k h^{kn}]$
$\underline{\Delta z}_k^{(a)(rs)}$	$[h^{kl} A^j_l]$	$[h^{kl}]$	$[h^{kn}]$
$\underline{\Delta z}_m^{(a)(rs)}$	$[h^{ml} A^j_l]$	$[h^{ml}]$	$[h^{mn}]$

. . . . (9.7)

according to (8.22) to be compared with the criterion matrix*)

	$\underline{\Delta z}_j^{\prime(rs)}$	$\underline{\Delta z}_l^{\prime(rs)}$	$\underline{\Delta z}_n^{\prime(rs)}$
$\underline{\Delta z}_i^{\prime(rs)}$	$[h^{ij}]$	$[h^{il}]$	$[h^{in}]$
$\underline{\Delta z}_k^{\prime(rs)}$	$[h^{kj}]$	$[h^{kl}]$	$[h^{kn}]$
$\underline{\Delta z}_m^{\prime(rs)}$	$[h^{mj}]$	$[h^{ml}]$	$[h^{mn}]$

. . . . (9.8)

From (9.7) compared with (9.8) appears the significance of including the given pointsP_m when considering the reconnaissance. The newly determined points P_i are not only considered in their relative positions with respect to the given points P_k with P_r, P_s, but also with respect to the given points P_m, which do not directly appear in the adjustment problem.

---

*) In the computation of eigenvalues from (8.6″) it can theoretically be proved that the value of $\{\lambda_i\}_{\max}$ increases with the order of the matrices $G^{(rs)}$ and $H^{(rs)}$ as has also been shown numerically.
In densification networks there is a large difference between the number of "given points" and the total number of points so that, on account of the foregoing, often in (9.8) elements $\bar{h}$ of a "larger" criterium matrix $\bar{H}^{(rs)}$ will have to be used instead of the elements h of the matrix $H^{(rs)}$ in (9.5) and (9.7). In this case however the matrix $(H^{(rs)} - \bar{H}^{(rs)})$ must be negative semi-definite, as already indicated in [11].
The line of thought expressed in section 10 is not affected by this.

The correlation of $\underline{\Delta z}_{i.\text{II}}^{(rs)}$ with $\underline{\Delta z}_m^{(a)(rs)}$ is thus caused by introducing $\underline{\Delta z}_k^{(a)(rs)}$ into the adjustment. The reconnaissance, by comparing (9.7) with (9.8), will have to show to what extent the new network should be surrounded by a ring of given points P_k with P_r, P_s.

It is always possible, according to (3.15) or (3.16), to go back to the $\underset{v,w}{\text{S}}$-system, from which we started in (9.2), and whose criteria are then fulfilled by the extended network. In this way one obtains a consistent system of criteria, independent of the stage of the densification one is working in.

Finally, it will have to be decided how many points P_m have to be introduced in (9.7) and which points are to be chosen.

The method proposed in this section implies that new densification networks need not have an unlimited size. A minimum size, such as for closed-traverse networks will be aimed at, but this aim is more concerned with reliability {error checks} than with precision. This problem area can only be investigated by the analysis of practical examples.

10 THE NECESSITY OF S-SYSTEMS FOR TESTING ON PRECISION

In sections 8 and 9, the comparison of covariance matrices has always been executed in S-systems. In section 9, the data contain the indication of an (a)-system, but this needed no further attention, because there a transformation to an S-system was made. Therefore it now appears to be important to investigate whether the line of thought developed in section 8 is also possible if the (a)-system mentioned in the data of section 9 is further used. In this investigation, the application of (4.5) can give the connection. For simplicity, the number of points with given coordinates will be restricted to two, so that the network considered need not be adjusted to given coordinates.

The equivalent of (9.1) then becomes:

measured	*given*	refers to criterion matrix	range of indices
$\underline{z}_i$		$\underline{\Delta z}'_i$	$i,j = \ldots$
$\underline{z}_r$	$\underline{z}_r^{(a)}$	$\underline{\Delta z}'_r$	
$\underline{z}_s$	$\underline{z}_s^{(a)}$	$\underline{\Delta z}'_s$	

. . . . (10.1)

Column 3 of (10.1) differs from the corresponding column of (9.1), because it is possible to establish a kind of "absolute" criterion matrix, from which by an S-transformation according to (2.5) the criterion matrix for the $\underset{r,s}{\text{S}}$-system can be derived.

Consequently we have in fact three different systems. Similarity transformation to the (a)-system gives (4.5):

$$\underline{\Delta z}_i^{(a)} = \underline{\Delta z}_i^{(rs)} + \frac{z^0_{si}}{z^0_{sr}} \underline{\Delta z}_r^{(a)} + \frac{z^0_{ri}}{z^0_{rs}} \underline{\Delta z}_s^{(a)} \quad \ldots \ldots \ldots \ldots \ldots \quad (10.2)$$

$$\underline{\Delta z}_i'^{(a)} = \underline{\Delta z}_i'^{(rs)} + \frac{z^0_{si}}{z^0_{sr}} \underline{\Delta z}_r^{(a)} + \frac{z^0_{ri}}{z^0_{rs}} \underline{\Delta z}_s^{(a)} \quad \ldots \ldots \ldots \ldots \ldots \quad (10.3)$$

The application of the non-singular transformation (2.8) then gives the covariance matrix, in a somewhat more careful notation than (9.5):

	$\underline{\Delta z}_j^{(rs)}$	$\underline{\Delta z}_j^{\prime(rs)}$	$\underline{\Delta z}_r^{(a)}, \underline{\Delta z}_r^{\prime(a)}$	$\underline{\Delta z}_s^{(a)}, \underline{\Delta z}_s^{\prime(a)}$
$\underline{\Delta z}_i^{(rs)}$	$(g^{ij})^{(rs)}$	0	0	0
$\underline{\Delta z}_i^{\prime(rs)}$	0	$(h^{ij})^{(rs)}$	0	0
$\underline{\Delta z}_r^{(a)}, \underline{\Delta z}_r^{\prime(a)}$	0	0	$(g^{rr})^{(a)}$	$(g^{rs})^{(a)}$
$\underline{\Delta z}_s^{(a)}, \underline{\Delta z}_s^{\prime(a)}$	0	0	$(g^{sr})^{(a)}$	$(g^{ss})^{(a)}$

. . . . . (10.4)

Or, if:

$$(\Lambda_i)\left[(g^{ij})^{(rs)}-(h^{ij})^{(rs)}\right](\Lambda_j)^* \leqslant 0 \quad \ldots\ldots \quad (10.5)$$

then also:

$$(\Lambda_i\Lambda_r\Lambda_s)\cdot\left[\begin{pmatrix}(g^{ij})^{(rs)} & 0 & 0\\ 0 & (g^{rr})^{(a)} & (g^{rs})^{(a)}\\ 0 & (g^{sr})^{(a)} & (g^{ss})^{(a)}\end{pmatrix}-\begin{pmatrix}(h^{ij})^{(rs)} & 0 & 0\\ 0 & (g^{rr})^{(a)} & (g^{rs})^{(a)}\\ 0 & (g^{sr})^{(a)} & (g^{ss})^{(a)}\end{pmatrix}\right]\cdot(\Lambda_j\Lambda_r\Lambda_s)^* \leqslant 0 \quad (10.6)$$

Hence also the vector: $\begin{pmatrix}\underline{\Delta z}_i^{(a)}\\ \underline{\Delta z}_r^{(a)}\\ \underline{\Delta z}_s^{(a)}\end{pmatrix}$

satisfies the precision requirements. (10.7)

In (9.5), a pseudo covariance matrix for $\underline{\Delta z}^{(a)^{(rs)}}$ was actually introduced. And in accordance with this, one might be tempted to do the same with (10.1). But in that case one obtains instead of (10.4):

	$\underline{\Delta z}_j^{(rs)}$	$\underline{\Delta z}_j^{\prime(rs)}$	$\underline{\Delta z}_r^{\prime}$	$\underline{\Delta z}_s^{\prime}$
$\underline{\Delta z}_i^{(rs)}$	$(g^{ij})^{(rs)}$	0	0	0
$\underline{\Delta z}_i^{\prime(rs)}$	0	$(h^{ij})^{(rs)}$	$\neq 0$	$\neq 0$
$\underline{\Delta z}_r^{\prime}$	0	$\neq 0$	h^{rr}	h^{rs}
$\underline{\Delta z}_s^{\prime}$	0	$\neq 0$	h^{sr}	h^{ss}

. . . . (10.8)

N.B.:
no S-system$_{r,s}$ as in (9.5).

Now (10.5) may be fulfilled, but the matrix established on the analogy of (10.6):

$$\left[\begin{pmatrix} (g^{ij})^{(rs)} & 0 & 0 \\ 0 & h^{rr} & h^{rs} \\ 0 & h^{sr} & h^{ss} \end{pmatrix} - \begin{pmatrix} (h^{ij})^{(rs)} & \neq 0 & \neq 0 \\ \neq 0 & h^{rr} & h^{rs} \\ \neq 0 & h^{sr} & h^{ss} \end{pmatrix}\right] \quad \ldots\ldots\ldots \quad (10.9)$$

is *indefinite*;

The conclusion analogous to (10.6) is therefore not valid here because of the non-zero covariances in the extended criterion matrix.

The situation of (10.9), starting from (10.5) in the limiting case when the equality sign is valid, can be shown most clearly in a two-dimensional picture of standard ellipses. See Fig. 10-1.

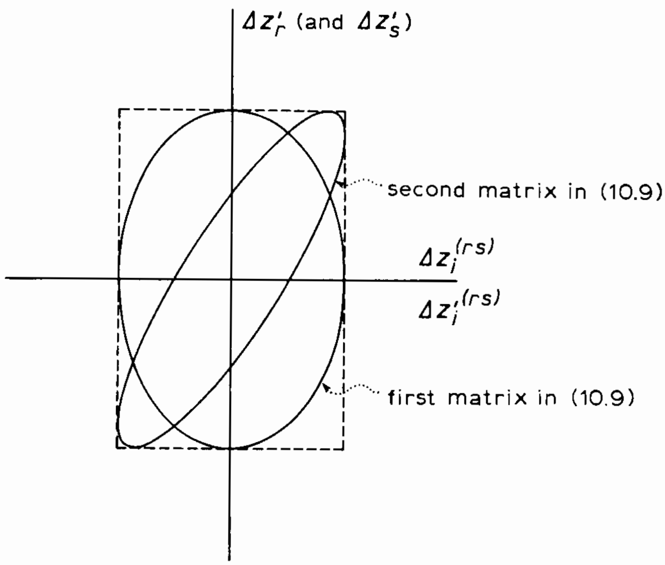


Fig. 10-1

Consequently one has to conclude that the introduction of pseudo variances *outside* an S-system impairs the consistency of a system of test conclusions and therefore should be *forbidden.*

This applies also to the contents of section 9, where the analogous extension also gives rise to difficulties.

Indeed, it must be asked if it is at all possible to find an (a)-system that is not an S-system. For the existing network must necessarily have two base points, although one can of course always more or less artificially, assign a probability distribution to the coordinates of these two base points. The question then is, what is the meaning of this probability distribution!

In any case the testing on precision by means of criterion matrices will have to be restricted to S-systems. For one is concerned here with criteria of form and *not* with criteria of location and orientation!

Reference is made to an alternative line of thought given in [11], which leads to the same conclusion.

11 THE PROBLEM OF LEVELLING NETWORKS TREATED AS AN INTRODUCTION TO THE CONSTRUCTION OF CRITERION MATRICES

A good insight may be obtained by studying the heights $\underset{\sim}{x}_i$ of points P_i in levelling networks.

In order to facilitate the comparison with the previously treated theory, the numbers of equivalent earlier formulae are shown to the left of the formulae, wherever this is possible. This also shortens the explanation.

Transformation from an x-system to an $x^{(a)}$-system:

$$\text{(2.1)} \qquad \tilde{x}_i^{(a)} = \tilde{x}_i + \tilde{\delta}^{(a)}; \qquad i, j = \ldots \qquad \text{(11.1)}$$

$$\text{(2.2)} \qquad x_i^{(a)0} = x_i^0 + \delta^{(a)0} \qquad \text{(11.2)}$$

Difference equations:

$$\text{(2.3)} \qquad (\tilde{x}_i^{(a)} - x_i^{(a)0}) = (\tilde{x}_i - x_i^0) + (\tilde{\delta}^{(a)} - \delta^{(a)0})$$

$$\boxed{\widetilde{\Delta x}_i^{(a)} = \widetilde{\Delta x}_i + \widetilde{\Delta\delta}^{(a)}} \qquad \text{(11.3)}$$

Now add a "base point" P_r:

$$\widetilde{\Delta x}_r^{(a)} = \widetilde{\Delta x}_r + \widetilde{\Delta\delta}^{(a)} \qquad \text{(11.4)}$$

Elimination of the model parameter $\widetilde{\Delta\delta}^{(a)}$ can be done by introducing difference quantities in the relative $\underset{r}{\text{S}}$-system:

$$\text{(2.5)} \qquad \begin{array}{|l|l|} \hline \widetilde{\Delta x}_i^{(a)(r)} = \widetilde{\Delta x}_i^{(a)} - \widetilde{\Delta x}_r^{(a)} & \widetilde{\Delta x}_r^{(a)(r)} = 0 \\ \hline \widetilde{\Delta x}_i^{(r)} = \widetilde{\Delta x}_i - \widetilde{\Delta x}_r & \widetilde{\Delta x}_r^{(r)} = 0 \\ \hline \end{array} \qquad \text{(11.5)}$$

Or, introducing a unit matrix (δ_i^i) and a column vector (δ_0^i) with elements 1:

$$\text{(2.8)} \qquad \begin{pmatrix} \widetilde{\Delta x}_i^{(a)(r)} \\ \widetilde{\Delta x}_r^{(a)} \end{pmatrix} = \begin{pmatrix} \delta_i^i & -\delta_0^i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Delta x}_i^{(a)} \\ \widetilde{\Delta x}_r^{(a)} \end{pmatrix} \qquad \text{(11.6')}$$

and, similarly,

quantities without the index (a) . . . . (11.6'')

With (11.3) and (11.4) one obtains:

(2.11)
$$\boxed{\begin{array}{ll} \widetilde{\Delta x}_i^{(a)(r)} = \widetilde{\Delta x}_i^{(r)} \\ \hline \widetilde{\Delta x}_r^{(r)} = 0 \\ \hline \widetilde{\Delta x}_r^{(a)} = \widetilde{\Delta x}_r + \widetilde{\Delta \delta}^{(a)} \end{array}} \qquad \begin{array}{r} \ldots\ldots\ldots (11.7') \\ \ldots\ldots\ldots (11.7'') \\ \ldots\ldots\ldots (11.7''') \end{array}$$

In practice one will usually put:

(2.10)
$$\boxed{\begin{array}{l} \delta^{(a)0} = 0 \\ \hline x_i^{(a)0} = x_i^0 \end{array}} \quad \ldots\ldots\ldots \quad (11.8)$$

Now add an alternative base point P_v:

(3.1)
$$\left.\begin{array}{l} \widetilde{\Delta x}_i^{(r)} = \widetilde{\Delta x}_i - \widetilde{\Delta x}_r \\ \widetilde{\Delta x}_v^{(r)} = \widetilde{\Delta x}_v - \widetilde{\Delta x}_r \end{array}\right\} \quad \ldots\ldots\ldots \quad (11.9)$$

Using (11.9) one obtains:

(3.3), (3.5)
$$\left.\begin{array}{l} \widetilde{\Delta x}_i^{(v)} = \widetilde{\Delta x}_i - \widetilde{\Delta x}_v = \widetilde{\Delta x}_i^{(r)} - \widetilde{\Delta x}_v^{(r)} = \widetilde{\Delta x}_i^{(r)(v)} \\ \widetilde{\Delta x}_r^{(v)} = \widetilde{\Delta x}_r - \widetilde{\Delta x}_v = 0 \qquad - \widetilde{\Delta x}_v^{(r)} = \widetilde{\Delta x}_r^{(r)(v)} \end{array}\right\} \quad \ldots\ldots\ldots \quad (11.10)$$

The transformation from the $\underset{r}{\text{S}}$-system to the $\underset{v}{\text{S}}$-system can therefore be written in the form:

(3.6)
$$\boxed{\begin{pmatrix} \widetilde{\Delta x}_i^{(v)} \\ \widetilde{\Delta x}_r^{(v)} \end{pmatrix} = \underbrace{\begin{pmatrix} \delta_i^i & -\delta_0^i \\ 0 & -1 \end{pmatrix}}_{\text{denote by: } (S_{(r)}^{(v)})} \cdot \begin{pmatrix} \widetilde{\Delta x}_i^{(r)} \\ \widetilde{\Delta x}_v^{(r)} \end{pmatrix}} \quad \ldots\ldots\ldots \quad (11.11)$$

with further considerations analogous to section 3.

From (11.7) it follows that:

(4.1)
$$\widetilde{\Delta x}_i^{(a)(r)} - \widetilde{\Delta x}_i^{(r)} = 0 \quad \ldots\ldots\ldots \quad (11.12)$$

From (11.12) with (11.5) follows:

$$(\widetilde{\Delta x}_i^{(a)} - \widetilde{\Delta x}_i) - (\widetilde{\Delta x}_r^{(a)} - \widetilde{\Delta x}_r) = 0$$

or, for each i:

$$\boxed{(\tilde{x}_i^{(a)} - \tilde{x}_i) = (\tilde{x}_r^{(a)} - \tilde{x}_r)} \tag{4.2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (11.13)$$

Similarly one obtains from (11.12):

$$\widetilde{\Delta x}_i^{(a)} - \widetilde{\Delta x}_r^{(a)} - \widetilde{\Delta x}_i^{(r)} = 0$$

$$\boxed{\widetilde{\Delta x}_i^{(a)} = \widetilde{\Delta x}_i^{(r)} + \widetilde{\Delta x}_r^{(a)}} \tag{4.5} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (11.14)$$

Remark

For height {or potential} variates it is not possible to introduce a scale parameter $\tilde{\gamma}^{(a)}$ in (11.1). Because then we need a pair of "base points" P_r and P_s instead of only one base point P_r, with in many cases $x_{rs}^0 = 0$. According to (4.4), replacing z by x, $\widetilde{\Delta\gamma}^{(a)}$ *and* $\widetilde{\Delta\delta}^{(a)}$ will then be indefinite.

Nevertheless in the domain of potential variates there are certain model parameters one should like to eliminate, more than one acting as scale parameter with the danger of conflicting scaling problems. Other ways must then be found to suppress these undesired parameters. An indication of this has already been given on page 7, lines 10 and 11 from bottom.

12 CRITERION MATRICES FOR LEVELLING NETWORKS

Consider an arbitrary x-system and make the transformation to an $\underset{r}{\text{S}}$-system, applying the law of propagation of *variances*:

$$\left.\begin{aligned} \underline{\Delta x}_i^{(r)} &= \underline{\Delta x}_i - \underline{\Delta x}_r \\ \underline{\Delta x}_j^{(r)} &= \underline{\Delta x}_j - \underline{\Delta x}_r \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \tag{12.1}$$

Hence:

$$\begin{aligned} \overline{x_i^{(r)}, x_j^{(r)}} &= \overline{x_i, x_j} - \overline{x_i, x_r} - \overline{x_j, x_r} + \overline{x_r, x_r} = \\ &= -(\overline{x_r, x_r} - \overline{x_i, x_j}) + (\overline{x_r, x_r} - \overline{x_i, x_r}) + (\overline{x_r, x_r} - \overline{x_j, x_r}) \end{aligned} \tag{12.2}$$

Now $\underline{\Delta x}_i^{(r)}$ is a difference variate, whose variance is positive and the analysis of levelling networks shows that this variance mainly depends on the distance l_{ir}. If an artificial variance matrix is to be constructed, it will be a reasonable point of departure to describe the variance of $\underline{\Delta x}_i^{(r)}$ as a positive function of l_{ir}.

This gives a connection to the ideas of BAARDA in the HTW-1956, whose consequences were further studied by J. E. ALBERDA in 1963.

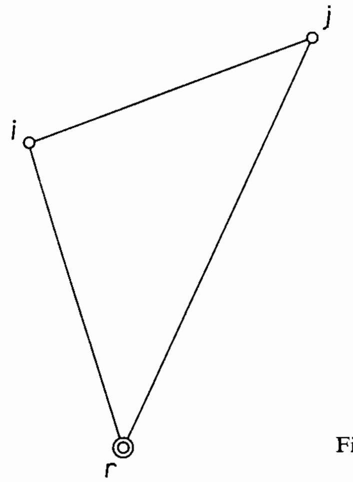


Fig. 12-1

In a further generalization we introduce here a polynomial in the distance l, in which indices and exponents p can be rational numbers.

$$(\overline{x_r, x_r} - \overline{x_i, x_j}) = \sum_p c_p (l_{ij})^p \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \tag{12.3}$$

(12.2) with (12.3):

$$\begin{aligned} \overline{x_i^{(r)}, x_j^{(r)}} &= \sum_p c_p \{-(l_{ij})^p + (l_{ir})^p + (l_{jr})^p\} = \\ &= c_0 + \sum_{p \neq 0} c_p \{-(l_{ij})^p + (l_{ir})^p + (l_{jr})^p\} \quad \ldots \ldots \ldots \end{aligned} \tag{12.4}$$

Now one must have:

$$\overline{x_r^{(r)}, x_r^{(r)}} \equiv 0, \quad \text{hence:} \boxed{c_0 = 0} \ ^{*)} \qquad (12.5)$$

and also: $i \neq r$:

$$\overline{x_i^{(r)}, x_i^{(r)}} = 2 \sum_{p \neq 0} c_p (l_{ir})^p > 0$$

Therefore for safety, choose: $\boxed{c_p \geqslant 0}$ (12.6)

Consequently the following notation is chosen:

$$\boxed{\begin{array}{c|c} \multicolumn{2}{l}{(\overline{x_r, x_r} - \overline{x_i, x_j}) = d_{ij}^2} \\ \hline c_p \geqslant 0 & d_{ij}^2 = \sum\limits_{p \neq 0} c_p (l_{ij})^p \\ \hline d_{ii}^2 = 0 & d_{ir}^2 = \sum\limits_{p \neq 0} c_p (l_{ir})^p \end{array}} \qquad (12.7)$$

In view of section 10 we compute:

$$\overline{x_i^{(r)}, x_j} = \overline{x_i, x_j} - \overline{x_r, x_j} = -(\overline{x_r, x_r} - \overline{x_i, x_j}) + (\overline{x_r, x_r} - \overline{x_j, x_r}) \qquad (12.8)$$

(12.2) and (12.8) give with (12.7), and, for the sake of completeness:

$$\boxed{x_r, x_r = d^2} \qquad (12.9)$$

the criterion matrix with bordering {for elements $\times$, see (12.12)}:

	$\underline{\Delta x}_i^{(r)}$	$\underline{\Delta x}_j^{(r)}$	$\underline{\Delta x}_i$	$\underline{\Delta x}_j$	$\underline{\Delta x}_r$
$\underline{\Delta x}_i^{(r)}$	$2d_{ir}^2$	$(-d_{ij}^2 + d_{ir}^2 + d_{jr}^2)$	d_{ir}^2	$(-d_{ij}^2 + d_{jr}^2)$	$-d_{ir}^2$
$\underline{\Delta x}_j^{(r)}$	$(-d_{ij}^2 + d_{ir}^2 + d_{jr}^2)$	$2d_{jr}^2$	$(-d_{ij}^2 + d_{ir}^2)$	d_{jr}^2	$-d_{jr}^2$
$\underline{\Delta x}_i$	d_{ir}^2	$(-d_{ij}^2 + d_{ir}^2)$	$\times$	$\times$	$\times$
$\underline{\Delta x}_j$	$(-d_{ij}^2 + d_{jr}^2)$	d_{jr}^2	$\times$	$\times$	$\times$
$\underline{\Delta x}_r$	$-d_{ir}^2$	$-d_{jr}^2$	$\times$	$\times$	d^2

(12.10)

*) We shall revert to this in (13.11).

(12.7) gives a "family" of criterion matrices, every member of this family is characterized by the choice of the coefficients c_p. Not every member is acceptable, a necessary condition is:

$$\boxed{\overline{(\Delta x_i^{(r)}), (\Delta x_j^{(r)})^*} \text{ positive definite}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12.11)$$

This question is further investigated in the note to section 13.

On examining (12.2) and (12.8) it appears to be possible to construct a kind of "absolute" criterion matrix, whose significance is, however, only abstract-theoretical, as shown in section 10. From (12.7) and (12.8) one obtains:

$$\overline{x_i, x_j} = \overline{x_r, x_r} - d_{ij}^2 = d^2 - d_{ij}^2$$

and hence:

$$\begin{array}{c|ccc} & \underline{\Delta x_i} & \underline{\Delta x_j} & \underline{\Delta x_r} \\ \hline \underline{\Delta x_i} & d^2 & (d^2 - d_{ij}^2) & (d^2 - d_{ir}^2) \\ \underline{\Delta x_j} & (d^2 - d_{ji}^2) & d^2 & (d^2 - d_{jr}^2) \\ \underline{\Delta x_r} & (d^2 - d_{ri}^2) & (d^2 - d_{rj}^2) & d^2 \end{array} \quad \ldots\ldots\ldots\ldots\ldots \quad (12.12)$$

Application of (12.1) to (12.12) produces again (12.10). {See also (13.22)–(13.25)}.

(12.12) is the type of criterion matrix – with only $c_1 \neq 0$ – as was used by BAARDA for each of the coordinates x and y separately in the treatment of traverse networks and area calculations in the HTW-1956, and also by ALBERDA in his investigation in 1963. The interpretation of d^2 in the HTW-1956 was not very satisfactory, and the author considers its introduction in the present theory as an eliminable "nuisance parameter" as an elegant solution for the difficulties encountered.

Since (12.12) only has a theoretical significance and may only be used to derive (12.10), and only the positive definiteness (12.11) has a practical significance; the discussion about giving numerical values to d^2 is irrelevant for problems of practice.

13 INTRODUCING OF Δd^2-TERMS IN CRITERION MATRICES FOR LEVELLING NETWORKS

One of the most interesting aspects of the HTW-1956 was the introduction of the circular point error ellipse with radius Δd, which essentially served to translate the uncertainty in the definition of a terrain point into operational stochastic concepts.

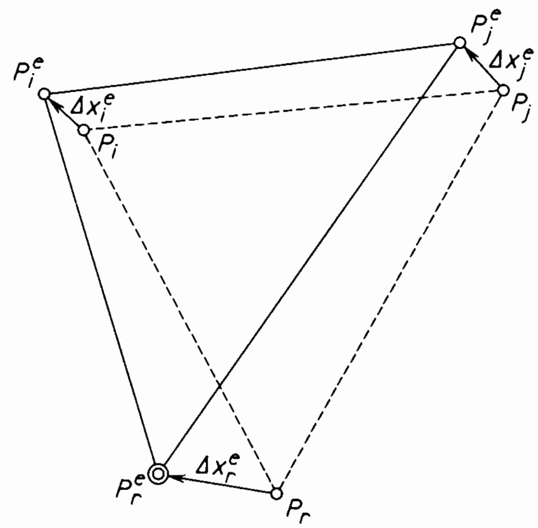


Fig. 13-1

Mathematically, this can be interpreted as follows:

Consider points P_i as in sections 11 and 12, and add points P_i^e characterizing the uncertainty mentioned; in the present case it is the uncertainty in its height.

	height variates
P_i	$\underset{\sim}{x}_i$
P_i^e	$\underset{\sim}{y}_i = \underset{\sim}{x}_i + \underline{\Delta x_i^e}$
	$\widetilde{\Delta x_i^e} = 0$
covariances	$\overline{\Delta x_i^e, \Delta x_i^e} = \Delta d_i^2 \equiv \{\Delta d_i\}^2$
	$\overline{\Delta x_i^e, \Delta x_j^e} = 0, \quad i \neq j$
	$\overline{\Delta x_i^e, \Delta x_j} = 0$

. . . . (13.1)

In order to develop the formulae as carefully as possible, Δd_i^2 is assumed to have a different value for each point P_i. In practice one will take an average value for groups of points. Well-marked points of high order will have a small or negligible value for Δd^2, this is important for the choice of the base point P_r or for the choice of the given points to which the network is connected in a second step. Therefore the value of Δd_r^2 will be small as compared with a value of Δd_i^2 when P_i is an arbitrary non-marked or poorly marked terrain point.

The assumption of (13.1) is such, that all relations of section 11 are also valid for the y-variates.

(12.1) becomes:

$$\left.\begin{aligned}
\underline{\Delta y}_i^{(r)} &= \underline{\Delta x}_i + \underline{\Delta x}_i^e - \underline{\Delta x}_r - \underline{\Delta x}_r^e = \underline{\Delta x}_i^{(r)} + \underline{\Delta x}_i^e - \underline{\Delta x}_r^e \\
\underline{\Delta y}_j^{(r)} &= \underline{\Delta x}_j + \underline{\Delta x}_j^e - \underline{\Delta x}_r - \underline{\Delta x}_r^e = \underline{\Delta x}_j^{(r)} + \underline{\Delta x}_j^e - \underline{\Delta x}_r^e \\
\underline{\Delta y}_r^{(r)} &= \underline{\Delta x}_r + \underline{\Delta x}_r^e - \underline{\Delta x}_r - \underline{\Delta x}_r^e = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (13.2)$$

Or, with (13.1):

$$\boxed{\begin{aligned}
&\overline{y_i^{(r)}, y_i^{(r)}} = \overline{x_i^{(r)}, x_i^{(r)}} + \Delta d_i^2 + \Delta d_r^2 \\
&\overline{y_i^{(r)}, y_j^{(r)}} = \overline{x_i^{(r)}, x_j^{(r)}} \qquad\quad + \Delta d_r^2 \\
&\overline{y_i^{(r)}, y_r^{(r)}} = 0 \\[4pt]
\hline
&\overline{y_i^{(r)}, y_i} \;\; = \overline{x_i^{(r)}, x_i} \;\; + \Delta d_i^2 \\
&\overline{y_i^{(r)}, y_j} \;\; = \overline{x_i^{(r)}, x_j} \\
&\overline{y_i^{(r)}, y_r} \;\; = \overline{x_i^{(r)}, x_r} \qquad\quad - \Delta d_r^2 \\[4pt]
\hline
&\overline{y_i \;, y_i} \;\; = \overline{x_i \;, x_i} \;\; + \Delta d_i^2 \\
&\overline{y_i \;, y_j} \qquad \overline{x_i \;, x_j} \\
&\overline{y_i \;, y_r} \;\; = \overline{x_i \;, x_r}
\end{aligned}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.3)$$

Next, consider the situation in (12.7), for which we obtain, with (13.1) and (13.3):

$$\boxed{\begin{aligned}
&\overline{y_r, y_r} - \overline{y_i, y_j} = \overline{x_r, x_r} + \Delta d_r^2 - \overline{x_i, x_j} = d_{ij}^2 + \Delta d_r^2; \quad i \neq j \\
&\overline{y_r, y_r} - \overline{y_i, y_i} = \overline{x_r, x_r} + \Delta d_r^2 - \overline{x_i, x_i} - \Delta d_i^2 = \\
&\qquad\qquad = [d_{ii}^2 = 0] - (\Delta d_i^2 - \Delta d_r^2) = -(\Delta d_i^2 - \Delta d_r^2) \\
&\overline{y_r, y_r} - \overline{y_r, y_r} = -(\Delta d_r^2 - \Delta d_r^2) = 0; \quad \text{check}
\end{aligned}} \quad \ldots\ldots \quad (13.4)$$

With (13.3) the most important part of (12.10) is then replaced by:

	$\underline{\Delta y}_i^{(r)}$	$\underline{\Delta y}_j^{(r)}$	$\underline{\Delta y}_r$
$\underline{\Delta y}_i^{(r)}$	$[2(d_{ir}^2+\Delta d_r^2)+(\Delta d_i^2-\Delta d_r^2)]$	$[-(d_{ij}^2+\Delta d_r^2)+(d_{ir}^2+\Delta d_r^2)+(d_{jr}^2+\Delta d_r^2)]$	$-(d_{ir}^2+\Delta d_r^2)$
$\underline{\Delta y}_j^{(r)}$	$[-(d_{ij}^2+\Delta d_r^2)+(d_{ir}^2+\Delta d_r^2)+(d_{jr}^2+\Delta d_r^2)]$	$[2(d_{jr}^2+\Delta d_r^2)+(\Delta d_j^2-\Delta d_r^2)]$	$-(d_{jr}^2+\Delta d_r^2)$
$\underline{\Delta y}_r$	$-(d_{ir}^2+\Delta d_r^2)$	$-(d_{jr}^2+\Delta d_r^2)$	d'^2

(13.5′)

in which we have introduced the notation:

$\overline{\Delta y_r, \Delta y_r} = d'^2$, with in view of (13.16),*) to be safe:

$$d'^2 \geqslant d^2 + \Delta d_r^2 \quad \ldots \quad (13.5'')$$

On the analogy of (12.12) one can again construct with (13.4) a theoretical – in the sense of: non-realistic – solution:

	$\underline{\Delta y}_i$	$\underline{\Delta y}_j$	$\underline{\Delta y}_r$
$\underline{\Delta y}_i$	$d'^2+(\Delta d_i^2-\Delta d_r^2)$	$d'^2-(d_{ij}^2+\Delta d_r^2)$	$d'^2-(d_{ir}^2+\Delta d_r^2)$
$\underline{\Delta y}_j$	$d'^2-(d_{ji}^2+\Delta d_r^2)$	$d'^2+(\Delta d_j^2-\Delta d_r^2)$	$d'^2-(d_{jr}^2+\Delta d_r^2)$
$\underline{\Delta y}_r$	$d'^2-(d_{ri}^2+\Delta d_r^2)$	$d'^2-(d_{rj}^2+\Delta d_r^2)$	d'^2

. . . (13.6)

As a check one can derive (13.5) from (13.6):

$$\overline{y_i^{(r)}, y_i^{(r)}} = \overline{y_i, y_i} - 2\cdot\overline{y_i, y_r} + \overline{y_r, y_r} = 2(d_{ir}^2+\Delta d_r^2)+(\Delta d_i^2-\Delta d_r^2)$$

$$\overline{y_i^{(r)}, y_j^{(r)}} = \overline{y_i, y_j} - \overline{y_i, y_r} - \overline{y_j, y_r} + \overline{y_r, y_r} = -(d_{ij}^2+\Delta d_r^2)+(d_{ir}^2+\Delta d_r^2)+(d_{jr}^2+\Delta d^2)$$

In (13.6) d′^{2} and Δd_r^2 can be taken together; one obtains:

	$\underline{\Delta y}_i$	$\underline{\Delta y}_j$	$\underline{\Delta y}_r$
$\underline{\Delta y}_i$	$d''^2+\Delta d_i^2$	$d''^2-d_{ij}^2$	$d''^2-d_{ir}^2$
$\underline{\Delta y}_j$	$d''^2-d_{ji}^2$	$d''^2+\Delta d_j^2$	$d''^2-d_{jr}^2$
$\underline{\Delta y}_r$	$d''^2-d_{ri}^2$	$d''^2-d_{rj}^2$	$d''^2+\Delta d_r^2$
$d''^2 \neq d'^2-\Delta d_r^2 \geqslant d^2$; see (13.5″)			

. . . . (13.7)

*) (13.16) to be used for the case $\Delta d_i^2 = \Delta d_r^2$, $i = \ldots$

The form (13.6) is connected to the notation in (13.4) and (13.5); Δd_r^2 is introduced as a kind of c_0-term (cf. section 12), but there is also an addition to the terms of the main diagonal if $\Delta d_i^2 \neq \Delta d_r^2$ for $i = \ldots$

The comparison of (13.7) with (12.12) illustrates the peculiarity of these matrices. The rather complicated addition of Δd^2-effects in (13.3) means – apart from a possible difference between d''^2 and d^2 – the addition of a Δd^2-term only to the elements of the main diagonal of (12.12).

It is interesting to note that the addition of Δd^2-terms does not impair the positive definiteness of the matrices constructed. This is easily seen from (13.5′) and (13.7). Use is made of the theorem*):

$$\begin{vmatrix} 1+a_i & a_j & a_r \\ a_i & 1+a_j & a_r \\ a_i & a_j & 1+a_r \end{vmatrix} = 1+a_i+a_j+a_r$$

For the d^2-addition in (13.5′) one obtains:

$$\begin{vmatrix} \Delta d_r^2+\Delta d_i^2 & \Delta d_r^2 & -\Delta d_r^2 \\ \Delta d_r^2 & \Delta d_r^2+\Delta d_j^2 & -\Delta d_r^2 \\ -\Delta d_r^2 & -\Delta d_r^2 & \Delta d_r^2+(d''^2-d^2) \end{vmatrix} =$$

$$= \left\{(d''^2-d^2)\cdot\prod_{i=1}^{n} \Delta d_i^2\right\}\left\{1+\frac{\Delta d_r^2}{d''^2-d^2}+\sum_{i=1}^{n}\frac{\Delta d_r^2}{\Delta d_i^2}\right\} =$$

$$= \left\{\prod_{i=1}^{n,r} \Delta d_i^2\right\}\left\{1+\sum_{i=1}^{n,r}\frac{d''^2-d^2}{\Delta d_i^2}\right\}; \quad i,j = 1,\ldots,n \tag{13.8′}$$

and for the Δd^2-addition in (13.7) similarly:

$$\begin{vmatrix} (d''^2-d^2)+\Delta d_i^2 & (d''^2-d^2) & (d''^2-d^2) \\ (d''^2-d^2) & (d''^2-d^2)+\Delta d_j^2 & (d''^2-d^2) \\ (d''^2-d^2) & (d''^2-d^2) & (d''^2-d^2)+\Delta d_r^2 \end{vmatrix} =$$

$$= \left\{\prod_{i=1}^{n,r} \Delta d_i^2\right\}\left\{1+\sum_{i=1}^{n,r}\frac{d''^2-d^2}{\Delta d_i^2}\right\} \quad \ldots\ldots\ldots\ldots \tag{13.8″}$$

Hence certainly:

$\Delta d_i \neq 0$	determinant > 0 for $d''^2 \geqslant d^2$, $d'^2 \geqslant d^2+\Delta d_r^2$	. . . . . (13.8‴)

If one assumes the positive definiteness of the criterion matrices without Δd^2 terms, the

*) L. Mirsky – An Introduction to Linear Algebra – Oxford University Press, reprint 1961, page 35.

positive definiteness of the combination follows from the theorem*):

> If A and B are positive definite hermitean matrices $\{B$ may be positive semi-definite$\}$ whose order is greater than 1, then:
> $|A+B|>|A|+|B|$

If the matrix (12.10) is examined with respect to positive definiteness for each term in (12.7) separately – as will be executed in (13.12)-(13.21) – (13.9) can again be used for the proof of positive definiteness of the combination.

In this case one must introduce via (12.7), compare (12.12):

$\sum\limits_p \overline{(x_i,\ x_j)}_p = \sum\limits_p \overline{(x_r,\ x_r)}_p - \sum\limits_p c_p \cdot (l_{ij})^p$	$p \neq 0$
$\sum\limits_p \overline{(x_r,\ x_r)}_p = d^2 = \sum\limits_p (d^2)_p$	$p \neq 0$
$\overline{(x_i,\ x_j)}_p = (d^2)_p - c_p \cdot (l_{ij})^p$	$p \neq 0$
$\overline{(x_i,\ x_i)}_p = \overline{(x_r,\ x_r)}_p = (d^2)_p$	$p \neq 0$

. . . . (13.10)

If one is careful, the restriction $p \neq 0$ can be removed. From (13.3)–(13.8) follows:

$\sum\limits_p \overline{(y_i,\ y_j)}_p = \sum\limits_p \overline{(y_r,\ y_r)}_p - \sum\limits_p c_p \cdot (l_{ij})^p$		$i \neq j$
$\sum\limits_p \overline{(y_r,\ y_r)}_p = d'^2 = \sum\limits_p (d^2)_p$		
$\overline{(y_i,\ y_j)}_p = (d^2)_p - c_p \cdot (l_{ij})^p$		$i \neq j$
$\overline{(y_i,\ y_i)}_p = \overline{(y_r,\ y_r)}_p = (d^2)_p$		$p \neq 0$
$\overline{(y_i,\ y_i)}_p = (d^2)_p + (\Delta d_i^2 - \Delta d_r^2)$		$p = 0$
$c_0 = \Delta d_r^2$	$d'^2 \geqslant d^2 + \Delta d_r^2$	

. . . . (13.11)

As a rule one will omit the prime in d'^2.

From these last considerations it is again evident, like from the end of section 12, what a curious quantity d^2 from (12.9) or (13.5″) is!

Note to section 13. The positive definiteness of the criterion matrix of the type considered

Take from (12.10) – possibly with the extension according to (13.11) $\Delta d_i^2 = \Delta d_r^2$ – the criterion matrix for the points P_i, P_j, P_k with bordering for P_r, and consider the determinant D (with minor $\bar{D}$):

*) Cf. MIRSKY, page 420.

$$D \equiv \left|\begin{array}{ccc:c} 2d_{ir}^2 & (d_{ir}^2+d_{jr}^2-d_{ij}^2) & (d_{ir}^2+d_{kr}^2-d_{ik}^2) & -d_{ir}^2 \\ (d_{jr}^2+d_{ir}^2-d_{ji}^2) & 2d_{jr}^2 & (d_{jr}^2+d_{kr}^2-d_{jk}^2) & -d_{jr}^2 \\ (d_{kr}^2+d_{ir}^2-d_{ki}^2) & (d_{kr}^2+d_{jr}^2-d_{kj}^2) & 2d_{kr}^2 \quad \bar{D} & -d_{kr}^2 \\ \hdashline -d_{ir}^2 & -d_{jr}^2 & -d_{kr}^2 & d^2 \end{array}\right| \quad \ldots\ldots \quad (13.12)$$

Add the first three columns to the fourth and then the first three rows to the fourth:

$$D = \left|\begin{array}{c:l} & (d_{ir}^2+d_{jr}^2+d_{kr}^2)+3d_{ir}^2-(d_{ij}^2+d_{ik}^2+d_{ir}^2) \\ \bar{D} & (\quad \text{id.} \quad)+3d_{jr}^2-(d_{ji}^2+d_{jk}^2+d_{jr}^2) \\ & (\quad \text{id.} \quad)+3d_{kr}^2-(d_{ki}^2+d_{kj}^2+d_{kr}^2) \\ \hdashline \text{transpose} & d^2+2(3-1)(d_{ir}^2+d_{jr}^2-d_{kr}^2)+ \\ \text{of right} & \\ \text{upper part} & \qquad -2(d_{ij}^2+d_{jk}^2+d_{ki}^2) \end{array}\right| \quad \ldots\ldots \quad (13.13)$$

Divide 1st column and row by $d_{ir}\sqrt{2}$

2nd column and row by $d_{jr}\sqrt{2}$

3rd column and row by $d_{kr}\sqrt{2}$

4th column and row by $3\sqrt{2}\sqrt{\dfrac{d_{ir}^2+d_{jr}^2+d_{kr}^2}{3}}$

Then put:

$$\frac{d_{ir}^2+d_{jr}^2-d_{ij}^2}{2d_{ir}d_{jr}} = \cos\Gamma_{irj}\ ^{*})$$

$$\frac{\frac{1}{3}(d_{ir}^2+d_{jr}^2+d_{kr}^2)+d_{ir}^2-\frac{1}{3}(d_{ij}^2+d_{ik}^2+d_{ir}^2)}{2d_{ir}\cdot\sqrt{\frac{1}{3}(d_{ir}^2+d_{jr}^2+d_{kr}^2)}} = \cos\Gamma_{ir\Sigma} \quad \ldots\ldots\ldots\ldots \quad (13.14)$$

Then (13.13) becomes:

$$D = c^2\cdot\left|\begin{array}{ccc:c} 1 & \cos\Gamma_{irj} & \cos\Gamma_{irk} & \cos\Gamma_{ir\Sigma} \\ \cos\Gamma_{jri} & 1 & \cos\Gamma_{jrk} & \cos\Gamma_{jr\Sigma} \\ \cos\Gamma_{kri} & \cos\Gamma_{krj} & 1 & \cos\Gamma_{kr\Sigma} \\ \hdashline \cos\Gamma_{ir\Sigma} & \cos\Gamma_{jr\Sigma} & \cos\Gamma_{kr\Sigma} & \bar{d} \end{array}\right| \quad \ldots\ldots\ldots\ldots \quad (13.15')$$

---

*) Here it appears that the further development can also be applied if d_{ij}^2 is constant, e.g. $=\Delta d_r^2$, as investigated in (13.8). For in that case we would have: $\cos\Gamma_{irj}=\frac{1}{2}$. For the contrast between the corresponding applications to (13.16) and (13.8), see (13.21) and the accompanying text.

$$\bar{d} = \frac{1}{2\cdot 3}\,\frac{d^2}{d_{ir}^2+d_{jr}^2+d_{kr}^2} + \frac{3-1}{3} - \tfrac{1}{3}\,\frac{d_{ij}^2+d_{jk}^2+d_{ki}^2}{d_{ir}^2+d_{jr}^2+d_{kr}^2} \quad \ldots\ldots\ldots\ldots \text{(13.15'')}$$

$$\left.\begin{aligned} &\bar{d} \geqslant 1 \quad \text{hence for:} \\ &\frac{d^2}{3} \geqslant 2\left\{\frac{d_{ir}^2+d_{jr}^2+d_{kr}^2}{3} + \frac{d_{ij}^2+d_{jk}^2+d_{ki}^2}{3}\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(13.16')}$$

or, with $i, j = 1, \ldots, n$, a sufficient condition is:

$$\boxed{\frac{d^2}{n} \geqslant 2\left\{\frac{1}{n}\sum_{i=1}^{n} d_{ir}^2 + \frac{1}{n}\sum_{\substack{i,j=1\\ j>i}}^{n} d_{ij}^2\right\}} \quad \ldots\ldots\ldots\ldots \text{(13.16'')}$$

(13.14) is only possible if the triangle inequality is valid, with non-coincident "triangle sides", with

$$d_{ir} = +\sqrt{d_{ir}^2}, \quad d_{jr} = +\sqrt{d_{jr}^2} \quad \text{and} \quad d_{ij} = +\sqrt{d_{ij}^2}:$$

$$\boxed{d_{ir}+d_{jr}-d_{ij} > 0}; \quad i,j = 1,\ldots,n; \quad j \neq i \quad \ldots\ldots\ldots\ldots \text{(13.17')}$$

$$d_{ir} + \left\{\frac{1}{n}\sum_{j=1,\ldots,n} d_{jr}^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{\substack{j=1,\ldots,i-1,\\ r,i+1,\ldots,n}} d_{ij}^2\right\}^{\frac{1}{2}} > 0 \quad \ldots\ldots\ldots\ldots \text{(13.17'')}$$

With*):

$$\left\{\frac{1}{n}\sum_{j=1,\ldots,n} d_{jr}^2\right\}^{\frac{1}{2}} \geqslant \frac{1}{n}\sum_{j=1,\ldots,n} d_{jr}$$

(13.17'') becomes:

$$\left\{d_{ir}^2 + 2d_{ir}\left(\frac{\Sigma d_{jr}^2}{n}\right)^{\frac{1}{2}} + \frac{\Sigma d_{jr}^2}{n}\right\}^{\frac{1}{2}} - \left\{\frac{\Sigma d_{ij}^2}{n}\right\}^{\frac{1}{2}} =$$

$$\geqslant \left\{d_{ir}^2 + 2d_{ir}\frac{\Sigma d_{jr}}{n} + \frac{\Sigma d_{jr}^2}{n}\right\}^{\frac{1}{2}} - \left\{\frac{\Sigma d_{ij}^2}{n}\right\}^{\frac{1}{2}} =$$

$$= \frac{1}{\sqrt{n}}\left[\left\{\sum_{j=1,\ldots,n} (d_{ir}+d_{jr})^2\right\}^{\frac{1}{2}} - \left\{\sum_{\substack{j=1,\ldots,i-1,\\ r,i+1,\ldots,n}} d_{ij}^2\right\}^{\frac{1}{2}}\right] > 0$$

according to (13.17') and: $d_{ir}+d_{ir}-d_{ir} > 0$

Or:

(13.17'') is fulfilled if (13.17') is valid.

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*) D. S. Mitrinović – Elementary Inequalities – Noordhoff, Groningen, 1964; page 13.

With $\bar{d} = 1$ and (13.17), (13.15) can always be interpreted as the metric of an oblique cartesian coordinate system, for there are exactly sufficient elements d_{ir}, d_{ij} for its determination.

(13.12) is the determinant of a covariance matrix, therefore the angles Γ can be interpreted as the angles between the planes formed by the coordinate axes corresponding to the observed quantities in the standardized sample space.*)

(13.17′) implies that it must be possible to construct new triangles $\bar{P}_r$, $\bar{P}_i$, $\bar{P}_j$ from the elements determining the triangles P_r, P_i, P_j, the "sides" d_{ir}, d_{jr} and d_{ij} satisfying (13.17′), and not coinciding. For the proof, we follow again MITRINOVIĆ.**)

From triangle P_r, P_i, P_j it follows that:

$$l_{ij} \leqslant l_{ir} + l_{jr}\,, \; l_{ij} \geqslant |l_{ir} - l_{jr}|$$

Assume:

$$l_{ij}^{p/2} \geqslant l_{ir}^{p/2} + l_{jr}^{p/2}, \qquad 0 \leqslant p < 2$$

Raising to the power $2/p$ gives:

$$l_{ij} \geqslant l_{ir} + l_{jr} + C^2, \qquad \text{or with } l_{ir} \neq 0 \neq l_{jr}:$$

$$l_{ij} > l_{ir} + l_{jr} \qquad \text{contradicting the existence of triangle } P_r,\, P_i,\, P_j$$

Hence the other possibility is valid:

$$l_{ij}^{p/2} < l_{ir}^{p/2} + l_{jr}^{p/2}, \qquad 0 \leqslant p < 2 \qquad (13.18')$$

Similarly it is proved that:

$$l_{ij}^{p/2} > |l_{ir}^{p/2} - l_{jr}^{p/2}|, \qquad 0 \leqslant p < 2 \qquad (13.18'')$$

Or, triangle $\bar{P}_r$, $\bar{P}_i$, $\bar{P}_j$ is always constructable with $\bar{P}_r$, $\bar{P}_i$, $\bar{P}_j$ non-collinear.

From (13.18) follows, in particular:

$$l_{ir}^{p/2} + l_{jr}^{p/2} - l_{ij}^{p/2} > 0, \qquad 0 \leqslant p < 2$$

and hence, with (12.7) and in view of (13.11):

$$\begin{array}{|c|c|}\hline c_p^{\frac{1}{2}} l_{ir}^{p/2} + c_p^{\frac{1}{2}} l_{jr}^{p/2} - c_p^{\frac{1}{2}} l_{ij}^{p/2} > 0 & c_p^{\frac{1}{2}} > 0 \\ \hline (d_{ir})_p + (d_{jr})_p - (d_{ij})_p > 0 & 0 \leqslant p < 2 \\ \hline \end{array} \qquad (13.19)$$

*) See: J. M. TIENSTRA – Theory of the adjustment of normally distributed observations – Argus, Amsterdam, 1956 {chapter 4}.

**) loc. cit. page 119.

giving the conditions for the validity of (13.17′), for each non-zero p-term one wishes to introduce in (13.11).

$$\boxed{\begin{array}{l}\textit{Conclusion}\text{: With (13.19) the determinant } \bar{D} \text{ is positive,}\\ \text{hence: } \overline{(\Delta x_i^{(r)}), (\Delta x_j^{(r)})^*} \text{ positive definite.}\\ \text{Considering also (13.16), the determinant } D \text{ is positive,}\\ \text{hence: the matrix (13.12) is positive definite.}\\ \text{This is valid for each separate } p\text{-term in (13.11), hence with (13.9) this is}\\ \text{certainly valid for the combination of } p\text{-terms in (13.11); for the addition of}\\ \Delta d^2\text{-terms reference is made to (13.8).}\end{array}} \qquad (13.20)$$

(13.19) leaves enough possibilities for the measurement of levelling networks. For in these networks the variance of measured height differences proves to be practically proportional to the lengths of the sections, whereas in the criterion matrix according to (12.4) the variance of a height difference in view of (13.19) can increase to almost direct proportionality with the square of the distance between the benchmarks.

If in (13.16) a constant value is introduced for d_{ij}^2, e.g. Δd_r^2, then a much higher value for the corresponding increment of d^2 is found than the value following from (13.8). This shows clearly that (13.16) is only a *sufficient* condition.

Starting from (13.12) in abbreviated notation, orthogonalization gives:

$$D = \begin{vmatrix} (\bar{D}) & (-d_{ir}^2) \\ (-d_{rj}^2) & d^2 \end{vmatrix} = \begin{vmatrix} (\bar{D}) & 0 \\ 0 & d^2 - (-d_{rj}^2)(\bar{D})^{-1}(-d_{ir}^2) \end{vmatrix}$$

Hence a *necessary and sufficient* condition for d^2 is:

$$\boxed{\begin{array}{l}\text{If } \bar{D} > 0, \text{ then } D > 0 \text{ if:}\\ d^2 > (-d_{rj}^2)(\bar{D})^{-1}(-d_{ir}^2)\end{array}} \qquad (13.21)$$

According to (13.20) – also in view of the alternative possibility to calculate d^2 according to (13.21) – the matrix (13.12) can always be made positive definite.

According to (11.6), (12.1) can always be extended to a non-singular transformation:

$$\begin{pmatrix} \underline{\Delta x}_i^{(r)} \\ \underline{\Delta x}_j^{(r)} \\ \underline{\Delta x}_k^{(r)} \\ \underline{\Delta x}_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \underline{\Delta x}_i \\ \underline{\Delta x}_j \\ \underline{\Delta x}_k \\ \underline{\Delta x}_r \end{pmatrix} \qquad (13.22)$$

and the inverse transformation becomes:

$$\begin{pmatrix} \underline{\Delta x}_i \\ \underline{\Delta x}_j \\ \underline{\Delta x}_k \\ \underline{\Delta x}_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \underline{\Delta x}_i^{(r)} \\ \underline{\Delta x}_j^{(r)} \\ \underline{\Delta x}_k^{(r)} \\ \underline{\Delta x}_r \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (13.23)$$

The matrix (12.12), written as:

	$\underline{\Delta x}_i$	$\underline{\Delta x}_j$	$\underline{\Delta x}_k$	$\underline{\Delta x}_r$
$\underline{\Delta x}_i$	d^2	$(d^2-d_{ij}^2)$	$(d^2-d_{ik}^2)$	$(d^2-d_{ir}^2)$
$\underline{\Delta x}_j$	$(d^2-d_{ji}^2)$	d^2	$(d^2-d_{jk}^2)$	$(d^2-d_{jr}^2)$
$\underline{\Delta x}_k$	$(d^2-d_{ki}^2)$	$(d^2-d_{kj}^2)$	d^2	$(d^2-d_{kr}^2)$
$\underline{\Delta x}_r$	$(d^2-d_{ri}^2)$	$(d^2-d_{rj}^2)$	$(d^2-d_{rk}^2)$	d^2

. . . . (13.24)

can be considered as the result of applying the law of propagation of variances to (13.23) using the covariance matrix (13.12).

Conclusion:
if the matrix (13.12) is positive definite then the matrix (13.24) is also positive definite. (13.25)

14 THE LAW OF PROPAGATION OF VARIANCES FOR COMPLEX COORDINATE VARIATES. CIRCULAR STANDARD ELLIPSES

A complex coordinate variate and its conjugate variate:

$$\underline{z}_i = \underline{y}_i + \mathrm{i}\underline{x}_i$$

$$\underline{z}_i^T = \underline{y}_i - \mathrm{i}\underline{x}_i$$

can be written, cf. [7] section 4:

$$\begin{pmatrix} \underline{z}_i \\ \underline{z}_i^T \end{pmatrix} = \begin{pmatrix} 1 & +\mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \underline{y}_i \\ \underline{x}_i \end{pmatrix} \quad \ldots \quad (14.1)$$

$$\begin{pmatrix} \underline{y}_i \\ \underline{x}_i \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & +\mathrm{i} \end{pmatrix} \begin{pmatrix} \underline{z}_i \\ \underline{z}_i^T \end{pmatrix} \quad \ldots \quad (14.2)$$

Apply the law of propagation of variances to (14.2) in the form of symbolic multiplication according to J. M. Tienstra, cf. [1], [4], [8]:

$$\overline{\begin{pmatrix} y_i \\ x_i \end{pmatrix}, \begin{pmatrix} y_j \\ x_j \end{pmatrix}^*} = \tfrac{1}{4} \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & +\mathrm{i} \end{pmatrix} \cdot \overline{\begin{pmatrix} z_i \\ z_i^T \end{pmatrix}, \begin{pmatrix} z_j \\ z_j^T \end{pmatrix}^*} \cdot \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & +\mathrm{i} \end{pmatrix}^* \quad \ldots \quad (14.3')$$

or, worked out:

$$\begin{pmatrix} \overline{y_i, y_j} & \overline{y_i, x_j} \\ \overline{x_i, y_j} & \overline{x_i, x_j} \end{pmatrix} =$$

$$= \begin{pmatrix} [\overline{z_i, z_j} + \overline{z_i^T, z_j} + \overline{z_i, z_j^T} + \overline{z_i^T, z_j^T}] & \mathrm{i}[-\overline{z_i, z_j} - \overline{z_i^T, z_j} + \overline{z_i, z_j^T} + \overline{z_i^T, z_j^T}] \\ \mathrm{i}[-\overline{z_i, z_j} + \overline{z_i^T, z_j} - \overline{z_i, z_j^T} + \overline{z_i^T, z_j^T}] & [-\overline{z_i, z_j} + \overline{z_i^T, z_j} + \overline{z_i, z_j^T} - \overline{z_i^T, z_j^T}] \end{pmatrix} \quad \ldots \quad (14.3'')$$

From (14.3″) one can solve $\overline{z_i, z_j}, \ldots, \overline{z_i^T, z_j^T}$, and express them in $\overline{y_i, y_j}, \ldots, \overline{x_i, x_j}$; this in fact *defines* covariances of complex coordinate variates.

The same result, however, is obtained by applying the symbolic multiplication to (14.1):

$$\overline{\begin{pmatrix} z_i \\ z_i^T \end{pmatrix}, \begin{pmatrix} z_j \\ z_j^T \end{pmatrix}^*} = \begin{pmatrix} 1 & +\mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \cdot \overline{\begin{pmatrix} y_i \\ x_i \end{pmatrix}, \begin{pmatrix} y_j \\ x_j \end{pmatrix}^*} \cdot \begin{pmatrix} 1 & +\mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}^* \quad \ldots \quad (14.4')$$

Or, worked out:

$$\begin{pmatrix} \overline{z_i\,, z_j} & \overline{z_i\,, z_j^T} \\ \overline{z_i^T, z_j} & \overline{z_i^T, z_j^T} \end{pmatrix} =$$
$$= \begin{pmatrix} [\overline{y_i, y_j} - \overline{x_i, x_j} + \mathrm{i}\{\overline{y_i, x_j} + \overline{x_i, y_j}\}] & [\overline{y_i, y_j} + \overline{x_i, x_j} - \mathrm{i}\{\overline{y_i, x_j} - \overline{x_i, y_j}\}] \\ [\overline{y_i, y_j} + \overline{x_i, x_j} + \mathrm{i}\{\overline{y_i, x_j} - \overline{x_i, y_j}\}] & [\overline{y_i, y_j} - \overline{x_i, x_j} - \mathrm{i}\{\overline{y_i, x_j} + \overline{x_i, y_j}\}] \end{pmatrix} \quad \ldots \quad (14.4'')$$

(14.4) presents the law of propagation of variances for complex coordinate variates, expressed in the various real components.

From (14.4) one obtains:

$$\left.\begin{aligned} \overline{z_i^T, z_j^T} &= (\overline{z_i, z_j})^T \\ \overline{z_i^T, z_j} &= (\overline{z_i, z_j^T})^T \end{aligned}\right\} [i \text{ and } j \text{ also interchangeable}] \quad \ldots\ldots\ldots\ldots \quad (14.5)$$

$$\overline{z_i\,, z_i} = \overline{y_i, y_i} - \overline{x_i, x_i} + 2\mathrm{i} \cdot \overline{y_i, x_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14.6)$$

Remark

Of course, adjustment problems in the plane can be further treated by complex numbers. For the present publication the above results are sufficient.

Circular standard ellipses

In geodetic literature around 1930, emphasis was given to the requirement that point standard ellipses of coordinated points should be circles as near as possible. J. M. TIENSTRA made many studies – unfortunately most unpublished – on this problem area, and in this connection he designed his "cotangent method" for the assignment of weights to measured angles.*)

He based his investigations on complex numbers in order to get a better insight into the problems of plane surveying; this approach was also the starting point for the later publications by BAARDA. The following study on circularity of standard ellipses also goes back to TIENSTRA.

From the theory treated in section 2 it follows that point standard ellipses alone do not provide an absolute criterion, because they can only be defined in an S-system. Therefore the requirement of circularity must be formulated such that it is invariant with respect to an S-transformation. From formulae like (2.12) it appears that, apart from point standard ellipses, also relative standard ellipses must be considered. It will be shown that this is sufficient.

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*) J. M. TIENSTRA – Nets of Triangles consisting of Points with Circular Error-Curves – Kon. Akad. van Wetenschappen, Proceedings Vol. XXXVI, No. 6, Amsterdam, 1933.

We start from an arbitrary z-system.
For circular point standard ellipses we have:

$$\overline{y_i, y_i} = \overline{x_i, x_i}, \quad \overline{y_i, x_i} = 0 \quad \ldots \quad (14.7')$$

Hence from: (14.6):

$$\boxed{\overline{z_i, z_i} = 0}; \quad i = \ldots \quad \ldots \quad (14.7'')$$

For circular relative standard ellipses we have:

$$\overline{y_{ij}, y_{ij}} = \overline{x_{ij}, x_{ij}}, \quad \overline{y_{ij}, x_{ij}} = 0 \quad \ldots \quad (14.8')$$

Hence, again with (14.6) and in the analogy of (14.7''):

$$\overline{z_{ij}, z_{ij}} = 0 \quad \ldots \quad (14.8'')$$

or written out:

$$\overline{(z_j - z_i), (z_j - z_i)} = \overline{z_j, z_j} + \overline{z_i, z_i} - 2\,\overline{z_j, z_i} = 0$$

or, with (14.7):

$$\boxed{\overline{z_i, z_j} = \overline{z_j, z_i} = 0}; \quad i, j = \ldots \quad \ldots \quad (14.9')$$

or, in view of (14.4''):

$$\overline{y_i, y_j} = \overline{x_i, x_j}, \quad \overline{y_i, x_j} = -\,\overline{x_i, y_j} \quad \ldots \quad (14.9'')$$

Application of the law of propagation of variances to any linear function of variates $\underline{z}_i$ results always in circular standard ellipses. So we have for quantities $\underline{\Delta z}_i^{(rs)}$, $\underline{\Delta z}_i^{(vw)}$, $\underline{\Delta \Pi}_{jik} = \underline{\Delta \Lambda}_{ik} - \underline{\Delta \Lambda}_{ij}$, {cf. section 2 and [2], [4]}:

$$\boxed{\begin{array}{llll}
\text{If:} & \overline{z_i, z_i} & = \overline{z_{ij}, z_{ij}} & = 0; \quad i, j, k = \ldots \\
\text{Then:} & \overline{z_i^{(rs)}, z_i^{(rs)}} & = \overline{z_{ij}^{(rs)}, z_{ij}^{(rs)}} & = 0 \\
 & \overline{z_i^{(vw)}, z_i^{(vw)}} & = \overline{z_{ij}^{(vw)}, z_{ij}^{(vw)}} & = 0 \\
 & \overline{\Pi_{jik}, \Pi_{jik}} & = \overline{\Pi_{jik}, \Pi_{j'i'k'}} & = 0 \\
 & \overline{\Lambda_{ik}, \Lambda_{ik}} & = \overline{\Lambda_{ij}, \Lambda_{ij}} & = 0 \\
 & \text{etc.} & &
\end{array}} \quad \ldots \quad (14.10)$$

From (14.10) follows, in view of (14.4)–(14.9):

In the case of circular point and relative standard ellipses *then*: only $\overline{z_i, z_j^T}$ to be considered

$$\begin{aligned} \overline{z_i, z_j^T} &= 2(\overline{x_i, x_j} + \mathrm{i} \cdot \overline{x_i, y_j}) \\ &= 2(\overline{y_i, y_j} - \mathrm{i} \cdot \overline{y_i, x_j}) \end{aligned} \quad \ldots \quad (14.11)$$

(14.11) is the base on which in the following sections is built on for the construction of an artificial covariance matrix, the criterion matrix for problems in the plane. This matrix is constructed under the strict requirement that point- and relative standard ellipses are circles.

Note to section 14. Conformal mapping

It is interesting to compare the similarity transformation with the general conformal mapping of isometric coordinate systems by an analytic function:

$$\underline{w}_i = w(\underline{z}_i) \quad \ldots \quad (14.12')$$

or, assuming that linearization is admissible*):

$$\underline{\Delta w}_i = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_i \cdot \underline{\Delta z}_i \quad \ldots \quad (14.12'')$$

in which the value $(\mathrm{d}w/\mathrm{d}z)_i$ now may be different from point to point.

Then we have, in correspondence with (14.10):

$$\begin{array}{|ll|} \text{If:} & \overline{z_i, z_j} = 0 \\ \text{Then:} & \overline{w_i, w_j} = 0 \\ \end{array} \quad \ldots \quad (14.13)$$

From (14.12″) follows a characteristic difference with the similarity transformation when difference-variates are considered.

Assuming:

$$\left\{\text{average value of} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_i\right\} = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0 \quad \ldots \quad (14.14')$$

we have:

*) See e.g.: C. H. VAN Os – Inleiding tot de functietheorie {Introduction to the theory of functions} – Noordhoff, Groningen, 1935; page 84, 95, 100.

$$\left.\begin{aligned}\underline{\Delta w}_i &= \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0 \cdot \underline{\Delta z}_i + \left\{\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_i - \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0\right\} \cdot \underline{\Delta z}_i \\ \underline{\Delta w}_{ij} &= \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0 \cdot \underline{\Delta z}_{ij} + \left\{\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_j - \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0\right\} \cdot \underline{\Delta z}_j + \\ &\qquad - \left\{\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_i - \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)_0\right\} \cdot \underline{\Delta z}_i\end{aligned}\right\} \qquad (14.14'')$$

The point standard ellipses are thus submitted to an expansion and a rotation both changing per point. For relative standard ellipses this is still more complicated.

This matter may be important when the results of coordinate computations on the geo-reference-ellipsoid have to be compared with those of coordinate computations in the plane with the intervention of a map projection.

15 CRITERION MATRICES IN NETWORKS FOR HORIZONTAL CONTROL

Consider the points P_i and P_j with reference to the base points P_r, P_s:

$$\left.\begin{aligned} \underline{\Delta z}_i^{(rs)} &= \underline{\Delta z}_i - \frac{z^0_{si}}{z^0_{sr}}\underline{\Delta z}_r - \frac{z^0_{ri}}{z^0_{rs}}\underline{\Delta z}_s \\ \underline{\Delta z}_j^{(rs)} &= \underline{\Delta z}_j - \frac{z^0_{sj}}{z^0_{sr}}\underline{\Delta z}_r - \frac{z^0_{rj}}{z^0_{rs}}\underline{\Delta z}_s \end{aligned}\right\} \qquad (15.1)$$

or with (2.12):

$$\left.\begin{aligned} \underline{\Delta z}_i^{(rs)} &= -\frac{z^0_{ri}z^0_{is}}{z^0_{rs}}\underline{\Delta \Pi}_{ris} \\ \underline{\Delta z}_j^{(rs)} &= -\frac{z^0_{rj}z^0_{js}}{z^0_{rs}}\underline{\Delta \Pi}_{rjs} \end{aligned}\right\} \qquad (15.2)$$

In order to simplify the notation,

the superscript 0 will be omitted. . . . . (15.3)

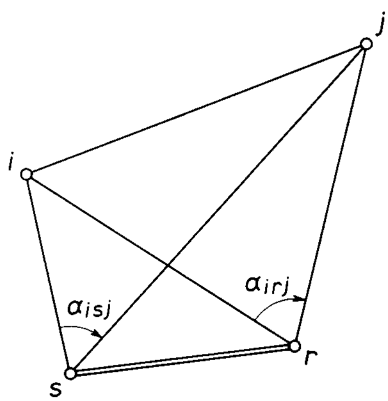


Fig. 15-1

If we introduce the notation:

$$\mathrm{Re}\{a \cdot b\} = \tfrac{1}{2}(a \cdot b + a^T \cdot b^T) \qquad (15.4)$$

It can be derived from (15.1) that*):

*) Cf. note to section 15.

$$\boxed{\begin{aligned}\operatorname{Re}\{e^{i\alpha_{irj}}\cdot\overline{z_i^{(rs)}, z_j^{(rs)T}}\} &= \frac{l_{ri}l_{js}}{l_{rs}}G_{rjs}+\frac{l_{rj}l_{is}}{l_{rs}}G_{ris}-l_{ij}G_{irj}+\\ &\quad-\frac{l_{ri}l_{rj}}{l_{rs}}\operatorname{Re}\left\{\frac{\overline{z_r, z_r^T}-\overline{z_s, z_s^T}}{l_{rs}}\right\}\end{aligned}} \quad \ldots\ldots\ldots \quad (15.5')$$

$$\boxed{\begin{aligned}\operatorname{Re}\{e^{i\alpha_{isj}}\cdot\overline{z_i^{(rs)}, z_j^{(rs)T}}\} &= \frac{l_{ri}l_{js}}{l_{rs}}G'_{ris}+\frac{l_{rj}l_{is}}{l_{rs}}G'_{rjs}-l_{ij}G'_{isj}+\\ &\quad-\frac{l_{si}l_{sj}}{l_{rs}}\operatorname{Re}\left\{\frac{\overline{z_s, z_s^T}-\overline{z_r, z_r^T}}{l_{rs}}\right\}\end{aligned}} \quad \ldots\ldots\ldots \quad (15.5'')$$

in which G and G' are symmetric functions of a triangle {the order of the indices of G is irrelevant} e.g.:

$$\boxed{\begin{aligned}G_{ris} = \operatorname{Re}\Bigg\{ & e^{i\alpha_{isr}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_i, z_r^T}}{l_{ir}}\right)+e^{i\alpha_{sri}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_s, z_i^T}}{l_{si}}\right)+\\ & +e^{i\alpha_{ris}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_r, z_s^T}}{l_{rs}}\right)\Bigg\}\end{aligned}} \quad \ldots\ldots \quad (15.6')$$

$$\boxed{\begin{aligned}G'_{ris} = \operatorname{Re}\Bigg\{ & e^{i\alpha_{isr}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_i, z_r^T}}{l_{ir}}\right)+e^{i\alpha_{sri}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_s, z_i^T}}{l_{si}}\right)+\\ & +e^{i\alpha_{ris}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_r, z_s^T}}{l_{rs}}\right)\Bigg\}\end{aligned}} \quad \ldots\ldots \quad (15.6'')$$

For the construction of criterion matrices, these formulae can be considerably simplified by introducing *):

$$\boxed{\overline{z_r, z_r^T}=\overline{z_s, z_s^T}} \;\; \boxed{\rightarrow} \;\; \boxed{G_{ris}=G'_{ris}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15.7)$$

For circular standard ellipses in any system we have, according to (14.11):

$$\left.\begin{aligned}\overline{z_i, z_j^T} &= 2(\overline{x_i, x_j}+i\cdot\overline{x_i, y_j})\\ \overline{z_i, z_i^T} &= 2\overline{x_i, x_i}>0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15.8)$$

Hence, with: $e^{i\alpha_{iri}}=e^{i\alpha_{isi}}=1$, from (15.5)*)

$$\left.\begin{aligned}&\overline{z_i^{(rs)}, z_i^{(rs)T}}=2\frac{l_{ri}l_{is}}{l_{rs}}G_{ris}>0\\ \text{Hence:}\quad & G_{ris}>0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15.9)$$

*) But see the text accompanying (15.21).

Now for the net conditions*) $N_{(r),i,s}$ and $N_{(i),s,r}$:

$$z_{ir}\cdot\underline{\Delta\Pi}_{ris}+z_{sr}\cdot\underline{\Delta\Pi}_{isr}=0$$

$$z_{si}\cdot\underline{\Delta\Pi}_{isr}+z_{ri}\cdot\underline{\Delta\Pi}_{sri}=0$$

or:

$$\frac{\underline{\Delta\Pi}_{ris}}{z_{rs}}=\frac{\underline{\Delta\Pi}_{isr}}{z_{ir}}=\frac{\underline{\Delta\Pi}_{sri}}{z_{si}} \qquad (15.10)$$

and hence the general validity of the theorem J. M. TIENSTRA derived for angles in his cotangent method for the adjustment of triangulation networks**):

$$\boxed{\begin{array}{l}\text{In triangle } r, i, s \text{ is:}\\ \dfrac{\overline{\Pi_{ris},\Pi^T_{ris}}}{l^2_{rs}}=\dfrac{\overline{\Pi_{isr},\Pi^T_{isr}}}{l^2_{ir}}=\dfrac{\overline{\Pi_{sri},\Pi^T_{sri}}}{l^2_{si}}\end{array}} \qquad (15.11)$$

With (15.2) follows from (15.9):

$$\overline{z_i^{(rs)},z_i^{(rs)T}}=\frac{l^2_{ri}l^2_{is}}{l^2_{rs}}\,\overline{\Pi_{ris},\Pi^T_{ris}}=2\frac{l_{ri}l_{is}}{l_{rs}}G_{ris}$$

or:

$$\frac{\overline{\Pi_{ris},\Pi^T_{ris}}}{l^2_{rs}}=\frac{2}{l_{rs}l_{si}l_{ir}}G_{ris} \qquad (15.12)$$

indeed a symmetric function of the triangle r, i, s, in agreement with (15.11).

(15.11) indicates a dependence of the variances of Π-variates on the lengths in corresponding triangles; from (15.12) it should then follow that G_{ris} is a function of lengths in triangle r, i, s, and according to (15.9) it should be a positive function.

Put therefore, analogous to (12.7), and with (15.7):

$$\left.\begin{array}{l}\overline{z_r,z_r^T}-\overline{z_r,z_i^T}=\overline{z_s,z_s^T}-\overline{z_r,z_i^T}=2(d^2_{ri}-i\bar{d}_{ri})\\ \overline{z_r,z_r^T}-\overline{z_i,z_j^T}=\overline{z_s,z_s^T}-\overline{z_i,z_j^T}=2(d^2_{ij}-i\bar{d}_{ij})\end{array}\right\} \qquad (15.13')$$

*) In section 2.2 of [4] was discussed the linking of observational results to a mathematical model which consists of a probability distribution of corresponding variates, between whose means a number of functional relationships {"laws of nature"} are assumed to exist. The relationships are derived from a consistent mathematical model. In section 2 of [9] the set of "laws of nature" has been indicated as "condition model". Using this model in an adjustment problem the means of variates in the "laws of nature" are replaced by estimators {of the means}. After this replacement the "laws of nature" are labelled as "condition equations". In section 15, 15.a and 16 a consistent network in building up artificial matrices is always used. That means that in these networks the variates introduced correspond to estimators in condition equations. Therefore we shall indicate functional relationships between these variates as "conditions".

**) J. M. TIENSTRA – Nets of Triangles Consisting of Points with Circular Error-Curves – Kon. Akad. van Wetenschappen, Proceedings Vol. XXXVI, No. 6, Amsterdam, 1933.

$\bar{d}_{ij}$ can be a positive or negative function of l_{ij}, with:

$$\bar{d}_{ji} = -\bar{d}_{ij} \qquad (15.13'')$$

because according to (15.13′):

$$\overline{z_r, z_r^T} - \overline{z_j, z_i^T} = 2(d_{ji}^2 - \mathrm{i}\cdot\bar{d}_{ji}) = (\overline{z_r, z_r^T} - \overline{z_i, z_j^T})^T = 2(d_{ij}^2 + \mathrm{i}\bar{d}_{ij})$$

If:

$$\left.\begin{aligned} \cos\gamma &= \tfrac{1}{2}(e^{\mathrm{i}\gamma} + e^{-\mathrm{i}\gamma}) \\ \sin\gamma &= -\tfrac{1}{2}\mathrm{i}(e^{\mathrm{i}\gamma} - e^{-\mathrm{i}\gamma}) \end{aligned}\right\} \qquad (15.14)$$

Then it follows from (15.6) with (15.13):

$$G_{ris} = 2\left\{\left(\cos\alpha_{isr}\cdot\frac{d_{ir}^2}{l_{ir}} + \cos\alpha_{sri}\cdot\frac{d_{si}^2}{l_{si}} + \cos\alpha_{ris}\cdot\frac{d_{rs}^2}{l_{rs}}\right) + \right.$$
$$\left. + \left(\sin\alpha_{isr}\cdot\frac{\bar{d}_{ir}}{l_{ir}} + \sin\alpha_{sri}\cdot\frac{\bar{d}_{si}}{l_{si}} + \sin\alpha_{ris}\cdot\frac{\bar{d}_{rs}}{l_{rs}}\right)\right\} \qquad (15.15)$$

Consider now, e.g., the situation in Fig. 15-2.

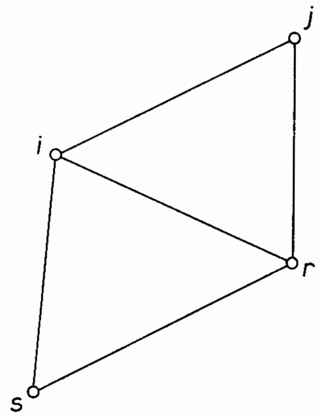


Fig. 15.2

A safe requirement for $G_{ris} > 0$ would be:

$$\sin\alpha_{isr}\frac{\bar{d}_{ir}}{l_{ir}} > 0$$

But a requirement for $G_{rji} > 0$ would be:

$$\sin\alpha_{rji}\frac{\bar{d}_{ri}}{l_{ri}} = -\sin\alpha_{rji}\frac{\bar{d}_{ir}}{l_{ir}} > 0$$

but: $\sin\alpha_{isr} > 0$ and $\sin\alpha_{rji} > 0$, so the requirements for G_{ris} and G_{rjs} are contradictory.

Therefore, for safety, one must put:

$$\overline{d}_{ri} = \overline{d}_{ij} = \ldots = 0 \quad \ldots\ldots \quad (15.16)$$

(15.13)–(15.16) then lead to the assumption, analogous to (12.7):

$$\boxed{\begin{array}{l|l} \overline{z_r, z_r^T} = \overline{z_s, z_s^T} = 2d^2 & c_p \geqslant 0 \\ \overline{z_r, z_r^T} - \overline{z_r, z_i^T} = 2d_{ri}^2 = & d_{ri}^2 = \sum\limits_p c_p \cdot (l_{ri})^p \\ = \overline{z_r, z_r^T} - \overline{z_i, z_r^T} = 2d_{ir}^2 & \\ \overline{z_r, z_r^T} - \overline{z_i, z_j^T} = 2d_{ij}^2 = & d_{ij}^2 = \sum\limits_p c_p \cdot (l_{ij})^p \\ = \overline{z_r, z_r^T} - \overline{z_j, z_i^T} = 2d_{ji}^2 & \end{array}} \quad \ldots\ldots \quad (15.17)$$

$$\boxed{G_{ris} = 2\left(\cos\alpha_{isr} \cdot \frac{d_{ir}^2}{l_{ir}} + \cos\alpha_{sri} \cdot \frac{d_{si}^2}{l_{si}} + \cos\alpha_{ris} \cdot \frac{d_{rs}^2}{l_{rs}}\right)} \quad \ldots\ldots \quad (15.18)$$

But (15.16) means, according to (15.8):

$$\boxed{\overline{d}_{ij} = 0} \rightarrow \boxed{\overline{x_i, y_j} = 0 = -\overline{y_i, x_j}} \quad \ldots\ldots \quad (15.19)^{*)}$$

so that (15.17) can be written:

$$\boxed{\begin{array}{l} \overline{y_r, y_r} = \overline{x_r, x_r} = \overline{y_s, y_s} = \overline{x_s, x_s} = d^2 \\ \hline \overline{y_r, y_r} - \overline{y_r, y_i} = \overline{x_r, x_r} - \overline{x_r, x_i} = d_{ri}^2 \\ \overline{y_r, y_r} - \overline{y_i, y_j} = \overline{x_r, x_r} - \overline{x_i, x_j} = d_{ij}^2 \end{array}} \quad \ldots\ldots \quad (15.20)$$

(15.20) means that for y-coordinates as well as for x-coordinates the system (12.7) with (12.9) is assumed. In addition, according to (15.19), the correlation between y- and x-coordinates is assumed to be zero.

Now the transition from (15.5) to (15.9) must be investigated somewhat closer. It is easily shown that: for $j \rightarrow i$, $G_{rjs} \rightarrow G_{ris}$, but:

---

*) The interpretation of (15.19) is difficult. It could be stated that the constructed criterion matrix refers to an infinite network. Then (15.19) agrees with a theorem of J. M. TIENSTRA dating from about 1936 and stating that in an infinite triangulation network x-coordinates are not correlated with y-coordinates. This study is connected with the cited paper of 1933 by TIENSTRA, but neither his manuscript nor personal notes made during a lecture enable a reconstruction of the proof. The difficulty of this interpretation lies in the fact that after an S-transformation (15.1) there does exist correlation between $\underline{x}_i^{(rs)}$ and $\underline{y}_j^{(rs)}$ $(i \neq j)$, see (15.38). And the question arises: how can an infinite network be computed in coordinates *without* assuming a computational base i.e. without working in an S-system?

$$l_{ij}\cdot G_{irj} = l_{ij}\cdot \mathrm{Re}\left\{ e^{i\alpha_{rji}} \cdot \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_i^T}}{l_{ri}}\right) + e^{i\alpha_{jir}}\cdot\left(\frac{\overline{z_r, z_r^T} - \overline{z_j, z_r^T}}{l_{jr}}\right) + \right.$$
$$\left. + e^{i\alpha_{irj}}\cdot\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_j^T}}{l_{ij}}\right)\right\}$$

with $j \to i$: $\alpha_{iri} = 0$, α_{rii} and α_{iir} undefined, although always mod $\{e^{i\alpha}\} = 1$. Hence:

$$\lim_{j \to i} l_{ij}\cdot G_{irj} = \mathrm{Re}\{\overline{z_r, z_r^T} - \overline{z_i, z_i^T}\} = 0, \text{ provided, like in (12.7):}$$

$$d_{ii}^2 = 0 \qquad \text{(15.21')}$$

by which (15.7) follows from (15.21).

Buth then (15.17) has to be complemented with:

$$\boxed{c_0 = 0} \qquad \text{(15.21'')}$$

Finally, from (13.18) and (13.19) in view of (15.21) follows:

$$\boxed{0 < p < 2} \qquad \text{(15.22)}$$

Now the introduction of Δd^2-terms from section 13 can be applied, with, among other things, the introduction, in spite of (15.21''), of c_0 in the form of $c_0 = \Delta d_r^2 = \Delta d_s^2$. But see for this also section 15.a, in which an entirely alternative approach to the present theory has been elaborated.

In an early stage of the theory, (15.22) was checked by H. C. VAN DER HOEK, who showed numerically:

$$G_{ris} > 0 \quad \text{for} \quad 0 \leqslant p < 2$$

In addition he proved that this is valid for the determinant of the whole criterium matrix. For the further construction of this criterion matrix we must, however, go further than (15.9).

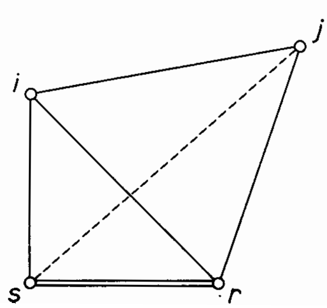
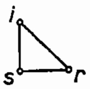
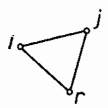


Fig. 15-3

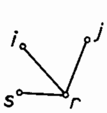
According to (15.9)–(15.12) this approach is consistent for one triangle at a time, but is it consistent for more than three points simultaneously?
Consider now, besides the previously introduced net conditions per triangle one of the net conditions in the quadrangle of Fig. 15-3:





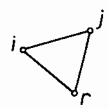
$$\left.\begin{array}{ll} N_{(i),s,r,j}: & z_{si}\cdot\underline{\Delta\Pi}_{isr}+z_{ri}\cdot\underline{\Delta\Pi}_{srj}+z_{ji}\cdot\underline{\Delta\Pi}_{rji}=0 \\ N_{(i),s,r}\ : & z_{si}\cdot\underline{\Delta\Pi}_{isr}+z_{ri}\cdot\underline{\Delta\Pi}_{sri}\qquad\qquad\quad\ \ =0 \\ N_{(i),r,j}\ : & \qquad\qquad\quad\ \ z_{ri}\cdot\underline{\Delta\Pi}_{irj}+z_{ji}\cdot\underline{\Delta\Pi}_{rji}=0 \end{array}\right\} \qquad (15.23)$$

From (15.23) follows a type of central condition:

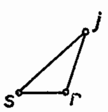


$$\boxed{\underline{\Delta\Pi}_{irj}-\underline{\Delta\Pi}_{srj}+\underline{\Delta\Pi}_{sri}=0} \qquad (15.24)$$

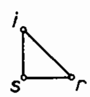
or with:



$$N_{(i),r,j}:\ \underline{\Delta\Pi}_{irj}=-\frac{z_{ji}}{z_{ri}}\cdot\underline{\Delta\Pi}_{rji}$$



$$N_{(s),r,j}:\ \underline{\Delta\Pi}_{srj}=-\frac{z_{js}}{z_{rs}}\cdot\underline{\Delta\Pi}_{rjs}$$



$$N_{(s),r,i}:\ \underline{\Delta\Pi}_{sri}=-\frac{z_{is}}{z_{rs}}\cdot\underline{\Delta\Pi}_{ris}$$

gives (15.24):

$$-\frac{z_{rj}z_{ji}}{z_{ri}}\cdot\underline{\Delta\Pi}_{rji}+\frac{z_{rj}z_{js}}{z_{rs}}\cdot\underline{\Delta\Pi}_{rjs}-\frac{z_{rj}}{z_{ri}}\frac{z_{ri}z_{is}}{z_{rs}}\cdot\underline{\Delta\Pi}_{ris}=0$$

or with (2.12):

$$\underline{\Delta z}_j^{(ri)}-\underline{\Delta z}_j^{(rs)}+\frac{z_{rj}}{z_{ri}}\underline{\Delta z}_i^{(rs)}=0$$

or, written with $\underline{\Delta z}_r^{(rs)}=0$:

$$\boxed{\underline{\Delta z}_j^{(ri)} = \underline{\Delta z}_j^{(rs)} - \frac{z_{ij}}{z_{ir}}\underline{\Delta z}_r^{(rs)} - \frac{z_{rj}}{z_{ri}}\underline{\Delta z}_i^{(rs)}} \qquad (15.25)$$

For the line of thought to be followed, the development of (15.25) from $N_{(i),s,r,j}$ in (15.23) is of importance.

Further, (15.25) gives the well-known expression of $\underline{\Delta z}_j$ in the $\underset{r,i}{\text{S}}$-system as derived from the $\underset{r,s}{\text{S}}$-system, see Fig. 15-3.

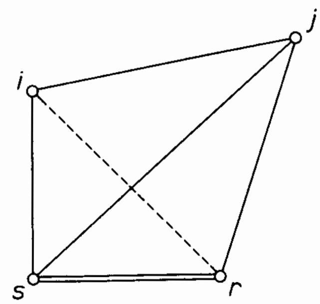
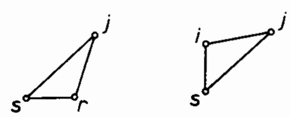


Fig. 15-4

Now consider the situation shown in Fig. 15-4. We examine a second net condition in the quadrangle, and also a different combination of net conditions in a triangle. Interchanging j and i and also r and s gives:

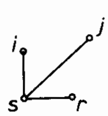
$$N_{(j),r,s,i} \quad \text{with} \quad N_{(j),r,s} \quad \text{and} \quad N_{(j),s,i} \qquad (15.26)$$



from (15.24):

$$\underline{\Delta \Pi}_{jsi} - \underline{\Delta \Pi}_{rsi} + \underline{\Delta \Pi}_{rsj} = 0$$

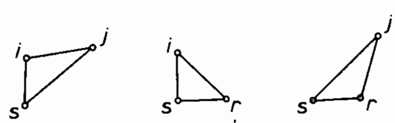
or:



$$\boxed{\underline{\Delta \Pi}_{isj} - \underline{\Delta \Pi}_{isr} + \underline{\Delta \Pi}_{jsr} = 0} \qquad (15.27)$$

or with:

$$N_{(j),si} \quad \text{en} \quad N_{(r),s,i} \quad \text{en} \quad N_{(r),s,j}:$$



analogous to (15.25), with $\underline{\Delta z}_s^{(sr)} = 0$:

$$\boxed{\underline{\Delta z}_i^{(sj)} = \underline{\Delta z}_i^{(sr)} - \frac{z_{ji}}{z_{js}}\underline{\Delta z}_s^{(sr)} - \frac{z_{si}}{z_{sj}}\underline{\Delta z}_j^{(sr)}} \quad \ldots\ldots\ldots\ldots\ldots \quad (15.28)$$

the well-known expression for $\underline{\Delta z}_i$ in the $\underset{s,j}{\text{S}}$-system, starting from the $\underset{s,r}{\text{S}}$-system which is identical to the $\underset{r,s}{\text{S}}$-system, see Fig. 15-4.

> *Conclusion*: Because $N_{(i),s,r,j}$ and $N_{(j),r,s,i}$ are independent conditions, the equations (15.25) and (15.28) are independent in the case of consistent {closed} triangles.

From (15.25) follows with, see (14.5), $\overline{z_j, z_i^T} = (\overline{z_i, z_j^T})^T$:

$$\overline{z_j^{(ri)}, z_j^{(ri)T}} = \overline{\left(z_j^{(rs)} - \frac{z_{rj}}{z_{ri}}\, z_i^{(rs)}\right), \left(z_j^{(rs)T} - \frac{z_{rj}^T}{z_{ri}^T}\, z_i^{(rs)T}\right)} =$$

$$= \overline{z_j^{(rs)}, z_j^{(rs)T}} - \left(\frac{z_{rj}}{z_{ri}}\,\overline{z_i^{(rs)}, z_j^{(rs)T}} + \frac{z_{rj}^T}{z_{ri}^T}\,\overline{z_j^{(rs)}, z_i^{(rs)T}}\right) + \frac{z_{rj}z_{rj}^T}{z_{ri}z_{ri}^T}\,\overline{z_i^{(rs)}, z_i^{(rs)T}} =$$

$$= \overline{z_j^{(rs)}, z_j^{(rs)T}} - 2\,\mathrm{Re}\left\{\frac{z_{rj}}{z_{ri}}\,\overline{z_i^{(rs)}, z_j^{(rs)T}}\right\} + \frac{l_{rj}^2}{l_{ri}^2}\,\overline{z_i^{(rs)}, z_i^{(rs)T}}$$

$$\mathrm{Re}\{e^{i\alpha_{irj}} \cdot \overline{z_i^{(rs)}, z_j^{(rs)T}}\} = \tfrac{1}{2}\frac{l_{ri}}{l_{rj}}\left(\overline{z_j^{(rs)}, z_j^{(rs)T}} + \frac{l_{rj}^2}{l_{ri}^2}\,\overline{z_i^{(rs)}, z_i^{(rs)T}} - \overline{z_j^{(ri)}, z_j^{(ri)T}}\right)$$

or with (15.9):

$$\boxed{\begin{aligned}\mathrm{Re}\{e^{i\alpha_{irj}}\,\overline{z_i^{(rs)}, z_j^{(rs)T}}\} &= \frac{l_{ri}l_{js}}{l_{rs}}G_{rjs} + \frac{l_{rj}l_{is}}{l_{rs}}G_{ris} - l_{ij}G_{irj} \\ &\underset{\text{denote by}}{=} q_{irj}^{(rs)}\end{aligned}} \quad \ldots\ldots\ldots \quad (15.30)$$

Because (15.30) is symmetric in i and j, (15.30) can also be derived from $\underline{\Delta z}_i^{(rj)}$ instead of $\underline{\Delta z}_j^{(ri)}$ in (15.25).

Similarly follows from (15.28):

$$\boxed{\begin{aligned}\mathrm{Re}\{e^{i\alpha_{isj}}\,\overline{z_i^{(rs)}, z_j^{(rs)T}}\} &= \frac{l_{ri}l_{js}}{l_{rs}}G_{ris} + \frac{l_{rj}l_{is}}{l_{rs}}G_{rjs} - l_{ij}G_{isj} \\ &\underset{\text{denote by}}{=} q_{isj}^{(rs)}\end{aligned}} \quad \ldots\ldots\ldots \quad (15.31)$$

This relation can also be derived via $\underline{\Delta z}_j^{(si)}$.

But (15.30) and (15.31) are identical to the equations (15.5′) and (15.5″) respectively – also in view of (15.7) – so that by (15.29) we have:

$$\boxed{\begin{array}{l}\textit{Conclusion}\text{: The equations}\\ \qquad\text{(15.5}'\text{ or (15.30)}\\ \qquad\text{and (15.15}'\text{) or (15.31)}\\ \qquad\text{are independent}\end{array}} \qquad (15.32)$$

Hence these equations can be solved for $\overline{z_i^{(rs)}, z_j^{(rs)T}}$:

To do this, introduce:

$$\left.\begin{aligned}\alpha_{irj} &= \tfrac{1}{2}(\alpha_{irj}+\alpha_{isj})+\tfrac{1}{2}(\alpha_{irj}-\alpha_{isj}) = \gamma_{ij}^{(rs)}+\delta_{ij}^{(rs)}\\ \alpha_{isj} &= \tfrac{1}{2}(\alpha_{irj}+\alpha_{isj})-\tfrac{1}{2}(\alpha_{irj}-\alpha_{isj}) = \gamma_{ij}^{(rs)}-\delta_{ij}^{(rs)}\end{aligned}\right\} \qquad (15.33)$$

Introduce temporarily the simpler notation:

$$\left.\begin{aligned}&\overline{z_i^{(rs)}, z_j^{(rs)T}} = \omega\\ &\alpha_{irj} = \alpha_r, \quad \alpha_{isj} = \alpha_s\\ &\gamma_{ij}^{(rs)} = \gamma\,, \quad \delta_{ij}^{(rs)} = \delta\end{aligned}\right\} \qquad (15.34)$$

From (15.30)–(15.34) with (15.14) follows then:

$$\left.\begin{aligned}e^{i\alpha_r}\cdot\omega+e^{-i\alpha_r}\cdot\omega^T &= 2q_r \quad \Big| \quad \alpha_r = \gamma+\delta\\ e^{i\alpha_s}\cdot\omega+e^{-i\alpha_s}\cdot\omega^T &= 2q_s \quad \Big| \quad \alpha_s = \gamma-\delta\end{aligned}\right\} \qquad (15.35)$$

sum:

$$\omega e^{i\gamma}(e^{i\delta}+e^{-i\delta})+\omega^T e^{-i\gamma}(e^{i\delta}+e^{-i\delta}) = 2(q_r+q_s)$$

difference:

$$\omega e^{i\gamma}(e^{i\delta}-e^{-i\delta})-\omega^T e^{-i\gamma}(e^{i\delta}-e^{-i\delta}) = 2(q_r-q_s)$$

$$\left.\begin{aligned}\{\omega e^{i\gamma}+(\omega e^{i\gamma})^T\}\cos\delta &= q_r+q_s\\ \{\omega e^{i\gamma}-(\omega e^{i\gamma})^T\}\sin\delta &= -i(q_r-q_s)\end{aligned}\right\} \qquad (15.36)$$

or, see also (15.46):

$$\left.\begin{aligned}&\cos\delta \neq 0, \quad \sin\delta \neq 0\\ &\omega = \tfrac{1}{2}e^{-i\gamma}\left(\frac{q_r+q_s}{\cos\delta} - i\frac{q_r-q_s}{\sin\delta}\right)\end{aligned}\right\} \qquad (15.37)$$

or, with (15.30)–(15.34) and (15.8):

$$\gamma_{ij}^{(rs)} = \tfrac{1}{2}(\alpha_{irj}+\alpha_{isj}) \qquad \delta_{ij}^{(rs)} = \tfrac{1}{2}(\alpha_{irj}-\alpha_{isj})$$

$$q_{irj}^{(rs)}+q_{isj}^{(rs)} = \frac{l_{ri}l_{js}+l_{rj}l_{is}}{l_{rs}}(G_{rjs}+G_{ris})-l_{ij}(G_{irj}+G_{isj})$$

$$q_{irj}^{(rs)}-q_{isj}^{(rs)} = \frac{l_{ri}l_{js}-l_{rj}l_{is}}{l_{rs}}(G_{rjs}-G_{ris})-l_{ij}(G_{irj}-G_{isj})$$

$$\overline{z_i^{(rs)}, z_j^{(rs)T}} = \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}\gamma_{ij}^{(rs)}}\left(\frac{q_{irj}^{(rs)}+q_{isj}^{(rs)}}{\cos\delta_{ij}^{(rs)}} - \mathrm{i}\,\frac{q_{irj}^{(rs)}-q_{isj}^{(rs)}}{\sin\delta_{ij}^{(rs)}}\right) \qquad (15.38)$$

$$\begin{aligned}\overline{z_i^{(rs)}, z_j^{(rs)T}} &= 2(\overline{x_i^{(rs)}, x_j^{(rs)}}+\mathrm{i}\cdot\overline{x_i^{(rs)}, y_j^{(rs)}})\\ &= 2(\overline{y_i^{(rs)}, y_j^{(rs)}}-\mathrm{i}\cdot\overline{y_i^{(rs)}, x_j^{(rs)}})\end{aligned}$$

$$\cos\delta_{ij}^{(rs)} \neq 0 \qquad \sin\delta_{ij}^{(rs)} \neq 0 \qquad (15.39)$$

It appears that the restriction (15.39) has only practical but no theoretical significance. Some limiting cases are therefore investigated.

Quadrilateral inscribable in a circle

It is mainly due to J. E. ALBERDA that this situation has been intensively studied. One of the results was that the incomplete and therefore incorrect approach of 1963 by BAARDA could be changed to the present one.

It proves to be sufficient to consider two cases:

Case A, see Fig. 15-5

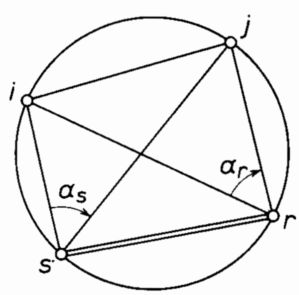


Fig. 15-5

$$\left.\begin{aligned}&\alpha_r = \alpha_s = \gamma; \qquad \delta = 0; \qquad \sin\delta = 0\\ &l_{ri}l_{js}-l_{rj}l_{is} = l_{ij}l_{rs}, \qquad \text{hence:}\\ &q_r - q_s = G_{rjs}-G_{ris}-G_{irj}+G_{isj} = 0\end{aligned}\right\} \qquad (15.40)$$

Case B, see Fig. 15-6

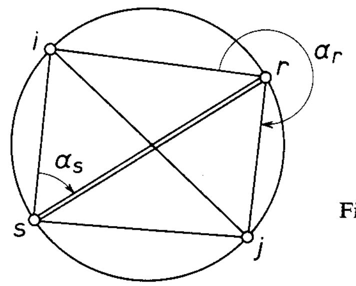


Fig. 15-6

$$\left.\begin{array}{l} \alpha_r = \alpha_s + \pi; \qquad \gamma = \alpha_r + \frac{\pi}{2}; \qquad \delta = \frac{\pi}{2}; \qquad \cos\delta = 0 \\ l_{ri}l_{js} + l_{rj}l_{is} = l_{ij}l_{rs}, \qquad \text{hence:} \\ q_r + q_s = G_{rjs} + G_{ris} - G_{irj} - G_{isj} = 0 \end{array}\right\} \quad \ldots\ldots\ldots \quad (15.41)$$

In both cases the equations (15.35) are dependent, and one of the equations (15.36) is lost. The question is if for these cases the solution for ω can be considered as a limit of (15.38).

Case A: This case of a quadrilateral inscribed in a circle is completely determined by $\delta \to 0$, hence $\sin\delta \to 0$, so that $\{q_r - q_s\}$ will approach zero, dependent on $\sin\delta$.

Put therefore, with f_A a real function:

$$q_r - q_s = f_A \cdot \sin\delta$$

Then from (15.37):

$$\boxed{\begin{array}{c|c} \multicolumn{2}{c}{\lim\limits_{\sin\delta\to 0} \omega = \frac{1}{2} e^{-i\gamma} \left\{ \dfrac{q_r + q_s}{\cos\delta} - i \cdot f_A \right\}} \\ \hline q_r = q_s & \cos\delta = 1 \end{array}} \quad \ldots\ldots\ldots \quad (15.42)$$

f_A has been left unspecified, but can be written out via formulae as indicated in the note to this section.

Case B: Here, the situation is entirely determined by $\cos\delta \to 0$, so that $(q_r + q_s)$ will approach zero dependent on $\cos\delta$.

Put therefore:

$$q_r + q_s = f_B \cdot \cos\delta$$

Then from (15.37):

$\lim\limits_{\cos\delta\to 0} \omega = \frac{1}{2} e^{-i\gamma}\left\{f_B - i\dfrac{q_r-q_s}{\sin\delta}\right\}$	
$q_r = -q_s$	$\sin\delta = 1$

. . . . (15.43)

f_B is unspecified, but can be written out.

Situation $j\to i$

$j\to i$ gives:

$$\alpha_r = \alpha_s = 0, \quad \gamma = 0,$$

$$\delta = 0, \quad \cos\delta = 1, \quad \sin\delta = 0,$$

$$\omega = 2\overline{x_i^{(rs)}, x_i^{(rs)}} \text{ is real, } \omega = \omega^T$$

Then from the second equation of (15.36):

$$\omega\cdot e^{i\gamma} - \omega^T\cdot e^{i\gamma^T} = -i\frac{q_r-q_s}{\sin\delta}$$

or $j\to i$ gives:

$$0 = \lim_{\delta\to 0}\left\{-i\frac{q_r-q_s}{\sin\delta}\right\} \quad \ldots\ldots\ldots \quad (15.44)$$

Then we have solution (15.42) with $f_A = 0$: $\omega = q_r$, or with (15.30):

$$\overline{z_i^{(rs)}, z_i^{(rs)T}} = 2\frac{l_{ri}l_{is}}{l_{rs}} G_{ris}, \quad \text{or} \quad (15.9).$$

Conclusion: (15.38) is generally valid, provided the appropriate limiting cases are included. This means that the approach followed produces a complete criterion matrix, be it that for numerical computations (15.38) will have to be replaced by a more suitable formula.

. . . . (15.45)

For this, reference is made to section 15.a.

Relation with the matrix of the HTW-1956

The d^2-matrix for given coordinates in the HTW-1956 can be interpreted as an incomplete criterion matrix. In the manual, the consequences of the approach chosen were more or less incidentically drawn in some cases where non-zero covariances were introduced.

It was J. E. ALBERDA*) who further developed the consequences of the HTW-1956

*) J. E. ALBERDA – Een vervangingsmatrix – Manuscript, autumn 1963.

system, alongside the approach by BAARDA which eventually led to the considerations in this section. Transcribed into the notation of (15.20) with (15.17), the base from which he started was:

$$d_{ij}^2 = c_1 \cdot l_{ij} \qquad (15.46)$$

or:

$$\left.\begin{aligned} &\overline{y_r, y_r} = \overline{x_r, x_r} = \overline{y_s, y_s} = \overline{x_s, x_s} = \\ &\qquad\qquad = \overline{y_i, y_i} = \overline{x_i, x_i} = d^2 \\ &\overline{y_i, y_j} = \overline{x_i, x_j} = d^2 - c_1 \cdot l_{ij} \\ &\overline{y_i, x_j} = \overline{x_i, y_j} = 0 \end{aligned}\right\} \qquad (15.47)$$

ALBERDA applied an S-transformation to the matrix (15.47) and so arrived at formulae in which the G-functions (15.18) could be recognized. But the meaning of the assumption (15.47) was not clear and could not be completely motivated. The same was indeed the case with the matrix used in the HTW-1956 from which the proportionality with l_{ij} was adopted; this proportionality is a consequence of the assumption, made in the HTW-1956, that d^2 is predominantly proportional to a mean value of distances between points considered.

The present study is based on an approach following the line of thought in the opposite direction. (15.5) results from the application of the law of propagation of variances to (15.1), and therefore actually to Π-variates.

The formulae (15.5) with (15.6) are now further given an artificial content via the clear line of thought through (15.17)–(15.22).

The parallelism with (12.7) means also that the d^2-matrix (12.12) is twice assumed, now with the inclusion of (13.7): once for $\overline{(y_i), (y_j)^*}$ and once for $\overline{(x_i), (x_j)^*}$, other correlation being zero. The construction via (15.5) guarantees that the application of (15.1) as an S-transformation gives a correct covariance matrix for S-variates.

The parallel with (12.12) consequently gives for coordinates $\underline{y}$ and $\underline{x}$:

$$\begin{array}{c|cccc|cccc}
 & \underline{\Delta y}_i & \underline{\Delta y}_j & \underline{\Delta y}_r & \underline{\Delta y}_s & \underline{\Delta x}_i & \underline{\Delta x}_j & \underline{\Delta x}_r & \underline{\Delta x}_s \\ \hline
\underline{\Delta y}_i & d^2 & (d^2-d_{ij}^2) & (d^2-d_{ir}^2) & (d^2-d_{is}^2) & & & & \\
\underline{\Delta y}_j & (d^2-d_{ji}^2) & d^2 & (d^2-d_{jr}^2) & (d^2-d_{js}^2) & & & & \\
\underline{\Delta y}_r & (d^2-d_{ri}^2) & (d^2-d_{rj}^2) & d^2 & (d^2-d_{rs}^2) & \multicolumn{4}{c}{0} \\
\underline{\Delta y}_s & (d^2-d_{si}^2) & (d^2-d_{sj}^2) & (d^2-d_{sr}^2) & d^2 & & & & \\ \hline
\underline{\Delta x}_i & & & & & & & & \\
\underline{\Delta x}_j & \multicolumn{4}{c|}{0} & \multicolumn{4}{c}{\text{same submatrix}} \\
\underline{\Delta x}_r & & & & & \multicolumn{4}{c}{\text{as left upper part}} \\
\underline{\Delta x}_s & & & & & & & &
\end{array} \qquad (15.48)$$

One notes the similarity between (15.48) and (15.47) but with (15.17) also the greater generality of (15.48). To this, the considerations around (13.11) then apply.

Although in this context the problem might have been considered as solved after (15.22), the theory further following from (15.5) has been pursued. This culminates in (15.38), taking account of the theory up to (15.45).

For according to the final conclusion of section 10 a criterion matrix can only refer to S-variates, and just this is true for (15.38). Besides, one is rid of the necessity to assume a value for d^2 in (15.48), a problem whose difficulty is evident from section 13, especially in view of the occurrence of P_s beside P_r.

In general we will therefore abstain from bordering the matrix from (15.38) with elements such as was done in (12.10) for the problem studied there.

For the sake of completeness however, (15.5) is complemented. One obtains:

$$\boxed{\begin{aligned}
\operatorname{Re}\{e^{i\alpha_{irj}}\overline{z_i^{(rs)}, z_j^T}\} &= \operatorname{Re}\left\{\frac{l_{ri}l_{js}}{l_{rs}}\left[e^{i\alpha_{rsj}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_r, z_j^T}}{l_{rj}}\right)+e^{i\alpha_{srj}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_s, z_j^T}}{l_{sj}}\right)\right]+\right.\\
&\qquad\left.-l_{ij}\left[e^{i\alpha_{irj}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_i, z_j^T}}{l_{ij}}\right)+e^{i\alpha_{rij}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_r, z_j^T}}{l_{rj}}\right)\right]\right\}\\
\hline
\operatorname{Re}\{e^{i\alpha_{isj}}\overline{z_i^{(rs)}, z_j^T}\} &= \operatorname{Re}\left\{\frac{l_{rj}l_{is}}{l_{rs}}\left[e^{i\alpha_{rsj}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_r, z_j^T}}{l_{rj}}\right)+e^{i\alpha_{srj}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_s, z_j^T}}{l_{sj}}\right)\right]+\right.\\
&\qquad\left.-l_{ij}\left[e^{i\alpha_{isj}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_i, z_j^T}}{l_{ij}}\right)+e^{i\alpha_{sij}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_s, z_j^T}}{l_{sj}}\right)\right]\right\}
\end{aligned}} \qquad (15.49)$$

Because Π_{irr} or Π_{iss} are undefined, one has to calculate separately from (15.1):

$$\boxed{\begin{aligned}
\overline{z_i^{(rs)}, z_r^T} &= l_{ir}\left\{-\left(\frac{\overline{z_r, z_r^T}-\overline{z_i, z_r^T}}{l_{ir}}\right)+e^{i\alpha_{sri}}\left(\frac{\overline{z_r, z_r^T}-\overline{z_s, z_r^T}}{l_{sr}}\right)\right\}\\
\hline
\overline{z_i^{(rs)}, z_r^T} &= l_{is}\left\{-\left(\frac{\overline{z_s, z_s^T}-\overline{z_i, z_s^T}}{l_{is}}\right)+e^{i\alpha_{rsi}}\left(\frac{\overline{z_s, z_s^T}-\overline{z_r, z_s^T}}{l_{rs}}\right)\right\}
\end{aligned}} \quad \ldots\ldots \quad (15.50)$$

The latter formulae worked out with (15.17) give:

$$\boxed{\begin{aligned}
\overline{z_i^{(rs)}, z_r^T} &= 2l_{ir}\left\{-\frac{d_{ir}^2}{l_{ir}}+\frac{d_{sr}^2}{l_{sr}}(\cos\alpha_{sri}+i\sin\alpha_{sri})\right\}\\
\hline
\overline{z_i^{(rs)}, z_s^T} &= 2l_{is}\left\{-\frac{d_{is}^2}{l_{is}}+\frac{d_{sr}^2}{l_{rs}}(\cos\alpha_{rsi}+i\sin\alpha_{rsi})\right\}
\end{aligned}} \quad \ldots\ldots\ldots\ldots \quad (15.51)$$

Reference to section 16

The criterion matrix was constructed successively each time for a quadrangle r, s, i, j, as in Fig. 15-1.

A question frequently raised in discussing this theory is: is the matrix constructed also valid for more than four points $r, s, i, j, k \ldots$?

This question will be treated seperately in section 16, because this gives a better insight into the system of S-transformations.

Remark

When discussing the connection with the theory on which the HTW-1956 was based, no reference has been made to another essential part of this theory, viz. the introduction of Δd^2-terms; this was only mentioned with (15.22). In the second part of section 15.a after (15.a.21) this matter will be extensively treated.

Note to section 15. Derivation of formulae (15.5)

Start from (15.1), referring to Fig. 15-1.

Write (15.1) in the form:

$$\left.\begin{aligned} \underline{\Delta z}_i^{(rs)} &= \underline{\Delta z}_i - e^{\Pi_{rsi}} \underline{\Delta z}_r - e^{\Pi_{sri}} \underline{\Delta z}_s \\ \underline{\Delta z}_j^{(rs)} &= \underline{\Delta z}_j - e^{\Pi_{rsj}} \underline{\Delta z}_r - e^{\Pi_{srj}} \underline{\Delta z}_s \end{aligned}\right\} \qquad (15.52)$$

Application of the law of propagation of variances gives, with section 14:

$$\begin{aligned} \overline{z_i^{(rs)}, z_j^{(rs)T}} = {} & \overline{z_i, z_j^T} - e^{\Pi^T_{rsj}} \overline{z_i, z_r^T} - e^{\Pi^T_{srj}} \overline{z_i, z_s^T} - e^{\Pi_{rsi}} \overline{z_r, z_j^T} - e^{\Pi_{sri}} \overline{z_s, z_j^T} + \\ & + e^{\Pi_{rsi}+\Pi^T_{rsj}} \overline{z_r, z_r^T} + e^{\Pi_{rsi}+\Pi^T_{srj}} \overline{z_r, z_s^T} + \\ & + e^{\Pi_{sri}+\Pi^T_{rsj}} \overline{z_s, z_r^T} + e^{\Pi_{sri}+\Pi^T_{srj}} \overline{z_s, z_s^T} \end{aligned} \qquad (15.53)$$

with, in general, see (14.5): $\overline{z_i, z_j^T} = \overline{z_j^T, z_i} = (\overline{z_j, z_i^T})^T$

Now, using [2]:

$$\boxed{\overline{z_i, z_j^T}} \qquad e^{i\alpha_{irj}} = +l_{ij} \frac{e^{i\alpha_{irj}}}{l_{ij}}$$

$$e^{-\Pi irj} e^{rsj} = e^{-\Pi_{irj}}(1 - e^{\Pi_{srj}}) = e^{-\Pi_{irj}} - e^{\Pi_{sri}} =$$

$$= 1 - e^{\Pi_{rji}} - 1 + e^{-\Pi_{isr}} = -e^{-\Pi_{ijr}} + e^{-\Pi_{isr}}, \quad \text{hence:}$$

$$e^{-i\alpha_{irj}} e^{\Pi_{rsj}} = v_{irj} e^{-\Pi_{irj}+\Pi_{rsj}} =$$

$$= -v_{irj} v_{rji} l_{ir} \frac{e^{-i\alpha_{ijr}}}{l_{ir}} + v_{irj} v_{rsi} l_{ir} \frac{e^{-i\alpha_{isr}}}{l_{ir}}, \quad \text{or:}$$

$$\boxed{\overline{z_i, z_r^T}} \qquad -\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi^T_{rsj}} = +l_{ij}\frac{\mathrm{e}^{\mathrm{i}\alpha_{ijr}}}{l_{ir}} - \frac{l_{is}l_{jr}}{l_{rs}}\frac{\mathrm{e}^{\mathrm{i}\alpha_{isr}}}{l_{ir}}$$

$$\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi^T_{srj}} = v_{srj}\mathrm{e}^{\mathrm{i}(\alpha_{irj}-\alpha_{srj})} = \frac{l_{jr}}{l_{rs}}\mathrm{e}^{-\mathrm{i}\alpha_{sri}}, \quad \text{hence:}$$

$$\boxed{\overline{z_i, z_s^T}} \qquad -\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi^T_{srj}} = -\frac{l_{is}l_{jr}}{l_{rs}}\frac{\mathrm{e}^{\mathrm{i}\alpha_{irs}}}{l_{is}}$$

$$\mathrm{e}^{\Pi_{irj}}\mathrm{e}^{\Pi_{rsi}} = \mathrm{e}^{\Pi_{irj}}(1-\mathrm{e}^{\Pi_{sri}}) = \mathrm{e}^{\Pi_{irj}} - \mathrm{e}^{\Pi_{srj}} =$$

$$= 1-\mathrm{e}^{-\Pi_{jir}} - 1 + \mathrm{e}^{-\Pi_{jsr}} = -\mathrm{e}^{\Pi_{rij}} + \mathrm{e}^{\Pi_{rsj}}, \quad \text{hence:}$$

$$\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi^T_{srj}} = v_{jri} = \mathrm{e}^{\Pi_{irj}+\Pi_{rsi}} =$$

$$= -v_{jri}v_{rij}l_{rj}\frac{\mathrm{e}^{\mathrm{i}\alpha_{rij}}}{l_{rj}} + v_{jri}v_{rsj}l_{rj}\frac{\mathrm{e}^{\mathrm{i}\alpha_{rsj}}}{l_{rj}}, \quad \text{or:}$$

$$\boxed{\overline{z_r, z_j^T}} \qquad -\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{rsi}} = +l_{ij}\frac{\mathrm{e}^{\mathrm{i}\alpha_{rij}}}{l_{rj}} - \frac{l_{ir}l_{js}}{l_{rs}}\frac{\mathrm{e}^{\mathrm{i}\alpha_{rsj}}}{l_{rj}}$$

$$\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{sri}} = v_{sri}\mathrm{e}^{\mathrm{i}(\alpha_{irj}+\alpha_{sri})} = \frac{l_{ir}}{l_{rs}}\mathrm{e}^{\mathrm{i}\alpha_{srj}}, \quad \text{hence:}$$

$$\boxed{\overline{z_s, z_j^T}} \qquad -\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{sri}} = -\frac{l_{ir}l_{js}}{l_{rs}}\frac{\mathrm{e}^{\mathrm{i}\alpha_{srj}}}{l_{sj}}$$

$$\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{rsi}+\Pi^T_{rsj}} = v_{rsi}v_{rsj}\mathrm{e}^{\mathrm{i}(\alpha_{irj}+\alpha_{rsi}-\alpha_{rsj})} = \frac{l_{is}l_{js}}{l_{rs}l_{rs}}\mathrm{e}^{\mathrm{i}(\alpha_{irj}-\alpha_{isj})},$$

but this result is not suitable for inclusion with the other eight results. A different way goes via the relations between coefficients in (15.52):

$$1-\mathrm{e}^{\Pi_{rsi}}-\mathrm{e}^{\Pi_{sri}} = 0$$

$$1-\mathrm{e}^{\Pi_{rsj}}-\mathrm{e}^{\Pi_{srj}} = 0,$$

hence:

$$(1-\mathrm{e}^{\Pi_{rsi}}-\mathrm{e}^{\Pi_{sri}})(1-\mathrm{e}^{\Pi^T_{rsj}}-\mathrm{e}^{\Pi^T_{srj}}) = 0$$

or, the sum of the nine coefficients in (15.53) is zero. Or:

$$\boxed{\overline{z_r, z_r^T}} \qquad \mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{rsi}+\Pi^T_{rsj}} = 0-(\text{the other eight results})$$

$$\mathrm{e}^{\mathrm{i}\alpha_{irj}}\mathrm{e}^{\Pi_{rsi}+\Pi^T_{srj}} = v_{rsi}v_{srj}\mathrm{e}^{\mathrm{i}(\alpha_{irj}+\alpha_{rsi}-\alpha_{srj})}$$

$$= v_{rsi}v_{srj}\mathrm{e}^{-\mathrm{i}(\alpha_{isr}+\alpha_{sri})} = -\frac{l_{is}l_{jr}}{l_{rs}l_{rs}}\mathrm{e}^{\mathrm{i}\alpha_{ris}}, \quad \text{hence:}$$

$$\boxed{\overline{z_r, z_s^T}} \qquad e^{i\alpha_{irj}} e^{\Pi_{rsi}+\Pi^T_{srj}} = -\frac{l_{is}l_{jr}}{l_{rs}} \frac{e^{i\alpha_{ris}}}{l_{rs}}$$

$$e^{i\alpha_{irj}} e^{\Pi_{sri}+\Pi^T_{rsj}} = v_{sri}v_{rsj}e^{i(\alpha_{irj}+\alpha_{sri}-\alpha_{rsj})} =$$

$$= v_{sri}v_{rsj}e^{i(\alpha_{srj}+\alpha_{jsr})} = -\frac{l_{ir}l_{js}}{l_{rs}l_{rs}}e^{-i\alpha_{rjs}}, \quad \text{hence:}$$

$$\boxed{\overline{z_s, z_r^T}} \qquad e^{i\alpha_{irj}} e^{\Pi_{sri}+\Pi^T_{rsj}} = -\frac{l_{ir}l_{js}}{l_{rs}} \frac{e^{i\alpha_{sjr}}}{l_{sr}}$$

$$e^{i\alpha_{irj}} e^{\Pi_{sri}+\Pi^T_{srj}} = v_{sri}v_{srj}e^{i(\alpha_{irj}+\alpha_{sri}-\alpha_{srj})} = \frac{l_{ir}l_{jr}}{l_{rs}l_{rs}}e^{i.0}, \quad \text{hence:}$$

$$\boxed{\overline{z_s, z_s^T}} \qquad e^{i\alpha_{irj}} e^{\Pi_{sri}+\Pi^T_{srj}} = \frac{l_{ir}l_{jr}}{l_{rs}} \frac{1}{l_{rs}}$$

From these nine results, it follows with (15.53) after some re-arrangement:

$$\begin{aligned}
& e^{i\alpha_{irj}} \cdot \overline{z_i^{(rs)}, z_j^{(rs)T}} = \\
& -\frac{l_{ir}l_{js}}{l_{rs}}\left[\left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_j^T}}{l_{rj}}\right)e^{i\alpha_{rsj}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_j^T}}{l_{sj}}\right)e^{i\alpha_{srj}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_r^T}}{l_{sr}}\right)e^{i\alpha_{sjr}}\right] + \\
& + \frac{l_{is}l_{jr}}{l_{rs}}\left[\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_r^T}}{l_{ir}}\right)e^{i\alpha_{isr}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_s^T}}{l_{is}}\right)e^{i\alpha_{irs}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_s^T}}{l_{rs}}\right)e^{i\alpha_{ris}}\right] + \\
& - l_{ij}\left[\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_j^T}}{l_{ij}}\right)e^{i\alpha_{irj}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_r^T}}{l_{ir}}\right)e^{i\alpha_{ijr}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_j^T}}{l_{rj}}\right)e^{i\alpha_{rij}}\right] + \\
& - \frac{l_{ir}l_{js}}{l_{rs}}\left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_s^T}}{l_{rs}}\right)
\end{aligned}$$

or, with (15.4), $\mathrm{Re}\{a \cdot b\} = \frac{1}{2}(a \cdot b + a^T \cdot b^T) = \mathrm{Re}\{a^T \cdot b^T\}$, it follows:

$$\boxed{\begin{aligned}
& \mathrm{Re}\{e^{i\alpha_{irj}} \cdot \overline{z_i^{(rs)}, z_j^{(rs)T}}\} = \\
& = \frac{l_{ri}l_{js}}{l_{rs}} \cdot \mathrm{Re}\left\{\left(\frac{\overline{z_r, z_r^T} - \overline{z_j, z_r^T}}{l_{jr}}\right)e^{i\alpha_{jsr}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_j^T}}{l_{sj}}\right)e^{i\alpha_{srj}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_s^T}}{l_{rs}}\right)e^{i\alpha_{rjs}}\right\} + \\
& + \frac{l_{rj}l_{is}}{l_{rs}} \cdot \mathrm{Re}\left\{\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_r^T}}{l_{ir}}\right)e^{i\alpha_{isr}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_i^T}}{l_{si}}\right)e^{i\alpha_{sri}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_s^T}}{l_{rs}}\right)e^{i\alpha_{ris}}\right\} + \\
& - l_{ij} . \mathrm{Re}\left\{\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_j^T}}{l_{ij}}\right)e^{i\alpha_{irj}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_i^T}}{l_{ri}}\right)e^{i\alpha_{rji}} + \left(\frac{\overline{z_r, z_r^T} - \overline{z_j, z_r^T}}{l_{jr}}\right)e^{i\alpha_{jir}}\right\} + \\
& - \frac{l_{ri}l_{rj}}{l_{rs}} \cdot \mathrm{Re}\left\{\frac{\overline{z_r, z_r^T} - \overline{z_s, z_s^T}}{l_{rs}}\right\}
\end{aligned}}$$

or, (15.5′) with (15.6′).

In an analogous way one can derive:

$$\mathrm{Re}\,\{e^{i\alpha_{isj}}\cdot\overline{z_i^{(rs)}, z_j^{(rs)T}}\}$$

or, (15.5″) with (15.6″).

These formulae degenerate into identities when the variances of $\underline{z}_r$, $\underline{z}_s$ are assumed to vanish, as should follow from (15.1). This is a {weak} check.

15.a CRITERION MATRICES IN SYSTEMS FOR HORIZONTAL CONTROL. Δd^2-TERMS. AN ALTERNATIVE DERIVATION

Start from (15.1) in the form (15.52):

$$\left.\begin{aligned}\underline{\Delta z}_i^{(rs)} &= \underline{\Delta z}_i - e^{\Pi_{rsi}}\cdot\underline{\Delta z}_r - e^{\Pi_{sri}}\cdot\underline{\Delta z}_s\\ \underline{\Delta z}_j^{(rs)T} &= \underline{\Delta z}_j^T - e^{\Pi^T_{rsj}}\cdot\underline{\Delta z}_r^T - e^{\Pi^T_{srj}}\cdot\underline{\Delta z}_s^T\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15.a.1)$$

and write this in two alternative forms, with:

$$\left.\begin{aligned}1 - e^{\Pi_{rsi}} - e^{\Pi_{sri}} &= 0\\ 1 - e^{\Pi^T_{rsj}} - e^{\Pi^T_{srj}} &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15.a.2)$$

as:

$$\left.\begin{aligned}\underline{\Delta z}_i^{(rs)} &= \underline{\Delta z}_{ri} - e^{\Pi_{sri}}\cdot\underline{\Delta z}_{rs}\\ \underline{\Delta z}_j^{(rs)T} &= \underline{\Delta z}_{rj}^T - e^{\Pi^T_{srj}}\cdot\underline{\Delta z}_{rs}^T\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15.a.3)$$

and:

$$\left.\begin{aligned}\underline{\Delta z}_i^{(rs)} &= \underline{\Delta z}_{si} - e^{\Pi_{rsi}}\cdot\underline{\Delta z}_{sr}\\ \underline{\Delta z}_j^{(rs)T} &= \underline{\Delta z}_{sj}^T - e^{\Pi^T_{rsj}}\cdot\underline{\Delta z}_{sr}^T\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15.a.4)$$

From (15.a.3).

$$\overline{z_i^{(rs)}, z_j^{(rs)T}} = \overline{z_{ri}, z_{rj}^T} - e^{\Pi^T_{srj}}\,\overline{z_{ri}, z_{rs}^T} - e^{\Pi_{sri}}\,\overline{z_{rs}, z_{rj}^T} + e^{\Pi_{sri}+\Pi^T_{srj}}\,\overline{z_{rs}, z_{rs}^T}$$

and with, for example:

$$\begin{aligned}\overline{z_{ri}, z_{rj}^T} &= \overline{z_i, z_j^T} - \overline{z_r, z_j^T} - \overline{z_i, z_r^T} + \overline{z_r, z_r^T} =\\ &= -(\overline{z_r, z_r^T} - \overline{z_i, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_r, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_i, z_r^T})\end{aligned}$$

becomes:

$$\begin{aligned}&\overline{z_i^{(rs)}, z_j^{(rs)T}} =\\ &\qquad [-(\overline{z_r, z_r^T} - \overline{z_i, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_r, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_i, z_r^T})] +\\ &\qquad - e^{\Pi^T_{srj}}[-(\overline{z_r, z_r^T} - \overline{z_i, z_s^T}) + (\overline{z_r, z_r^T} - \overline{z_r, z_s^T}) + (\overline{z_r, z_r^T} - \overline{z_i, z_r^T})] +\\ &\qquad - e^{\Pi_{sri}}[-(\overline{z_r, z_r^T} - \overline{z_s, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_r, z_j^T}) + (\overline{z_r, z_r^T} - \overline{z_s, z_r^T})] +\\ &\qquad + e^{\Pi_{sri}+\Pi^T_{srj}}[-(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) + (\overline{z_r, z_r^T} - \overline{z_r, z_s^T}) + (\overline{z_r, z_r^T} - \overline{z_s, z_r^T})]\end{aligned}$$

and, with (15.a.2):

$$\boxed{\begin{aligned}
&\overline{z_i^{(rs)}, z_j^{(rs)T}} = \\
&-(\overline{z_r, z_r^T} - \overline{z_i, z_j^T}) + e^{\Pi^T_{rsj}}(\overline{z_r, z_r^T} - \overline{z_i, z_r^T}) + e^{\Pi^T_{srj}}(\overline{z_r, z_r^T} - \overline{z_i, z_s^T}) + \\
&\qquad\qquad + e^{\Pi_{rsi}}(\overline{z_r, z_r^T} - \overline{z_r, z_j^T}) + e^{\Pi_{sri}}(\overline{z_r, z_r^T} - \overline{z_s, z_j^T}) + \\
&- e^{\Pi_{rsi}+\Pi^T_{srj}}(\overline{z_r, z_r^T} - \overline{z_r, z_s^T}) - e^{\Pi_{sri}+\Pi^T_{rsj}}(\overline{z_r, z_r^T} - \overline{z_s, z_r^T}) + \\
&\qquad\qquad - e^{\Pi_{sri}+\Pi^T_{srj}}(\overline{z_r, z_r^T} - \overline{z_s, z_s^T})
\end{aligned}} \quad . \; . \; . \quad (15.a.5)$$

Analogous, from (15.a.4):

$$\boxed{\begin{aligned}
&\overline{z_i^{(rs)}, z_j^{(rs)T}} = \\
&-(\overline{z_s, z_s^T} - \overline{z_i, z_j^T}) + e^{\Pi^T_{rsj}}(\overline{z_s, z_s^T} - \overline{z_i, z_r^T}) + e^{\Pi^T_{srj}}(\overline{z_s, z_s^T} - \overline{z_i, z_s^T}) + \\
&\qquad\qquad + e^{\Pi_{rsi}}(\overline{z_s, z_s^T} - \overline{z_r, z_j^T}) + e^{\Pi_{sri}}(\overline{z_s, z_s^T} - \overline{z_s, z_j^T}) + \\
&- e^{\Pi_{rsi}+\Pi^T_{srj}}(\overline{z_s, z_s^T} - \overline{z_r, z_s^T}) - e^{\Pi_{sri}+\Pi^T_{rsj}}(\overline{z_s, z_s^T} - \overline{z_s, z_r^T}) + \\
&\qquad\qquad - e^{\Pi_{rsi}+\Pi^T_{rsj}}(\overline{z_s, z_s^T} - \overline{z_r, z_r^T})
\end{aligned}} \quad . \; . \; . \quad (15.a.6)$$

Remarkable are the equal coefficients in both equations except for the last term. The difference of these last terms is:

$$\begin{aligned}
&e^{\Pi_{sri}+\Pi^T_{srj}}(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) - e^{\Pi_{rsi}+\Pi^T_{rsj}}(\overline{z_s, z_s^T} - \overline{z_r, z_r^T}) = \\
&= (e^{\Pi_{sri}+\Pi^T_{srj}} + e^{\Pi_{rsi}+\Pi^T_{rsj}})(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) = \\
&= \left(\frac{l_{ri}l_{rj}}{l_{rs}^2} e^{i\alpha_{jri}} + \frac{l_{si}l_{sj}}{l_{rs}^2} e^{i\alpha_{jsi}}\right)(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) = \\
&= \left\{\frac{l_{ri}l_{rj}}{l_{rs}^2}\left(\frac{l_{ri}}{l_{rj}}\right)^{-1} e^{\Pi_{jri}} + \frac{l_{si}l_{sj}}{l_{rs}^2}\left(\frac{l_{si}}{l_{sj}}\right)^{-1} e^{\Pi_{jsi}}\right\}(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) = \\
&\left.\begin{aligned}
&= \left\{\left(\frac{l_{ri}}{l_{rs}}\right)^2 e^{\Pi_{jri}} + \left(\frac{l_{si}}{l_{sr}}\right)^2 e^{\Pi_{jsi}}\right\}(\overline{z_r, z_r^T} - \overline{z_s, z_s^T}) \\
&\text{in general} \neq 0, \quad \text{if} \quad (z_r, z_r^T - z_s, z_s^T) \neq 0
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15.a.7)
\end{aligned}$$

With $j = i$, and:

$$
\left.\begin{aligned}
&\{\mathrm{e}^{\Pi^T_{rsi}}(\overline{z_r, z_r^T} - \overline{z_i, z_r^T})\}^T = \mathrm{e}^{\Pi_{rsi}}(\overline{z_r, z_r^T} - \overline{z_r, z_i^T})\\
&-\mathrm{e}^{\Pi_{rsi}+\Pi^T_{sri}} = -\frac{l_{ri} l_{is}}{l_{rs}^2}\mathrm{e}^{\mathrm{i}(-\alpha_{isr}-\alpha_{sri})} = +\frac{l_{ri} l_{is}}{l_{rs}^2}\mathrm{e}^{\mathrm{i}\alpha_{ris}}\\
&\mathrm{e}^{\Pi_{sri}+\Pi^T_{sri}} = \left(\frac{l_{ri}}{l_{rs}}\right)^2\\
&\text{etc.}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15.a.8)
$$

follows from (15.a.5):

$$
\begin{aligned}
\overline{z_i^{(rs)}, z_i^{(rs)T}} = {} & -(\overline{z_r, z_r^T} - \overline{z_i, z_i^T}) + 2\,\mathrm{Re}\,\{\mathrm{e}^{\Pi_{rsi}}(\overline{z_r, z_r^T} - \overline{z_r, z_i^T})] + \\
& + 2\,\mathrm{Re}\,\{\mathrm{e}^{\Pi_{sri}}(\overline{z_r, z_r^T} - \overline{z_s, z_i^T})\} + \\
& - 2\,\mathrm{Re}\,\{\mathrm{e}^{\Pi_{rsi}+\Pi^T_{sri}}(\overline{z_r, z_r^T} - \overline{z_r, z_s^T})\} - \mathrm{e}^{\Pi_{sri}+\Pi^T_{sri}}(\overline{z_r, z_r^T} - \overline{z_s, z_s^T})
\end{aligned}
$$

or with:

$$
\{\mathrm{e}^{\mathrm{i}\alpha_{rsi}}(\overline{z_r, z_r^T} - \overline{z_r, z_i^T})\}^T = \mathrm{e}^{\mathrm{i}\alpha_{isr}}(\overline{z_r, z_r^T} - \overline{z_i, z_r^T}):
$$

$$
\boxed{\begin{aligned}
&\overline{z_i^{(rs)}, z_i^{(rs)T}} = -(\overline{z_r, z_r^T} - \overline{z_i, z_i^T}) + 2\frac{l_{ri} l_{is}}{l_{rs}} \cdot G_{ris} - \left(\frac{l_{ri}}{l_{rs}}\right)^2 (\overline{z_r, z_r^T} - \overline{z_s, z_s^T})\\
&G_{ris} = \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\alpha_{isr}}\left(\frac{\overline{z_r, z_r^T} - \overline{z_i, z_r^T}}{l_{ir}}\right) + \mathrm{e}^{\mathrm{i}\alpha_{sri}} \cdot \left(\frac{\overline{z_r, z_r^T} - \overline{z_s, z_i^T}}{l_{si}}\right) + \right.\\
&\qquad\left. + \mathrm{e}^{\mathrm{i}\alpha_{ris}} \cdot \left(\frac{\overline{z_r, z_r^T} - \overline{z_r, z_s^T}}{l_{rs}}\right)\right\}
\end{aligned}} \quad (15.a.9)
$$

Analogous, from (15.a.6):

$$
\boxed{\begin{aligned}
&\overline{z_i^{(rs)}, z_i^{(rs)T}} = -(\overline{z_s, z_s^T} - \overline{z_i, z_i^T}) + 2\frac{l_{ri} l_{is}}{l_{rs}} \cdot G'_{ris} - \left(\frac{l_{si}}{l_{sr}}\right)^2 (\overline{z_s, z_s^T} - \overline{z_r, z_r^T})\\
&G'_{ris} = \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\alpha_{isr}} \cdot \left(\frac{\overline{z_s, z_s^T} - \overline{z_i, z_r^T}}{l_{ir}}\right) + \mathrm{e}^{\mathrm{i}\alpha_{sri}} \cdot \left(\frac{\overline{z_s, z_s^T} - \overline{z_s, z_i^T}}{l_{si}}\right) + \right.\\
&\qquad\left. + \mathrm{e}^{\mathrm{i}\alpha_{ris}} \cdot \left(\frac{\overline{z_s, z_s^T} - \overline{z_r, z_s^T}}{l_{rs}}\right)\right\}
\end{aligned}} \quad (15.a.10)
$$

The following relation is generally valid in triangle r, i, s, see (15.10) and (15.11):

$$
\frac{\underline{\Delta\Pi}_{ris}}{z_{rs}} = \frac{\underline{\Delta\Pi}_{isr}}{z_{ir}} = \frac{\underline{\Delta\Pi}_{sri}}{z_{si}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15.a.11')
$$

hence, also, in general:

$$\frac{\overline{\Pi_{ris}, \Pi_{ris}^T}}{l_{rs}^2} = \frac{\overline{\Pi_{isr}, \Pi_{isr}^T}}{l_{ir}^2} = \frac{\overline{\Pi_{sri}, \Pi_{sri}^T}}{l_{si}^2} \quad \ldots \ldots \quad (15.a.11'')$$

Now with (2.12) we have:

$$\underline{\Delta z_i^{(rs)}} = -\frac{z_{ri}z_{is}}{z_{rs}} \cdot \underline{\Delta \Pi_{ris}} \quad \ldots \ldots \quad (15.a.12')$$

hence the general validity of:

$$\overline{z_i^{(rs)}, z_i^{(rs)T}} = \left(\frac{l_{ri}l_{is}}{l_{rs}}\right)^2 \cdot \overline{\Pi_{ris}, \Pi_{ris}^T} \quad \ldots \ldots \quad (15.a.12'')$$

From (15.a.11″) and (15.a.12″) one can draw the conclusion that $\overline{z_i^{(rs)}, z_i^{(rs)T}}$ is a function of the side lengths in triangle r, i, s, in particular a positive real function under the requirement of circular standard ellipses, see section 14.

If one wishes, moreover, to eliminate the difference $(\overline{z_r, z_r^T} - \overline{z_s, z_s^T})$ by putting it equal to zero, then one can introduce, on the analogy of (12.7):

$$\boxed{\begin{aligned} &\overline{z_r, z_r^T} - \overline{z_i, z_j^T} = \overline{z_s, z_s^T} - \overline{z_i, z_j^T} = \{\overline{z_r, z_r^T} - \overline{z_j, z_i^T}\}^T = \\ &\quad = 2\cdot\overline{x_r, x_r} - 2(\overline{x_i, x_j} + \mathrm{i}\cdot\overline{x_i, y_j}) = 2\cdot\overline{x_s, x_s} - 2(\overline{x_i, x_j} + \mathrm{i}\cdot\overline{x_i, y_j}) = \\ &\quad = 2(d_{ij}^2 - \mathrm{i}\,\bar{d}_{ij}) = 2(d_{ji}^2 - \mathrm{i}\,\bar{d}_{ji})^T \\ \hline &d_{ij}^2 = d_{ji}^2 = \sum_{p\neq 0} c_p (l_{ij})^p, \quad c_p \geqslant 0 \\ &\bar{d}_{ij} = -\bar{d}_{ji}, \quad \text{not further specified} \end{aligned}} \quad (15.a.13)$$

With (15.a.13), formulae (15.a.9) and (15.a.10) become:

$$\boxed{\begin{aligned} &\overline{z_i^{(rs)}, z_i^{(rs)T}} = 2\frac{l_{ri}l_{is}}{l_{rs}} G_{ris} > 0, \qquad i \neq r, s \\ \hline &G_{ris} = G'_{rsi} = \\ &\quad = 2\left\{\left(\cos\alpha_{isr}\cdot\frac{d_{ir}^2}{l_{ir}} + \cos\alpha_{sri}\cdot\frac{d_{si}^2}{l_{si}} + \cos\alpha_{ris}\cdot\frac{d_{rs}^2}{l_{rs}}\right) + \right. \\ &\quad\quad \left. + \left(\sin\alpha_{isr}\cdot\frac{\bar{d}_{ir}}{l_{ir}} + \sin\alpha_{sri}\cdot\frac{\bar{d}_{si}}{l_{si}} + \sin\alpha_{ris}\cdot\frac{\bar{d}_{rs}}{l_{rs}}\right)\right\} \end{aligned}} \quad \ldots \ldots \quad (15.a.14)$$

Now it is necessary that: $\overline{z_i^{(rs)}, z_i^{(rs)T}} > 0$ for $i \neq r$, s.

Consider:

$$A = \cos\alpha_{isr} \cdot \frac{d_{ir}^2}{l_{ir}} + \cos\alpha_{sri} \cdot \frac{d_{si}^2}{l_{si}} + \cos\alpha_{ris} \cdot \frac{d_{rs}^2}{l_{rs}} \quad \ldots\ldots\ldots\ldots \quad (15.a.15')$$

in which at most one term can be negative, e.g. the term with $\cos\alpha_{ris}$. Then assume:

$$0 \leqslant \alpha_{isr} + \alpha_{sri} \leqslant \frac{\pi}{2}, \quad \frac{\pi}{2} < \alpha_{ris} \leqslant \pi, \quad \text{hence} \quad l_{rs} > l_{ir}, l_{si}$$

Then:

$$A = \left\{\frac{d_{ir}^2}{l_{ir}} + \frac{d_{si}^2}{l_{si}} - \frac{d_{rs}^2}{l_{rs}}\right\} + 2\left\{-\frac{d_{ir}^2}{l_{ir}} \cdot \sin^2\frac{\alpha_{isr}}{2} - \frac{d_{si}^2}{l_{si}} \cdot \sin^2\frac{\alpha_{sri}}{2} + \right.$$

$$\left. + \frac{d_{rs}^2}{l_{rs}} \cdot \sin^2\frac{\alpha_{isr} + \alpha_{sri}}{2}\right\} =$$

$$= \left\{\frac{d_{ir}^2}{l_{ir}} + \frac{d_{si}^2}{l_{si}} - \frac{d_{rs}^2}{l_{rs}}\right\} + 2\left\{\left(\frac{d_{rs}^2}{l_{rs}} - \frac{d_{ir}^2}{l_{ir}}\right)\sin^2\frac{\alpha_{isr}}{2} + \right.$$

$$\left. + \left(\frac{d_{rs}^2}{l_{rs}} - \frac{d_{si}^2}{l_{si}}\right)\sin^2\frac{\alpha_{sri}}{2} + \frac{d_{rs}^2}{l_{rs}}\left(\sin^2\frac{\alpha_{isr} + \alpha_{sri}}{2} - \sin^2\frac{\alpha_{isr}}{2} - \sin^2\frac{\alpha_{sri}}{2}\right)\right\} =$$

$$= \left\{\frac{d_{ir}^2}{l_{ir}} + \frac{d_{si}^2}{l_{si}} - \frac{d_{rs}^2}{l_{rs}}\right\} + 2\left\{\left(\frac{d_{rs}^2}{l_{rs}} - \frac{d_{ir}^2}{l_{ir}}\right)\sin^2\frac{\alpha_{isr}}{2} + \right.$$

$$\left. + \left(\frac{d_{rs}^2}{l_{rs}} - \frac{d_{si}^2}{l_{si}}\right)\sin^2\frac{\alpha_{sri}}{2} + \frac{d_{rs}^2}{l_{rs}} \cdot 2\sin\frac{\alpha_{isr}}{2} \cdot \sin\frac{\alpha_{sri}}{2} \cdot \cos\frac{\alpha_{isr} + \alpha_{sri}}{2}\right\}$$

Now choose one term from d_{ij}^2 in (15.a.13), then:

$$A_p = c_p\left[\{l_{ir}^{p-1} + l_{si}^{p-1} - l_{rs}^{p-1}\} + 2\left\{(l_{rs}^{p-1} - l_{ir}^{p-1})\sin^2\frac{\alpha_{isr}}{2} + \right.\right.$$

$$\left.\left. + (l_{rs}^{p-1} - l_{si}^{p-1})\sin^2\frac{\alpha_{sri}}{2} + l_{rs}^{p-1} \cdot 2\sin\frac{\alpha_{isr}}{2} \cdot \sin\frac{\alpha_{sri}}{2} \cdot \cos\frac{\alpha_{isr} + \alpha_{sri}}{2}\right\}\right]$$

Under the assumptions made, all trigonometric coefficients are $\geqslant 0$. Apply (13.19):

$$c_p\{l_{ir}^{p-1} + l_{si}^{p-1} - l_{rs}^{p-1}\} > 0; \qquad c_p > 0, \quad 0 \leqslant p < 2$$

or:

$A_p > 0$	$c_p > 0$	$0 \leqslant p < 2$
$A > 0$	$c_p \geqslant 0$	not all $c_p = 0$

. . . . (15.a.15″)

Next consider:

$$B = \sin\alpha_{isr} \cdot \frac{\bar{d}_{ir}}{l_{ir}} + \sin\alpha_{sri} \cdot \frac{\bar{d}_{si}}{l_{si}} + \sin\alpha_{ris} \cdot \frac{\bar{d}_{rs}}{l_{rs}} \quad \ldots\ldots\ldots\ldots \quad (15.a.16')$$

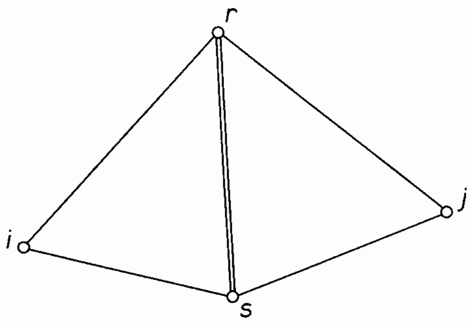


Fig. 15.a.1

Consider also the situation symmetric with respect to P_s, P_r:

$$B' = \sin\alpha_{jsr}\cdot\frac{\bar{d}_{jr}}{l_{jr}} + \sin\alpha_{srj}\cdot\frac{\bar{d}_{si}}{l_{si}} + \sin\alpha_{rjs}\cdot\frac{\bar{d}_{rs}}{l_{rs}}$$

A safe requirement is: $B \geqslant 0$, $B' \geqslant 0$. But if the trigonometric coefficients in $B > 0$, then those in B' are < 0, since all trigonometric coefficients in the same equation have the same sign.

Assume: $\bar{d}_{ij} = \text{function } \{l_{ij}\}$

Then, in particular:

from $B > 0$: $\bar{d}_{rs} \geqslant 0$

from $B' > 0$: $\bar{d}_{rs} \leqslant 0$

This is only possible if function $\{l_{rs}\} \equiv 0$. Hence:

$$\boxed{B = 0} \; \boxed{\bar{d}_{ij} = 0} \; \boxed{\overline{x_i, y_j} = 0 = -\overline{y_i, x_j}} \quad \ldots\ldots\ldots\ldots \quad (15.a.16'')$$

From (15.a.13)–(15.a.16) follows the construction of the criterion matrix from the elements:

$$\boxed{\begin{aligned} \overline{z_r, z_r^T} - \overline{z_i, z_j^T} &= \overline{z_s, z_s^T} - \overline{z_i, z_j^T} = \overline{z_r, z_r^T} - \overline{z_j, z_i^T} = \\ &= 2(\overline{x_r, x_r} - \overline{x_i, x_j}) = 2(\overline{x_s, x_s} - \overline{x_i, x_j}) = \\ &= 2d_{ij}^2 = 2d_{ji}^2 \\ d_{ij}^2 = \sum c_p (l_{ij})^p \quad & c_p \geqslant 0 \quad 0 < p < 2 \end{aligned}} \quad \ldots\ldots\ldots \quad (15.a.17)$$

From (15.a.5) or (15.a.6) follows then*):

*) Different ways of elaborating the formulae are possible; this one was chosen by J. C. P. de Kruif as a base for computer programming.

$$\boxed{\begin{aligned} \tfrac{1}{2}\cdot\overline{z_i^{(rs)}, z_j^{(rs)T}} &= \overline{x_i^{(rs)}, x_j^{(rs)}} + \mathrm{i}\cdot\overline{x_i^{(rs)}, y_j^{(rs)}} = \\ &= \overline{y_i^{(rs)}, y_j^{(rs)}} - \mathrm{i}\cdot\overline{y_i^{(rs)}, x_j^{(rs)}} = \\ \hline \tfrac{1}{2}\cdot\overline{z_i^{(rs)}, z_j^{(rs)T}} &= -d_{ij}^2 - \frac{1}{l_{rs}^2}\left[z_{rs}z_{sj}^T\cdot d_{ri}^2 + z_{sr}z_{rj}^T\cdot d_{si}^2 + \right. \\ &\qquad + z_{rs}^T z_{si}\cdot d_{rj}^2 + z_{sr}^T z_{ri}\cdot d_{sj}^2 + \\ &\qquad \left. + (z_{ri}z_{js}^T + z_{rj}^T z_{is})d_{rs}^2\right] \end{aligned}} \quad \ldots\ldots\ldots \quad (15.a.18)$$

and from this or from (15.a.14):

$$\boxed{\begin{aligned} \tfrac{1}{2}\cdot\overline{z_i^{(rs)}, z_i^{(rs)T}} &= \overline{x_i^{(rs)}, x_i^{(rs)}} = \overline{y_i^{(rs)}, y_i^{(rs)}} = \\ &= \frac{l_{ri}l_{is}}{l_{rs}} G_{ris} > 0, \quad i \neq r, s \\ \hline G_{ris} &= 2\left\{\cos\alpha_{isr}\cdot\frac{d_{ir}^2}{l_{ir}} + \cos\alpha_{sri}\cdot\frac{d_{si}^2}{l_{si}} + \cos\alpha_{ris}\cdot\frac{d_{rs}^2}{l_{rs}}\right\} \end{aligned}} \quad \ldots\ldots\ldots \quad (15.a.19)$$

The parallelism of (15.a.17) with (12.7) also means a double use of the matrix (12.12), once for $\overline{(y_i), (y_j)^*}$ and once for $\overline{(x_i), (x_j)^*}$, with $\overline{(x_i), (y_j)^*} = (0)$.

This explains why the requirements for positive definiteness also agree. One obtains, with:

$$\overline{y_r, y_r} = \overline{y_s, y_s} = \overline{x_r, x_r} = \overline{x_s, x_s} = d^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15.a.20')$$

$$\begin{array}{c|cccc|cccc} & \underline{\Delta y_i} & \underline{\Delta y_j} & \underline{\Delta y_r} & \underline{\Delta y_s} & \underline{\Delta x_i} & \underline{\Delta x_j} & \underline{\Delta x_r} & \underline{\Delta x_s} \\ \hline \underline{\Delta y_i} & d^2 & (d^2-d_{ij}^2) & (d^2-d_{ir}^2) & (d^2-d_{is}^2) & & & & \\ \underline{\Delta y_j} & (d^2-d_{ji}^2) & d^2 & (d^2-d_{jr}^2) & (d^2-d_{js}^2) & & 0 & & \\ \underline{\Delta y_r} & (d^2-d_{ri}^2) & (d^2-d_{rj}^2) & d^2 & (d^2-d_{rs}^2) & & & & \\ \underline{\Delta y_s} & (d^2-d_{si}^2) & (d^2-d_{sj}^2) & (d^2-d_{sr}^2) & d^2 & & & & \\ \hline \underline{\Delta x_i} & & & & & & & & \\ \underline{\Delta x_j} & & & & & & \text{same submatrix} & & \\ \underline{\Delta x_r} & & 0 & & & & \text{as left upper part} & & \\ \underline{\Delta x_s} & & & & & & & & \end{array} \quad (15.a.20'')$$

Since the requirements under which (13.24) is positive definite are, according to (15.a.17), the same as for (15.a.20″) because it is possible to extend the matrix (15.a.18) with (15.a.19), by bordering it with terms $\overline{z_i^{(rs)}, z_r^T}$, $\overline{z_i^{(rs)}, z_s^T}$, $\overline{z_r, z_s^T}$ and $\overline{z_r, z_r^T}$ and $\overline{z_s, z_s^T}$ to obtain a matrix analogous to (13.12),

and since, because of the way of constructing, it is always possible to consider this extended matrix as having been established by applying the law of propagation of variances to the non-singular transformation (2.8) using the variance matrix (15.a.20″),

the conclusion (13.25) can be applied in the reverse way:

Conclusion: If the matrix (15.a.20″) is positive definite then: the extended matrix (15.a.18), (15.a.19) is positive definite

Introduction of Δd^2-terms

Introduce „centering variates" like in (13.1):

Instead of $\underline{y}_i$ Instead of $\underline{x}_i$	$y_i + \underline{\Delta y}_i^e$ $\underline{x}_i + \underline{\Delta x}_i^e$	$\Big\}\ \underline{z}_i + \underline{\Delta z}_i^e$
means	$\widetilde{\Delta y_i^e} = \widetilde{\Delta x_i^e} = 0$	
covariances	$\overline{\Delta y_i^e, \Delta y_i^e} = \overline{\Delta x_i^e, \Delta x_i^e} = \Delta d_i^2$ $\overline{\Delta y_i^e, \Delta y_j^e} = \overline{\Delta y_i^e, \Delta x_j^e} = \overline{\Delta x_i^e, x_j^e} = 0$ $\overline{\Delta y_i, \Delta y_j^e} = \overline{\Delta y_i, \Delta x_j^e} = 0$ $\overline{\Delta x_i, \Delta y_j^e} = \overline{\Delta x_i, \Delta x_j^e} = 0$	. . . . (15.a.21)

Then follows from (15.a.17), compare (13.4):

$$\text{If: } \Delta d_r^2 = \Delta d_s^2 \underset{\text{denote by}}{=} \Delta d_{r,s}^2 \qquad (15.a.22')$$

$$\overline{(z_r+\Delta z_r^e), (z_r+\Delta z_r^e)} - \overline{(z_i+\Delta z_i^e), (z_j+\Delta z_j^e)} = 2(d_{ij}^2 + \Delta d_{r,s}^2); \quad i \neq j$$

$$\overline{(z_r+\Delta z_r^e), (z_r+\Delta z_r^e)} - \overline{(z_i+\Delta z_i^e), (z_i+\Delta z_i^e)} = -2(\Delta d_i^2 - \Delta d_{r,s}^2) \qquad (15.a.22'')$$

$$\overline{(z_r+\Delta z_r^e), (z_r+\Delta z_r^e)} - \overline{(z_s+\Delta z_s^e), (z_s+\Delta z_s^e)} = 0$$

(15.a.22′) is a restriction introduced to keep the elaboration of (15.a.5) or (15.a.6) manageable in practice, see the last relation in (15.a.22″). This restriction will be hardly noticeable in practice, because usually one will take as base points P_r, P_s well-marked points {e.g. in the form of „given points"}, in view of the connection of a network to points with known {„given"} coordinates.

Since the choice of numerical values for Δd_i^2 is usually guess-work {e.g. agreed to in the „compromise" between client and executing geodesist} one will take together as many points as possible with roughly equal Δd_i^2 to form one area with $\Delta d_i^2 = \Delta d'^2$, another with $\Delta d''^2$, etc.

For unmarked terrain points one then arrives at a division of the area to be surveyed into Δd-areas, as advised already in the HTW-1956. The new theory has the attractive feature that a difference can be maintained between marked and unmarked points*).

With (15.a.22), the formulae (15.a.18), and (15.a.19), must then be replaced by:

$$\boxed{\begin{aligned} &\tfrac{1}{2}\cdot\overline{(z_i+\Delta z_i^e)^{(rs)},(z_j+\Delta z_j^e)^{(rs)T}} = \\ &= -(d_{ij}^2+\Delta d_{r,s}^2) - \frac{1}{l_{rs}^2}[z_{rs}z_{sj}^T(d_{ri}^2+\Delta d_{r,s}^2)+z_{sr}z_{rj}^T(d_{si}^2+\Delta d_{r,s}^2)+ \\ &\qquad + z_{rs}^T z_{si}(d_{rj}^2+\Delta d_{r,s}^2)+z_{sr}^T z_{ri}(d_{sj}^2+\Delta d_{r,s}^2)+ \\ &\boxed{i \neq j} \qquad +(z_{ri}z_{js}^T+z_{rj}^T z_{is})(d_{rs}^2+\Delta d_{r,s}^2)] \end{aligned}} \quad . \; . \quad (15.a.23)$$

$$\boxed{\begin{aligned} &\tfrac{1}{2}\cdot\overline{(z_i+\Delta z_i^e)^{(rs)},(z_i+\Delta z_i^e)^{(rs)T}} = \frac{l_{ri}l_{is}}{l_{rs}}(G_{ris}+\Delta G_{ris})+(\Delta d_i^2-\Delta d_{r,s}^2) \\ &(G_{ris}+\Delta G_{ris}) = 2\left\{\cos\alpha_{isr}\cdot\frac{d_{ir}^2+\Delta d_{r,s}^2}{l_{ir}}+\cos\alpha_{sri}\cdot\frac{d_{si}^2+\Delta d_{r,s}^2}{l_{si}}+\cos\alpha_{ris}\cdot\frac{d_{rs}^2+\Delta d_{r,s}^2}{l_{rs}}\right\} \end{aligned}} \quad (15.a.24)$$

With a view to the application of computers, the following „rule of thumb" can be given.

Replace (15.a.17) by:

$$\boxed{\begin{aligned} &\tfrac{1}{2}(\overline{z_r z_r^T}-\overline{z_i, z_j^T}) = \tfrac{1}{2}(\overline{z_r, z_r^T}-\overline{z_j, z_i^T}) = d_{ij}^2 = d_{ji}^2 \\ &d_{ij}^2 = \sum_p c_p\cdot(l_{ij})^p \qquad c_p \geqslant 0 \qquad 0 \leqslant p < 2 \\ &\text{also: } i,j \to r,s \qquad c_0 = \Delta d_{r,s}^2 = \lim_{j\to i} d_{ij}^2 \\ &\text{Always apply (15.a.18), with the following} \\ &\text{increment to the elements of the main diagonal:} \\ &-(d_{ii}^2-\lim_{j\to i} d_{ij}^2) = \Delta d_i^2 \quad \text{, with the result:} \\ &-(d_{rr}^2-\lim_{j\to r} d_{rj}^2) = \Delta d_r^2 = \Delta d_{r,s}^2 = c_0 \\ &-(d_{ss}^2-\lim_{j\to s} d_{sj}^2) = \Delta d_s^2 = \Delta d_{r,s}^2 = c_0 \end{aligned}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15.a.25)$$

*) But sometimes one has to take into account larger values of $\Delta d_{r,s}^2$. This may occur when given coordinates have been determined by photogrammetry. See also the last part of the section.

A check from (15.a.18) with (15.a.25):

$$\tfrac{1}{2}\cdot\overline{z_r^{(rs)}, z_r^{(rs)T}} = (\Delta d_r^2 - c_0) +$$
$$-\frac{1}{l_{rs}^2}[z_{rs}z_{sr}^T(-\Delta d_r^2 + c_0) + z_{sr}z_{rr}^T\cdot d_{sr}^2 +$$
$$+ z_{rs}^T z_{sr}(-\Delta d_r^2 + c_0) + z_{sr}^T z_{rr}\cdot d_{sr}^2 +$$
$$+ (z_{rr}z_{rs}^T + z_{rr}^T z_{rs})d_{rs}^2] = 0$$

$$\tfrac{1}{2}\cdot\overline{z_r^{(rs)}, z_s^{(rs)T}} = -d_{rs}^2 - \frac{1}{l_{rs}^2}[z_{rs}z_{ss}^T(-\Delta d_r^2 + c_0) + z_{sr}z_{rs}^T\cdot d_{sr}^2 +$$
$$+ z_{rs}^T z_{sr}\cdot d_{rs}^2 + z_{sr}^T z_{rr}(-\Delta d_s^2 + c_0) +$$
$$+ (z_{rr}z_{ss}^T + z_{rs}^T z_{rs})d_{rs}^2] =$$
$$= -d_{rs}^2 - \frac{1}{l_{rs}^2}(-l_{rs}^2 - l_{rs}^2 + l_{rs}^2)d_{rs}^2 = 0$$

Parameters of the criterion matrix

In [5] the relation between the executing geodesist and his client was described as a producer-consumer relation, such as occurring in many statistical studies. A balance must be found between the technical and economical possibilities of the producer and the requirements and economical limitations of the consumer. This balance will probably always be a compromise, where *agreements* are made on, firstly, internal and external reliability of networks as defined in [9], and, secondly, the precision of networks as indicated in (8.22) and section 9. The latter concerns an agreement on the values to be chosen for the parameters of the criterion matrix:

$$\left.\begin{array}{ll} \Delta d_i^2, & i = \dots \\ \Delta d_{r,s}^2 = c_0 & \\ c_p, & 0 < p < 2 \end{array}\right\} \qquad (15.a.26)$$

The Δd^2-parameter values must be clearly distinguished from:

$$\{\Delta d_i^2\}_T, \quad i = \dots \qquad (15.a.27)$$

which as the variance in an arbitrary direction describes the process of the definition of terrain spots as „points", and is independent of a geodetic network but can be estimated by experimental measurements. Since the refinement of every measuring process depends on the objective it has to serve, the definition of „terrain points" always depends on this objective and therefore it is part of the producer-consumer compromise.

A network provides coordinates whose precision must meet the requirements of the criterion matrix. But according to section 10 this can only be formulated in one of the possible S-systems.

This underlines the fact that coordinates can never be an objective in themselves; in an intermediate stage they serve as a description of a set of terrain points for the preparation of technical projects, afterwards they serve again as a means to realize these projects by setting out terrain points.

The concept ,,terrain point" is mentioned twice, hence one is twice confronted with $\{d_i^2\}_T$. As in the HTW-1956, it must be possible in the producer-consumer compromise to connect the parameter values of the criterion matrix with the values for $\{\Delta d_i^2\}_T$. A start for such a theory of ,,reconnaissance" has already been made; the elaboration, however, will take a considerable time.

In a way, the criterion matrix can be considered as the description of the precision of the network established as a result of the producer-consumer compromise. As such, the criterion matrix {possibly with different parameter values which are experimentally determined} can be used as a substitute matrix for the {unknown or unavailable} covariance matrix of the coordinates of ,,given points" in densification networks, as studied in sections 5, 7 and 9. This must of course also be considered in the producer-consumer compromise.

16 AXIOMATIC APPROACH TO THE MODEL OF S-SYSTEMS

In connection with the question raised in the reference at the end of section 15, one may ask how an S-transformation like (15.1) is connected with the relation between S-variate and Π-variate in (15.2), with the conditions*) in a triangle like (15.10) to which (15.9) is connected, and with the conditions in a quadrangle such as (15.24)+(15.25) and (15.27)+(15.28) to which (15.38) is connected.

It will be tried to answer this question by sketching an axiomatic approach within a linearized model.

Definition $\underline{\Delta z}_i^{(rs)}$, see (12.2) or (15.2):

$$\boxed{\underline{\Delta z}_i^{(rs)} \underset{\text{def}}{=} -\frac{z_{ri}z_{is}}{z_{rs}}\underline{\Delta \Pi}_{ris}} \qquad (16.1)$$

Consequences:

$$\underline{\Delta z}_i^{(sr)} = -\frac{z_{si}z_{ir}}{z_{sr}}\underline{\Delta \Pi}_{sir} =$$

$$= -\frac{z_{ri}z_{is}}{z_{rs}}\underline{\Delta \Pi}_{ris} = \underline{\Delta z}_i^{(rs)} \qquad (16.2)$$

$$\underline{\Delta z}_r^{(rs)} = -\frac{z_{rr}z_{rs}}{z_{rs}}\underline{\Delta \Pi}_{rrs}$$

Π_{rrs} has not been defined in [2], but $z_{rr} = 0$, hence, cf (2.7) and (2.11):

$$\underline{\Delta z}_r^{(rs)} = \underline{\Delta z}_s^{(rs)} = 0 \qquad (16.3)$$

$$\underline{\Delta z}_{ij}^{(rs)} = \underline{\Delta z}_j^{(rs)} - \underline{\Delta z}_i^{(rs)} = \underline{\Delta z}_{ij}^{(sr)} = -\underline{\Delta z}_{ji}^{(rs)} \qquad (16.4)$$

Δ-conditions for consistent triangle

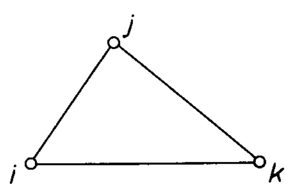


Fig. 16-1

*) Concerning the concept condition, see note on page 83.

To direct thoughts, think of the following special case of a general S-transformation, worked out with (16.1)–(16.4):

$$\underline{\Delta z}_k^{(ji)} = \underline{\Delta z}_k^{(ik)} - \frac{z_{jk}}{z_{ji}} \underline{\Delta z}_i^{(ik)} - \frac{z_{ik}}{z_{ij}} \underline{\Delta z}_j^{(ik)}$$

$$= -\frac{z_{ik}}{z_{ij}} \underline{\Delta z}_j^{(ik)}$$

Require, for each consistent triangle jik, that *two* of the following three conditions are fulfilled:

A_i	$\underline{\Delta z}_k^{(ji)} = -\frac{z_{ik}}{z_{ij}} \underline{\Delta z}_j^{(ik)}$	(16.5)
A_k	$\underline{\Delta z}_j^{(ik)} = -\frac{z_{kj}}{z_{ki}} \underline{\Delta z}_i^{(kj)}$	(16.6)
A_j	$\underline{\Delta z}_i^{(kj)} = -\frac{z_{ji}}{z_{jk}} \underline{\Delta z}_k^{(ji)}$	(16.7)

From (16.5) and (16.6):

$$\underline{\Delta z}_k^{(ji)} = -\frac{z_{ik}}{z_{ij}} \underline{\Delta z}_j^{(ik)} = \frac{z_{ik}}{z_{ij}} \frac{z_{kj}}{z_{ki}} \underline{\Delta z}_i^{(kj)} =$$

$$= -\frac{z_{jk}}{z_{ji}} \underline{\Delta z}_i^{(kj)} \qquad \rightarrow (16.7) .$$

Hence: (A_j) dependent on (A_i) and (A_k) . . . . (16.8)

(16.5) with (16.1):

$$-\frac{z_{jk} z_{ki}}{z_{ji}} \underline{\Delta \Pi}_{jki} + \frac{z_{ik}}{z_{ij}} \left(-\frac{z_{ij} z_{jk}}{z_{ik}} \underline{\Delta \Pi}_{ijk} \right) = 0$$

$$z_{ki} \cdot \underline{\Delta \Pi}_{jki} + z_{ji} \cdot \underline{\Delta \Pi}_{ijk} = 0 \qquad \rightarrow N_{(i),j,k}$$

Hence:

$$\left. \begin{array}{l} (A_i) \text{ with (16.1) gives net condition } N_{(i),j,k} \\ (A_k) \text{ with (16.1) gives net condition } N_{(k),i,j} \\ (A_j) \text{ with (16.1) gives net condition } N_{(j),k,i} \end{array} \right\} \quad \ldots\ldots\ldots\ldots \quad (16.9)$$

(16.9) according to [2] explains the dependence (16.8), hence the requirement according to (16.5)–(16.7) is also necessary and sufficient for a consistent triangle.

B-conditions for consistent quadrangle

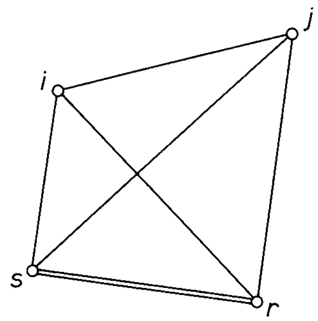


Fig. 16-2

Require, apart from two A-conditions for each triangle, per quadrangle *ijrs at most two* B-conditions, usually related to the choice of the computational base *rs*:

B_r	$\underline{\Delta z}_j^{(ri)} = \underline{\Delta z}_j^{(rs)} - \frac{z_{rj}}{z_{ri}} \underline{\Delta z}_i^{(rs)}$	(16.10)
B_s	$\underline{\Delta z}_i^{(sj)} = \underline{\Delta z}_i^{(sr)} - \frac{z_{si}}{z_{sj}} \underline{\Delta z}_j^{(sr)}$	(16.11)

(16.10) gives a relation between the triangle *jri*, *jrs* and *irs*, *not* triangle *isj*;
(16.11) gives a relation between the triangles *isj*, *isr* and *jsr*, *not* triangle *jri*;
or (B_r) and (B_s) cannot be derived from each other by means of A-conditions.

But B-conditions can be transformed by means of A-conditions, for example, with:

$$\underline{\Delta z}_i^{(rj)} = -\frac{z_{ri}}{z_{rj}} \underline{\Delta z}_j^{(ri)}$$

$$\underline{\Delta z}_j^{(si)} = -\frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sj)}$$

we obtain:

(16.10): $$\underline{\Delta z}_i^{(rj)} = \underline{\Delta z}_i^{(rs)} - \frac{z_{ri}}{z_{rj}} \underline{\Delta z}_j^{(rs)} \qquad (16.12)$$

(16.11): $$\underline{\Delta z}_j^{(si)} = \underline{\Delta z}_j^{(sr)} - \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sr)} \qquad (16.13)$$

But a further transformation is also possible, for example, with:

$$\underline{\Delta z}_j^{(ri)} = -\frac{z_{ij}}{z_{ir}} \underline{\Delta z}_r^{(ij)}$$

$$\underline{\Delta z}_j^{(rs)} = -\frac{z_{sj}}{z_{sr}} \underline{\Delta z}_r^{(sj)}$$

$$\underline{\Delta z}_i^{(rs)} = -\frac{z_{si}}{z_{sr}} \underline{\Delta z}_r^{(si)}$$

we obtain:

$$(16.10): \quad -\frac{z_{ij}}{z_{ir}}\underline{\Delta z}_r^{(ij)} + \frac{z_{sj}}{z_{sr}}\underline{\Delta z}_r^{(sj)} - \frac{z_{rj}}{z_{ri}}\frac{z_{si}}{z_{sr}}\underline{\Delta z}_r^{(si)} = 0$$

or with (16.1):

$$z_{rj}(\underline{\Delta \Pi}_{irj} - \underline{\Delta \Pi}_{srj} + \underline{\Delta \Pi}_{sri}) = 0 \qquad \rightarrow W_r \text{ *)}$$

Hence:

$$\left.\begin{array}{l} (B_r) \text{ with } A\text{-conditions and (16.1) gives central condition } W_r \\ (B_s) \text{ with } A\text{-conditions and (16.1) gives central condition } W_s \end{array}\right\} \quad \ldots\ldots \quad (16.14)$$

Now, with the A-conditions:

$$\underline{\Delta z}_j^{(ri)} = -\frac{z_{ij}}{z_{ir}}\underline{\Delta z}_r^{(ij)}$$

$$\underline{\Delta z}_i^{(sj)} = -\frac{z_{ji}}{z_{js}}\underline{\Delta z}_s^{(ij)}$$

$$(16.10): \quad \underline{\Delta z}_r^{(ij)} = -\frac{z_{ir}}{z_{ij}}\underline{\Delta z}_j^{(rs)} + \frac{z_{jr}}{z_{ij}}\underline{\Delta z}_i^{(rs)}$$

$$(16.11): \quad \underline{\Delta z}_s^{(ij)} = -\frac{z_{is}}{z_{ij}}\underline{\Delta z}_j^{(rs)} + \frac{z_{js}}{z_{ij}}\underline{\Delta z}_i^{(rs)}$$

The two equations give as their difference:

$$\boxed{B_{rs} \quad \underline{\Delta z}_{rs}^{(ij)} = -\frac{z_{rs}}{z_{ij}}\underline{\Delta z}_{ij}^{(rs)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16.15)$$

(B_{rs}) can replace either (B_r) or (B_s). This type of B-conditions can be very useful in derivations; it can also be considered as an extension of the type of the A-conditions, in the case of the opposite sides of a quadrangle. To see this, one can e.g. write (16.5) with (16.3) in the form:

$$(16.5): \quad \underline{\Delta z}_{ik}^{(ij)} = -\frac{z_{ik}}{z_{ij}}\underline{\Delta z}_{ij}^{(ik)}$$

From (16.10), several B-conditions can be derived by cyclic permutation of indices, see Fig. 16-2:

---

*) W_r indicates a central condition, see [13], chapter 3. W is the first letter of the Dutch word "waaier", meaning "fan" in English.

B_r	$\underline{\Delta z}_j^{(ri)} = \underline{\Delta z}_j^{(rs)} - \frac{z_{rj}}{z_{ri}} \underline{\Delta z}_i^{(rs)}$	$\rightarrow W_r$	(16.16a)
	$i \rightarrow j, j \rightarrow r, r \rightarrow s, s \rightarrow i$, gives:		
B_s	$\underline{\Delta z}_r^{(sj)} = \underline{\Delta z}_r^{(si)} - \frac{z_{sr}}{z_{sj}} \underline{\Delta z}_j^{(si)}$	$\rightarrow W_s$	(16.16b)
	$j \rightarrow r, r \rightarrow s, s \rightarrow i, i \rightarrow j$, gives:		
B_i	$\underline{\Delta z}_s^{(ir)} = \underline{\Delta z}_s^{(ij)} - \frac{z_{is}}{z_{ir}} \underline{\Delta z}_r^{(ij)}$	$\rightarrow W_i$	(16.16c)
	$r \rightarrow s, s \rightarrow i, i \rightarrow j, j \rightarrow r$, gives:		
B_j	$\underline{\Delta z}_i^{(js)} = \underline{\Delta z}_i^{(jr)} - \frac{z_{ji}}{z_{js}} \underline{\Delta z}_s^{(jr)}$	$\rightarrow W_j$	(16.16d)
	$s \rightarrow i, i \rightarrow j, j \rightarrow r, r \rightarrow s$, gives:	(B_r)	

With A-conditions follows from (16.16b), as a check:

$$- \frac{z_{sr}}{z_{sj}} \underline{\Delta z}_j^{(sr)} = - \frac{z_{sr}}{z_{si}} \underline{\Delta z}_i^{(sr)} + \frac{z_{sr}}{z_{sj}} \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sj)}$$

or:

$$\underline{\Delta z}_i^{(sj)} = \underline{\Delta z}_i^{(sr)} - \frac{z_{si}}{z_{sj}} \underline{\Delta z}_j^{(sr)}, \quad \text{or} \quad (16.11)$$

Now eliminate $\underline{\Delta z}_j^{(rs)}$ from (B_r) and (B_s), for which it is easiest to take the forms (16.10) and (16.11):

$$(16.10): \quad \underline{\Delta z}_j^{(ri)} + \frac{z_{rj}}{z_{ri}} \underline{\Delta z}_i^{(rs)} = \underline{\Delta z}_j^{(rs)}$$

$$(16.11): \quad \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sj)} - \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sr)} = - \underline{\Delta z}_j^{(rs)}$$

Addition gives, with:

$$\frac{z_{rj}}{z_{ri}} - \frac{z_{sj}}{z_{si}} = \frac{z_{ri} + z_{ij}}{z_{ri}} - \frac{z_{si} + z_{ij}}{z_{si}} = z_{ij} \frac{z_{si} - z_{ri}}{z_{ri} z_{si}} = \frac{z_{ij} z_{sr}}{z_{ri} z_{si}}$$

$$\underline{\Delta z}_j^{(ri)} + \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sj)} + \frac{z_{ij} z_{sr}}{z_{ri} z_{si}} \underline{\Delta z}_i^{(rs)} = 0$$

or, with A-conditions:

$$-\frac{z_{ij}}{z_{ir}}\underline{\Delta z_r^{(ij)}}-\frac{z_{sj}}{z_{si}}\frac{z_{ji}}{z_{js}}\underline{\Delta z_s^{(ij)}}-\frac{z_{ij}z_{sr}}{z_{ri}z_{si}}\frac{z_{ri}}{z_{rs}}\underline{\Delta z_s^{(ir)}}=0$$

or:

$$\underline{\Delta z_s^{(ir)}}=\underline{\Delta z_s^{(ij)}}-\frac{z_{is}}{z_{ir}}\underline{\Delta z_r^{(ij)}}, \quad \text{hence} \quad (16.16c)$$

In a similar way, (16.16d) can be derived, or:

In a quadrangle, besides two A-conditions per triangle, only two B-conditions are independent . . . . (16.17)

As a check, a comparison with the theory of section 8.6 in [2], although a closer investigation will be given later, cf. the note to section 16.

Every vertex of a quadrangle is the centre of three rays {"radiate variates" in [11]}, determined by two Π-variates, out of the three possible combinations of two radiate variates into Π-variates. Consistency requires one W-condition per vertex {think of station adjustment}.

According to (8.64) in [2]:	$3\times 2=$	6 N-conditions
In four vertices:	$4\times 1=$	4 W-conditions
Total	$=10$	conditions

Now there are four triangles in a quadrangle, and hence $4\times 2=8$ N-conditions can be established.

The difference, $8-6=2$ N-conditions can be used to replace two of the four W-conditions. Then one obtains:

Per triangle	2 N- (or A-) conditions $\rightarrow$	$4\times 2=8$
Per quadrangle	2 W- (or B-) conditions $\rightarrow$	$=2$
	Total	$=10$

. . . . (16.18)

From quadrangle to pentagon

The computation of variances according to (15.38) means that all possible quadrangles {and hence, for i and j coinciding, all triangles} on a computational base rs are considered.

We shall follow the same line here; Fig. 16.2 is extended with the point k.

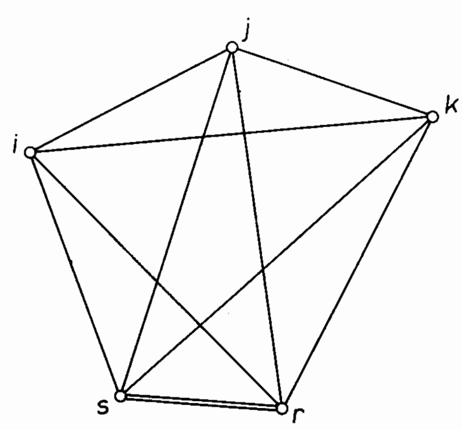


Fig. 16-3

Now consider the B_r- and B_s-conditions in each of the four quadrangles on the base sr:

Quadrangle	*Independent* B-relations		
$i\ s\ r\ j$	B_r	$\underline{\Delta z}_j^{(ri)} = \underline{\Delta z}_j^{(rs)} - \frac{z_{rj}}{z_{ri}} \underline{\Delta z}_i^{(rs)}$	(16.19)
{see (16.13)}	B_s	$\underline{\Delta z}_j^{(si)} = \underline{\Delta z}_j^{(sr)} - \frac{z_{sj}}{z_{si}} \underline{\Delta z}_i^{(sr)}$	(16.19′)
$i\ s\ r\ k$	B_r	$\underline{\Delta z}_k^{(ri)} = \underline{\Delta z}_k^{(rs)} - \frac{z_{rk}}{z_{ri}} \underline{\Delta z}_i^{(rs)}$	(16.20)
	B_s	$\underline{\Delta z}_k^{(si)} = \underline{\Delta z}_k^{(sr)} - \frac{z_{sk}}{z_{si}} \underline{\Delta z}_i^{(sr)}$	(16.20′)
$j\ s\ r\ k$	B_r	$\underline{\Delta z}_k^{(rj)} = \underline{\Delta z}_k^{(rs)} - \frac{z_{rk}}{z_{rj}} \underline{\Delta z}_j^{(rs)}$	(16.21)
	B_s	$\underline{\Delta z}_k^{(sj)} = \underline{\Delta z}_k^{(sr)} - \frac{z_{sk}}{z_{sj}} \underline{\Delta z}_j^{(sr)}$	(16.21′)

Substitution of $\underline{\Delta z}_k^{(rs)}$ from (16.20) and of $\underline{\Delta z}_j^{(rs)}$ from (16.19) into (16.21) gives:

$$\underline{\Delta z}_k^{(rj)} = \underline{\Delta z}_k^{(ri)} - \frac{z_{rk}}{z_{rj}} \underline{\Delta z}_j^{(ri)} + \underbrace{\left(- \frac{z_{rk}}{z_{ri}} + \frac{z_{rk} z_{rj}}{z_{rj} z_{ri}} \right)}_{=0} \underline{\Delta z}_i^{(rs)}$$

Similarly (16.21′) with (16.20′) and (16.19′). Hence:

Quadrangle	Dependent B-relations		
$j\ i\ r\ k$	B_r	$\underline{\Delta z}_k^{(rj)} = \underline{\Delta z}_k^{(ri)} - \frac{z_{rk}}{z_{rj}} \underline{\Delta z}_j^{(ri)}$	(16.22)
$j\ i\ s\ k$	B_s	$\underline{\Delta z}_k^{(sj)} = \underline{\Delta z}_k^{(si)} - \frac{z_{sk}}{z_{sj}} \underline{\Delta z}_j^{(si)}$	(16.23)

This is immediately clear, because the 4 rays in r and s make possible 6 combinations of 2 {i.e. a Π-variate}, whereas only 3 are needed.

From (16.19) and (16.20), respectively (16.19′) and (16.20′) follows by subtraction:

$$(16.20) - (16.19):\ \underline{\Delta z}_{jk}^{(ri)} = \underline{\Delta z}_{jk}^{(rs)} - \frac{z_{jk}}{z_{ri}} \underline{\Delta z}_i^{(rs)} \qquad (16.24)$$

$$(16.20') - (16.19'):\ \underline{\Delta z}_{jk}^{(si)} = \underline{\Delta z}_{jk}^{(rs)} - \frac{z_{jk}}{z_{si}} \underline{\Delta z}_i^{(rs)} \qquad (16.25)$$

Elimination of $\underline{\Delta z}_i^{(rs)}$ gives after some rearrangement:

$$\frac{z_{ri}}{z_{jk}}\underline{\Delta z}_{jk}^{(ri)} - \frac{z_{si}}{z_{jk}}\underline{\Delta z}_{jk}^{(si)} = \frac{z_{rs}}{z_{jk}}\underline{\Delta z}_{jk}^{(rs)} \qquad (16.26)$$

But in quadrangle *jsrk*, already completely covered by (16.21) and (16.21′) we have then a dependent B_{jk}-condition, compare (16.15):

$$\begin{aligned}\frac{z_{rs}}{z_{jk}}\underline{\Delta z}_{jk}^{(rs)} &= -\underline{\Delta z}_{rs}^{(jk)} \\ &= -\underline{\Delta z}_{ri}^{(jk)} + \underline{\Delta z}_{si}^{(jk)} \qquad (16.27)\end{aligned}$$

(16.26) with (16.27) gives:

$$\left(\underline{\Delta z}_{ri}^{(jk)} + \frac{z_{ri}}{z_{jk}}\underline{\Delta z}_{jk}^{(ri)}\right) - \left(\underline{\Delta z}_{si}^{(jk)} + \frac{z_{si}}{z_{jk}}\underline{\Delta z}_{jk}^{(si)}\right) = 0 \qquad (16.28)$$

In (16.28) one recognizes the B_{si}-condition in quadrangle *jirk* and the B_{si}-condition in quadrangle *jisk*, exactly the complement to (16.22) respectively (16.23) which is necessary to guarantee the consistency of both quadrangles. But from (16.28) follows that only one of the two conditions is dependent. Hence, for example;

Quadrangle	*Independent B*-condition	
j i r k	B_{ri}	$\underline{\Delta z}_{ri}^{(jk)} = -\frac{z_{ri}}{z_{jk}}\underline{\Delta z}_{jk}^{(ri)}$

(16.29)

Quadrangle	Dependent B-condition	
j i s k	B_{si}	$\underline{\Delta z}_{si}^{(jk)} = -\frac{z_{si}}{z_{jk}}\underline{\Delta z}_{jk}^{(si)}$

(16.30)

Starting from the three quadrangles in (16.19) to (16.21′), used for the computation of variances in (15.38), (16.22) with (16.29) and (16.23) with (16.30) give the necessary complement to five possible quadrangles {not considering the order of the vertices}. The pentagon is therefore consistent, and the computation of variances can be executed with respect to one of the sides as a computational base.

As a check again a comparison with the theory of section 8.6 in [2]. Every vertex of the pentagon is the centre of four rays, determined by three Π-variates, out of the possible six Π-variates. Hence, consistency requires three W-conditions per vertex.

According to (8.6.4) in [2]:	$4 \times 3 = 12$	N-conditions
In five vertices:	$5 \times 3 = 15$	W-conditions
Total	$= 27$	conditions

Now in a pentagon there are ten triangles, and therefore $10 \times 2 = 20$ N-conditions can be established. The difference $20-12=8$ N-conditions can be used to replace eight of the fifteen W-conditions.

Thus we obtain:

$$\left.\begin{array}{lr} \text{Per triangle } 2\ N\text{- (or } A\text{-) conditions} & 10\times 2=20 \\ \text{Per pentagon } 7\ W\text{- (or } B\text{-) conditions} & 7 \\ \hline \text{Total} & =27 \end{array}\right\} \qquad (16.31)$$

This investigation explains the requirement that in a consistent pentagon "at most two" B-conditions can be used {see above (16.10)}, in a pentagon with five quadrangles only 7 out of the $5 \times 2 = 10$ B-conditions are independent. In the present investigation the three dependent B-conditions are (16.22), (16.23) and (16.30).

The general S-transformation

After (16.28)-(16.30) one can continue with one of the two relations (16.24), (16.25). Choose, e.g., (16.24), and apply (16.29):

$$-\frac{z_{jk}}{z_{ri}}\underline{\Delta z}_{ri}^{(jk)} = \underline{\Delta z}_{jk}^{(rs)} - \frac{z_{jk}}{z_{ri}}\underline{\Delta z}_{i}^{(rs)}$$

$$\underline{\Delta z}_{ri}^{(jk)} = \underline{\Delta z}_{i}^{(rs)} - \frac{z_{ri}}{z_{jk}}\underline{\Delta z}_{jk}^{(rs)}$$

$$\underline{\Delta z}_{i}^{(jk)} = \underline{\Delta z}_{i}^{(rs)} - \frac{z_{ri}}{z_{jk}}\underline{\Delta z}_{jk}^{(rs)} + \underline{\Delta z}_{r}^{(jk)} \qquad (16.32)$$

With A-condition in triangle krj:

$$\underline{\Delta z}_{r}^{(jk)} = -\frac{z_{jr}}{z_{jk}}\underline{\Delta z}_{k}^{(rj)}$$

Or, with B_r-condition (16.21):

$$\underline{\Delta z}_{r}^{(jk)} = -\frac{z_{jr}}{z_{jk}}\underline{\Delta z}_{k}^{(rs)} + \frac{z_{kr}}{z_{jk}}\underline{\Delta z}_{j}^{(rs)} \qquad (16.33)$$

(16.33) into (16.32) gives:

$$\boxed{\underline{\Delta z}_{i}^{(jk)} = \underline{\Delta z}_{i}^{(rs)} - \frac{z_{ki}}{z_{kj}}\underline{\Delta z}_{j}^{(rs)} - \frac{z_{ji}}{z_{jk}}\underline{\Delta z}_{k}^{(rs)}} \qquad (16.34)$$

In (16.34) the general S-transformation (3.5) is recognized, because with:

$j \rightarrow v, \quad k \rightarrow w,$ we obtain:

$$\underline{\Delta z}_i^{(vw)} = \underline{\Delta z}_i^{(rs)} - \frac{z_{wi}}{z_{wv}} \underline{\Delta z}_v^{(rs)} - \frac{z_{vi}}{z_{rv}} \underline{\Delta z}_w^{(rs)} \quad \ldots \ldots \ldots \ldots \quad (16.35)$$

From the derivation of (16.32) the necessity of applying (16.29) is evident. This is understandable, because if one wishes to apply (15.38) directly for the computation of variances with respect to the computational base *jk*, one has to deal with the quadrangles:

rkjs, rkji, skji

or, differently arranged, with reference to formulae:

jsrk {(16.21), (16.21′)}, *jirk* {(16.22), (16.29)}, *jisk* {(16.23), (16.30)}

Extension to *n*-gon

To complete the proof that two *A*-conditions per triangle and at most two *B*-conditions per quadrangle suffice – so that the variance computation according to (15.38) is unique – the count according to (16.18) for the quadrangle and according to (16.31) for the pentagon should be continued for the *n*-gon with $n > 5$. This, however, is left to the diligent reader.

Note to section 16. Conditions in quadrangle and pentagon according to [2]

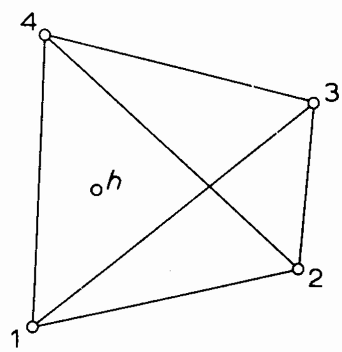


Fig. 16-4

Consider quadrangle 1,2,3,4 and refer to section 10.2 in [2] *).

The conditions in the four triangles and the four central conditions are, in a consistent quadrangle:

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*) For a summary in English of the Polygon Theory in the Complex Plane, see [13], chapter 3. Reference is also made to [4], section 4.

W indica·es a central condition, see note p. 114.

V indicates a polygon condition. *V* is the first letter of the Dutch word "veelhoek", meaning polygon in English.

N_h is derived from net- and polygon conditions, using an arbitrary auxiliary point.

$V_{1,2,4}$	$\underline{\Delta\Pi}_{412}+\underline{\Delta\Pi}_{124}+\underline{\Delta\Pi}_{241}=0$	. . . . . . (16.36a)
$V_{2,3,1}$	$\underline{\Delta\Pi}_{123}+\underline{\Delta\Pi}_{231}+\underline{\Delta\Pi}_{312}=0$	. . . . . . (16.36b)
$V_{3,4,2}$	$\underline{\Delta\Pi}_{234}+\underline{\Delta\Pi}_{342}+\underline{\Delta\Pi}_{423}=0$	. . . . . . (16.36c)
$V_{4,1,3}$	$\underline{\Delta\Pi}_{341}+\underline{\Delta\Pi}_{413}+\underline{\Delta\Pi}_{134}=0$	. . . . . . (16.36d)
$N_{(h),1,2,4}$	$z_{1h}\cdot\underline{\Delta\Pi}_{412}+z_{2h}\cdot\underline{\Delta\Pi}_{124}+z_{4h}\cdot\underline{\Delta\Pi}_{241}=0$	. . . . . . (16.36e)
$N_{(h),2,3,1}$	$z_{2h}\cdot\underline{\Delta\Pi}_{123}+z_{3h}\cdot\underline{\Delta\Pi}_{231}+z_{1h}\cdot\underline{\Delta\Pi}_{312}=0$	. . . . . . (16.36f)
$N_{(h),3,4,2}$	$z_{3h}\cdot\underline{\Delta\Pi}_{234}+z_{4h}\cdot\underline{\Delta\Pi}_{342}+z_{2h}\cdot\underline{\Delta\Pi}_{423}=0$	. . . . . . (16.36g)
$N_{(h),4,1,3}$	$z_{4h}\cdot\underline{\Delta\Pi}_{341}+z_{1h}\cdot\underline{\Delta\Pi}_{413}+z_{3h}\cdot\underline{\Delta\Pi}_{134}=0$	. . . . . . (16.36h)
W_1	$\underline{\Delta\Pi}_{413}+\underline{\Delta\Pi}_{312}-\underline{\Delta\Pi}_{412}=0$	. . . . . . (16.36i)
W_2	$\underline{\Delta\Pi}_{124}+\underline{\Delta\Pi}_{423}-\underline{\Delta\Pi}_{123}=0$	. . . . . . (16.36j)
W_3	$\underline{\Delta\Pi}_{231}+\underline{\Delta\Pi}_{134}-\underline{\Delta\Pi}_{234}=0$	. . . . . . (16.36k)
W_4	$\underline{\Delta\Pi}_{342}+\underline{\Delta\Pi}_{241}-\underline{\Delta\Pi}_{341}=0$	. . . . . . (16.36l)

Only 10 out of the 12 conditions (16.36) are independent, for, simplifying the numbering of the formulae, it is easily verified that:

$$\{(a)-(b)+(c)-(d)\}+\{(i)-(j)+(k)-(l)\}=0 \qquad \text{. . . (16.37a)}$$

$$\{(e)-(f)+(g)-(h)\}+\{z_{1h}\cdot(i)-z_{2h}\cdot(j)+z_{3h}\cdot(k)-z_{4h}\cdot(l)\}=0 \qquad \text{. . . (16.37b)}$$

In agreement with the approach followed in this section, the polygon- and net-conditions for each triangle are maintained. According to (16.37), *two central conditions* are then *dependent*. . . (16.38)

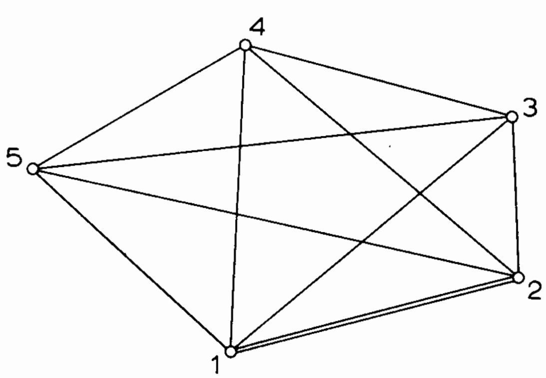

Fig. 16-5

Examine now pentagon 1,2,3,4,5 and apply (16.38) for the 5 quadrangles in this figure. In each quadrangle the following central conditions have been chosen as independent, with the addition of some dependent central conditions:

Quadrangle	*Independent* central conditions		
5, 1, 2, 3	W_1	$\underline{\Delta\Pi}_{513}+\underline{\Delta\Pi}_{312}-\underline{\Delta\Pi}_{512}=0$	. . . . (16.39a)
	W_2	$\underline{\Delta\Pi}_{125}+\underline{\Delta\Pi}_{523}-\underline{\Delta\Pi}_{123}=0$	. . . . (16.39b)
5, 1, 2, 4	W_1	$\underline{\Delta\Pi}_{514}+\underline{\Delta\Pi}_{412}-\underline{\Delta\Pi}_{512}=0$	. . . . (16.39c)
	W_2	$\underline{\Delta\Pi}_{125}+\underline{\Delta\Pi}_{524}-\underline{\Delta\Pi}_{124}=0$	. . . . (16.39d)
4, 1, 2, 3	W_1	$\underline{\Delta\Pi}_{413}+\underline{\Delta\Pi}_{312}-\underline{\Delta\Pi}_{412}=0$	. . . . (16.39e)
	W_2	$\underline{\Delta\Pi}_{124}+\underline{\Delta\Pi}_{423}-\underline{\Delta\Pi}_{123}=0$	. . . . (16.39f)
5, 1, 3, 4	W_1	$\underline{\Delta\Pi}_{514}+\underline{\Delta\Pi}_{413}-\underline{\Delta\Pi}_{513}=0$	. . . . (16.39g)
	W_3	$\underline{\Delta\Pi}_{135}+\underline{\Delta\Pi}_{534}-\underline{\Delta\Pi}_{134}=0$	. . . . (16.39h)
5, 2, 3, 4	W_2	$\underline{\Delta\Pi}_{524}+\underline{\Delta\Pi}_{423}-\underline{\Delta\Pi}_{523}=0$	. . . . (16.39i)
	W_3	$\underline{\Delta\Pi}_{235}+\underline{\Delta\Pi}_{534}-\underline{\Delta\Pi}_{234}=0$	. . . . (16.39j)
Quadrangle	Dependent central conditions		
5, 1, 2, 3	W_3	$\underline{\Delta\Pi}_{231}+\underline{\Delta\Pi}_{135}-\underline{\Delta\Pi}_{235}=0$	. . . . (16.39k)
4, 1, 2, 3	W_3	$\underline{\Delta\Pi}_{231}+\underline{\Delta\Pi}_{134}-\underline{\Delta\Pi}_{234}=0$	. . . . (16.39l)

But a calculation shows that the conditions (16.39) are related by:

Conditions	Relations between conditions	
W_1's	(a)−(c)−(e)+(g) = 0	. . . . (16.40a)
W_2's	(b)−(d)−(f)+(i) = 0	. . . . (16.40b)
W_3's	(h)−(j)−(k)+(l) = 0	. . . . (16.40c)

Hence the conclusion is: out of the 10 independent central conditions in (16.39), the combination of the 5 quadrangles in a pentagon results in only 7 independent ones *).

*) This result was first proved by J. VAN MIERLO in a different way.

In connection with (16.19) – (16.21′) and (16.29) one can choose, for example:

In pentagon	Quadrangle	*Independent* central conditions
(16.39a)	5, 1, 2, 3	W_1
(16.39b)		W_2
(16.39c)	5, 1, 2, 4	W_1
(16.39d)		W_2
(16.39e)	4. 1, 2, 3	W_1
(16.39f)		W_2
(16.39h)	5, 1, 3, 4	W_3

. . (16.41)

This provides a second sufficient proof of the contents of this section.

17 CONNECTION OF SEVERAL STOCHASTICALLY INDEPENDENT COORDINATE SYSTEMS BY MEANS OF S-TRANSFORMATIONS

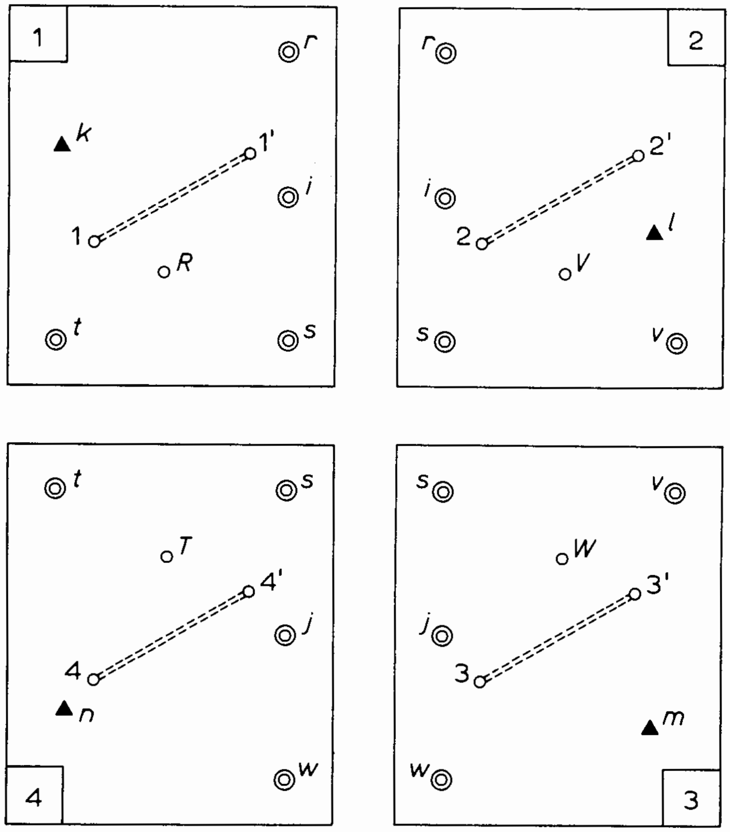

Fig. 17-1

Four networks have been measured and adjusted, hence there are four coordinate systems (1), (2), (3), (4), and it is assumed that the coordinate variates from different networks are stochastically independent. Some points belong to a fifth network of higher order, measured in an earlier period. The latter points have "given" coordinate variates in an (a)-system, stochastically independent of the coordinate variates mentioned before.

From Fig. 17-1 is evident:

base points of systems (1)–(4)	$P_1, P_{1'}; P_2, P_{2'}; P_3, P_{3'}; P_4, P_{4'}$
common points idem	$P_r, P_s, P_v, P_w, P_t, P_i, P_j$
"given points" also in (a)-system	P_k, P_l, P_m, P_n
remaining points of systems (1)–(4)	P_R, P_V, P_W, P_T

Since S-transformations have the character of difference equations, the five coordinate systems must be roughly coincident, which is achieved by some form of preliminary similarity transformation. This makes it possible to choose for every point, *once and for all*, an approximate coordinate value z^0. After this choice it becomes possible to consider the (1) system as an $\underset{1,1'}{S}$-system {or to transform it into this}; similarly, the systems (2)–(4) are specified. Thus one obtains the following survey of data:

Given	Working hypothesis: no correlation between systems
(a)-system	$\underline{\Delta z}_k^{(a)}, \underline{\Delta z}_l^{(a)}, \underline{\Delta z}_m^{(a)}, \underline{\Delta z}_n^{(a)}$
(1)-system	$\underline{\Delta z}_r^{(1,1')}, \underline{\Delta z}_s^{(1,1')}, \underline{\Delta z}_t^{(1,1')}, \underline{\Delta z}_i^{(1,1')}, \underline{\Delta z}_k^{(1,1')}, \underline{\Delta z}_R^{(1,1')}$
(2)-system	$\underline{\Delta z}_r^{(2,2')}, \underline{\Delta z}_s^{(2,2')}, \underline{\Delta z}_v^{(2,2')}, \underline{\Delta z}_i^{(2,2')}, \underline{\Delta z}_l^{(2,2')}, \underline{\Delta z}_V^{(2,2')}$
(3)-system	$\underline{\Delta z}_w^{(3,3')}, \underline{\Delta z}_s^{(3,3')}, \underline{\Delta z}_v^{(3,3')}, \underline{\Delta z}_j^{(3,3')}, \underline{\Delta z}_m^{(3,3')}, \underline{\Delta z}_W^{(3,3')}$
(4)-system	$\underline{\Delta z}_w^{(4,4')}, \underline{\Delta z}_s^{(4,4')}, \underline{\Delta z}_t^{(4,4')}, \underline{\Delta z}_j^{(4,4')}, \underline{\Delta z}_n^{(4,4')}, \underline{\Delta z}_T^{(4,4')}$

(17.1)

In (17.1) the variates whose value is zero according to (2.7) have been omitted, such as:

$$\underline{\Delta z}_1^{(1,1')} = \underline{\Delta z}_{1'}^{(1,1')} = \underline{\Delta z}_2^{(2,2')} = \ldots = 0$$

Computation of corrections to "free"*) variates

Under the working hypothesis mentioned in (17.1), viz. that coordinate variates from different systems are uncorrelated, the computations of corrections to "free" variates {(3.13) ff., (5.3), (6.1)} can be separated in a simple way from the actual adjustment problem, in which "tied" variates occur in the condition equations. For this, one needs the covariance matrix of the tied variates and the covariances between "free" and "tied" variates, both for each separate system. There will usually be large numbers of "free" variates; in (17.1) they are denoted by:

$$\underline{\Delta z}_R^{(1,1')}, \underline{\Delta z}_V^{(2,2')}, \underline{\Delta z}_W^{(3,3')}, \underline{\Delta z}_T^{(4,4')} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17.2')$$

As an example, the computation of the correction to the first variate in (17.2′) is given in schematic form, combining x, y coordinates into z-coordinates**):

$$(\varepsilon_R^{(1,1')}) = \overline{(z_R^{(1,1')}), \begin{pmatrix} z_r^{(1,1')} \\ z_s^{(1,1')} \\ z_t^{(1,1')} \\ z_i^{(1,1')} \\ z_k^{(1,1')} \end{pmatrix}} \cdot \left\{ \overline{\begin{pmatrix} z_r^{(1,1')} \\ z_s^{(1,1')} \\ z_t^{(1,1')} \\ z_i^{(1,1')} \\ z_k^{(1,1')} \end{pmatrix}, \begin{pmatrix} z_r^{(1,1')} \\ z_s^{(1,1')} \\ z_t^{(1,1')} \\ z_i^{(1,1')} \\ z_k^{(1,1')} \end{pmatrix}^*} \right\}^{-1} \cdot \begin{pmatrix} \varepsilon_r^{(1,1')} \\ \varepsilon_s^{(1,1')} \\ \varepsilon_t^{(1,1')} \\ \varepsilon_i^{(1,1')} \\ \varepsilon_k^{(1,1')} \end{pmatrix} \quad \ldots\ldots \quad (17.2'')$$

*) See note on page 33.

**) The combination as given here is actually inadmissible and is only used as a convenient symbolism.

The last vector in (17.2″) pertains to the corrections for "tied" variates; in an S-transformation all vectors and covariance matrices are transformed. For the computation of (17.2″) the $\underset{1,1'}{\mathrm{S}}$-system has been maintained; the choice of the system is determined by practical considerations with respect to the total computing system.

Connection of systems (1) and (2)

Choose temporary the $\underset{r,s}{\mathrm{S}}$-system:

$$\underline{\Delta z}_r^{(1,1')(r,s)} = \underline{\Delta z}_s^{(1,1')(r,s)} = 0, \qquad \text{and schematically:}$$

$$\begin{bmatrix} \underline{\Delta z}_i^{(1,1')(r,s)} \\ \underline{\Delta z}_1^{(r,s)} \\ \underline{\Delta z}_{1'}^{(r,s)} \end{bmatrix} = (S_{(1,1')}^{(r,s)}) \cdot \begin{bmatrix} \underline{\Delta z}_i^{(1,1')} \\ \underline{\Delta z}_r^{(1,1')} \\ \underline{\Delta z}_s^{(1,1')} \end{bmatrix} \qquad (17.3)$$

The $\underline{\varepsilon}$-variates of the Δz-variates always transform according to the same formulae, this is here left out of consideration, as being a matter of the total computing system.

$$\underline{\Delta z}_r^{(2,2')(r,s)} = \underline{\Delta z}_s^{(2,2')(r,s)} = 0$$

$$\begin{bmatrix} \underline{\Delta z}_i^{(2,2')(r,s)} \\ \underline{\Delta z}_v^{(2,2')(r,s)} \\ \underline{\Delta z}_l^{(2,2')(r,s)} \\ \underline{\Delta z}_2^{(r,s)} \\ \underline{\Delta z}_{2'}^{(r,s)} \end{bmatrix} = (S_{(2,2')}^{(r,s)}) \cdot \begin{bmatrix} \underline{\Delta z}_i^{(2,2')} \\ \underline{\Delta z}_v^{(2,2')} \\ \underline{\Delta z}_l^{(2,2')} \\ \underline{\Delta z}_r^{(2,2')} \\ \underline{\Delta z}_s^{(2,2')} \end{bmatrix} \qquad (17.4)$$

Contribution to the condition model, in $\underset{r,s}{\mathrm{S}}$:

$$\boxed{\underline{\widetilde{\Delta z}}_i^{(1,1')(r,s)} = \underline{\widetilde{\Delta z}}_i^{(2,2')(r,s)}} \qquad (17.5')$$

Of course one can make a combination of the left-hand vectors in (17.3) and (17.4), and return to $\underset{1\,1'}{\mathrm{S}}$:

$$\begin{pmatrix} \underline{\Delta z}_i^{(2,2')(1,1')} \\ \underline{\Delta z}_v^{(2,2')(1,1')} \\ \underline{\Delta z}_l^{(2,2')(1,1')} \\ \underline{\Delta z}_2^{(1,1')} \\ \underline{\Delta z}_{2'}^{(1,1')} \\ \underline{\Delta z}_r^{(1,1')} \\ \underline{\Delta z}_s^{(1,1')} \end{pmatrix} = (S_{(r,s)}^{(1,1')}) \cdot \begin{pmatrix} \underline{\Delta z}_i^{(2,2')(r,s)} \\ \underline{\Delta z}_v^{(2,2')(r,s)} \\ \underline{\Delta z}_l^{(2,2')(r,s)} \\ \underline{\Delta z}_2^{(r,s)} \\ \underline{\Delta z}_{2'}^{(r,s)} \\ \underline{\Delta z}_1^{(r,s)} \\ \underline{\Delta z}_{1'}^{(r,s)} \end{pmatrix} \quad \ldots\ldots \quad (17.6)$$

Then the contribution to the condition model, in $\underset{1,1'}{S}$ is:

$$\boxed{\underline{\tilde{\Delta z}}_i^{(1,1')} = \underline{\tilde{\Delta z}}_i^{(2,2')(1,1')}} \quad \ldots\ldots \quad (17.5'')$$

N.B. With a view to the computation of covariances, transformations like (17.3), (17.4), (17.6) will perhaps have to be established in a more complete form. This also is a matter of the total computing system.

Connection of systems (2) and (3)

Choose temporary the $\underset{s,v}{S}$-system:

$$\underline{\Delta z}_s^{(2,2')(s,v)} = \underline{\Delta z}_v^{(2,2')(s,v)} = 0, \quad \text{and for indication:}$$

$$\begin{pmatrix} \vdots \\ \underline{\Delta z}_r^{(2,2')(s,v)} \end{pmatrix} = (S_{(s,r)}^{(s,v)}) \cdot \begin{pmatrix} \vdots \\ \underline{\Delta z}_v^{(2,2')(s,r)} \end{pmatrix} \quad \ldots\ldots \quad (17.7)$$

$$\underline{\Delta z}_s^{(3,3')(s,v)} = \underline{\Delta z}_v^{(3,3')(s,v)} = 0$$

$$\begin{pmatrix} \underline{\Delta z}_j^{(3,3')(s,v)} \\ \underline{\Delta z}_w^{(3,3')(s,v)} \\ \underline{\Delta z}_m^{(3,3')(s,v)} \\ \underline{\Delta z}_3^{(s,v)} \\ \underline{\Delta z}_{3'}^{(s,v)} \end{pmatrix} = (S_{(3,3')}^{(s,v)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(3,3')} \\ \underline{\Delta z}_w^{(3,3')} \\ \underline{\Delta z}_m^{(3,3')} \\ \underline{\Delta z}_s^{(3,3')} \\ \underline{\Delta z}_v^{(3,3')} \end{pmatrix} \quad \ldots\ldots \quad (17.8)$$

Contribution to the condition model, in $\underset{s,v}{S}$: $\boxed{\text{none}}$ . . . . (17.9)

Combination of (17.7) and (17.8) back to $\underset{s,r}{S} \equiv \underset{r,s}{S}$:

$$\begin{pmatrix} \underline{\Delta z}_j^{(3,3')(s,r)} \\ \underline{\Delta z}_w^{(3,3')(s,r)} \\ \underline{\Delta z}_m^{(3,3')(s,r)} \\ \underline{\Delta z}_3^{(s,r)} \\ \underline{\Delta z}_{3'}^{(s,r)} \\ \underline{\Delta z}_v^{(2,2')(s,r)} \end{pmatrix} = (S_{(s,v)}^{(s,r)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(3,3')(s,v)} \\ \underline{\Delta z}_w^{(3,3')(s,v)} \\ \underline{\Delta z}_m^{(3,3')(s,v)} \\ \underline{\Delta z}_3^{(s,v)} \\ \underline{\Delta z}_{3'}^{(s,v)} \\ \underline{\Delta z}_r^{(2,2')(s,v)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (17.10)$$

A combination of (17.10) with (17.3) gives the possibility of a return to $\underset{1,1'}{S}$ -analogous to (17.6):

$$\begin{pmatrix} \underline{\Delta z}_j^{(3,3')(1,1')} \\ \vdots \\ \underline{\Delta z}_{3'}^{(1,1')} \\ \underline{\Delta z}_r^{(1,1')} \\ \underline{\Delta z}_s^{(1,1')} \end{pmatrix} = (S_{(r,s)}^{(1,1')}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(3,3')(r,s)} \\ \vdots \\ \underline{\Delta z}_{3'}^{(r,s)} \\ \underline{\Delta z}_1^{(r,s)} \\ \underline{\Delta z}_{1'}^{(r,s)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (17.11)$$

In the left-hand vector of (17.11) the indication of the S-transformations made is so far restricted that the original observation variates are still recognizable. This will be continued, although sometimes this simplification of notation may lead to misunderstandings. Compare (17.23) and (17.26) and see the accompanying remark.

Connection of systems (3) and (4)

Choose temporary the $\underset{v,w}{S}$ -system:

$$\underline{\Delta z}_s^{(3,3')(s,w)} = \underline{\Delta z}_w^{(3,3')(s,w)} = 0$$

$$\begin{pmatrix} \underline{\Delta z}_j^{(3,3')(s,w)} \\ \underline{\Delta z}_v^{(3,3')(s,w)} \end{pmatrix} = (S_{(s,v)}^{(s,w)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(3,3')(s,v)} \\ \underline{\Delta z}_w^{(3,3')(s,v)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (17.12)$$

$$\underline{\Delta z}_s^{(4,4')(s,w)} = \underline{\Delta z}_w^{(4,4')(s,w)} = 0$$

$$\begin{pmatrix} \underline{\Delta z}_j^{(4,4')(s,w)} \\ \underline{\Delta z}_t^{(4,4')(s,w)} \\ \underline{\Delta z}_n^{(4,4')(s,w)} \\ \underline{\Delta z}_4^{(s,w)} \\ \underline{\Delta z}_{4'}^{(s,w)} \end{pmatrix} = (S_{(4,4')}^{(s,w)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(4,4')} \\ \underline{\Delta z}_t^{(4,4')} \\ \underline{\Delta z}_n^{(4,4')} \\ \underline{\Delta z}_s^{(4,4')} \\ \underline{\Delta z}_w^{(4,4')} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (17.13)$$

Contribution to the condition model, in $\underset{s,w}{S}$:

$$\boxed{\widetilde{\Delta z}_j^{(3,3')(s,w)} = \widetilde{\Delta z}_j^{(4,4')(s,w)}} \qquad (17.14')$$

Combination of (17.12) and (17.13) back to $\underset{s,v}{S}$:

$$\begin{pmatrix} \underline{\Delta z}_j^{(4,4')(s,v)} \\ \underline{\Delta z}_t^{(4,4')(s,v)} \\ \vdots \\ \underline{\Delta z}_{4'}^{(s,v)} \\ \underline{\Delta z}_w^{(3,3')(s,v)} \end{pmatrix} = (S_{(s,w)}^{(s,v)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(4,4')(s,w)} \\ \underline{\Delta z}_t^{(4,4')(s,w)} \\ \vdots \\ \underline{\Delta z}_{4'}^{(s,w)} \\ \underline{\Delta z}_v^{(3,3')(s,w)} \end{pmatrix} \qquad (17.15)$$

Combination of (17.15) with (17.7) back to $\underset{s,r}{S}$:

$$\begin{pmatrix} \underline{\Delta z}_j^{(4,4')(s,r)} \\ \underline{\Delta z}_t^{(4,4')(s,r)} \\ \vdots \\ \underline{\Delta z}_{4'}^{(s,r)} \\ \underline{\Delta z}_v^{(2,2')(s,r)} \end{pmatrix} = (S_{(s,v)}^{(s,r)}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(4,4')(s,v)} \\ \underline{\Delta z}_t^{(4,4')(s,v)} \\ \vdots \\ \underline{\Delta z}_{4'}^{(s,v)} \\ \underline{\Delta z}_r^{(2,2')(s,v)} \end{pmatrix} \qquad (17.16)$$

Combination of (17.16) with (17.3) back to $\underset{1,1'}{S}$:

$$\begin{pmatrix} \underline{\Delta z}_j^{(4,4')(1,1')} \\ \underline{\Delta z}_t^{(4,4')(1,1')} \\ \vdots \\ \underline{\Delta z}_{4'}^{(1,1')} \\ \underline{\Delta z}_r^{(1,1')} \\ \underline{\Delta z}_s^{(1,1')} \end{pmatrix} = (S_{(r,s)}^{(1,1')}) \cdot \begin{pmatrix} \underline{\Delta z}_j^{(4,4')(r,s)} \\ \underline{\Delta z}_t^{(4,4')(r,s)} \\ \vdots \\ \underline{\Delta z}_{4'}^{(r,s)} \\ \underline{\Delta z}_1^{(r,s)} \\ \underline{\Delta z}_{1'}^{(r,s)} \end{pmatrix} \qquad (17.17)$$

Then contribution to the condition model, in $\underset{1,1'}{S}$:

$$\boxed{\widetilde{\Delta z}_j^{(3,3')(1,1')} = \widetilde{\Delta z}_j^{(4,4')(1,1')}} \qquad (17.14'')$$

and also contribution to the condition model, in $\underset{1,1'}{S}$:

$$\boxed{\widetilde{\Delta z}_t^{(1,1')} = \widetilde{\Delta z}_t^{(4,4')(1,1')}} \quad \ldots\ldots\ldots\ldots \quad (17.18)$$

Connection of systems (1)–(4)

It is now possible to connect systems (1) and (4) in a first step. The condition model is formed by (17.5), (17.9), (17.14), (17.18). The adjustment can, for example, be executed entirely in the $\underset{1,1'}{S}$-system. The condition model is restricted and simple of construction. All variates $\Delta z^{(1,1')}$ that do not occur in the condition equations are again "free" variates, the computation of corrections may be rather laborious because of correlation; consideration should therefore be given to executing the computation of corrections in different S-systems. This is again a matter which depends on the total computing system.

In a second step, the connection of systems (1) – (4) with the system (a) can be executed. Of course it is not necessary to execute the computations in different steps.

Connection of systems (1)–(4) with system (a)

Choose, e.g., the temporary $\underset{k,m}{S}$-system. Then, with (2.8), compare also (3.12):

$$\underline{\Delta z}_k^{(a)(k,m)} = \underline{\Delta z}_m^{(a)(k,m)} = 0$$

$$\begin{pmatrix} \underline{\Delta z}_l^{(a)(k,m)} \\ \underline{\Delta z}_n^{(a)(k,m)} \\ \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_m^{(a)} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & -\dfrac{\overset{0}{z}_{ml}}{\overset{0}{z}_{mk}} & -\dfrac{\overset{0}{z}_{kl}}{\overset{0}{z}_{km}} \\ 0 & 1 & -\dfrac{\overset{0}{z}_{mn}}{\overset{0}{z}_{mk}} & -\dfrac{\overset{0}{z}_{kn}}{\overset{0}{z}_{km}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{type of matrix, } \overset{\text{denote}}{\text{by}} : (\bar{S}_{(a)}^{(k,m)})} \cdot \begin{pmatrix} \underline{\Delta z}_l^{(a)} \\ \underline{\Delta z}_n^{(a)} \\ \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_m^{(a)} \end{pmatrix} \quad \ldots\ldots \quad (17.19)$$

$$\underline{\Delta z}_k^{(1,1')(k,m)} = \underline{\Delta z}_m^{(3,3')(k,m)} = 0$$

$$\begin{pmatrix} \underline{\Delta z}_l^{(2,2')(k,m)} \\ \underline{\Delta z}_n^{(4,4')(k,m)} \\ \vdots \\ \underline{\Delta z}_1^{(k,m)} \\ \underline{\Delta z}_{1'}^{(km)} \end{pmatrix} = (S_{(1,1')}^{(k,m)}) \cdot \begin{pmatrix} \underline{\Delta z}_l^{(2,2')(1,1')} \\ \underline{\Delta z}_n^{(4,4')(1,1')} \\ \vdots \\ \underline{\Delta z}_k^{(1,1')} \\ \underline{\Delta z}_m^{(3,3')(1,1')} \end{pmatrix} \quad \ldots\ldots\ldots \quad (17.20)$$

Contribution to the condition model, in $\underset{k,m}{S}$:

$$\boxed{\begin{pmatrix} \widetilde{\Delta z}_l^{(a)(k,m)} \\ \widetilde{\Delta z}_n^{(a)(k,m)} \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta z}_l^{(2,2')(k,m)} \\ \widetilde{\Delta z}_n^{(4,4')(k,m)} \end{pmatrix}} \quad \ldots\ldots \quad (17.21)$$

Of course (17.21) can be established in $\underset{1,1'}{S}$, but from a computational point of view this will be less advisable. If one does not wish to execute the computation in steps, all variates transformed to $\underset{1,1'}{S}$ can be transformed to $\underset{k,m}{S}$, as already suggested in (17.20). This applies also to (17.5), (17.14) and (17.18), forming together with (17.21) the condition model.

From the $\underset{k,m}{S}$-system, (17.20) can be transformed to the (a)-system, by means of the inverse transformation of (17.19), provided it is analogously extended. The formation of the inverse transformation matrix only entails changing the sign of the non-diagonal elements; as a notation can be introduced:

$$(\bar{S}^{(a)}_{(k,m)}) = (\bar{S}^{(k,m)}_{(a)})^{-1} \quad \ldots\ldots \quad (17.22')$$

$$\begin{pmatrix} \underline{\Delta z}_l^{(2,2')(a)} \\ \underline{\Delta z}_n^{(4,4')(a)} \\ \vdots \\ \underline{\Delta z}_1^{(a)} \\ \underline{\Delta z}_{1'}^{(a)} \\ \underline{\Delta z}_k^{(1,1')(a)} = \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_m^{(3,3')(a)} = \underline{\Delta z}_m^{(a)} \end{pmatrix} = (\bar{S}^{(a)}_{(k,m)}) \cdot \begin{pmatrix} \underline{\Delta z}_l^{(2,2')(k,m)} \\ \underline{\Delta z}_n^{(4,4')(k,m)} \\ \vdots \\ \underline{\Delta z}_1^{(k,m)} \\ \underline{\Delta z}_{1'}^{(k,m)} \\ \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_m^{(a)} \end{pmatrix} \quad \ldots\ldots \quad (17.22'')$$

The total condition model then becomes, see (17.5), (17.14), (17.18), (17.21):

$$\boxed{\begin{array}{l|l} \widetilde{\Delta z}_i^{(1,1')(a)} = \widetilde{\Delta z}_i^{(2,2')(a)} & \\ \hline \widetilde{\Delta z}_t^{(1,1')(a)} = \widetilde{\Delta z}_t^{(4,4')(a)} & \widetilde{\Delta z}_j^{(4,4')(a)} = \widetilde{\Delta z}_j^{(3,3')(a)} \\ \hline \multicolumn{2}{c}{\widetilde{\Delta z}_l^{(a)} = \widetilde{\Delta z}_l^{(2,2')(a)}} \\ \multicolumn{2}{c}{\widetilde{\Delta z}_n^{(a)} = \widetilde{\Delta z}_n^{(4,4')(a)}} \end{array}} \quad \ldots\ldots \quad (17.23)$$

The notation used is not entirely unambiguous; actually, all S-transformations applied should have been indicated in the superscripts {in view of the computation of covariansces and numerical values Δz or, more general: in view of the definition of the variates Δz}.

The notation then becomes very cumbersome. For example, from (17.13), (17.15)-(17.17), (17.20) and (17.22) follows that we should write:

$$\underline{\Delta z}_n^{(4,4')(a)} \rightarrow \underline{\Delta z}_n^{(4,4')(s,w)(s,v)(s,r)(1,1')(k,m)(a)}$$

Two or more "given points" in the systems (1)–(4)

Extend (17.1) with:

(a)-system	$\underline{\Delta z}_{k'}^{(a)}$	$\underline{\Delta z}_{l'}^{(a)}$	$\underline{\Delta z}_{m'}^{(a)}$	$\underline{\Delta z}_{n'}^{(a)}$
(1)-system	$\underline{\Delta z}_{k'}^{(1,1')}$			
(2)-system		$\underline{\Delta z}_{l'}^{(2,2')}$		
(3)-system			$\underline{\Delta z}_{l'}^{(3,3')}$	
(4)-system				$\underline{\Delta z}_{n'}^{(4,4')}$

. . . . (17.24)

then the procedure becomes much simpler by transforming each of the systems (1) – (4) directly to the (a) system by means of $\underset{k,k'}{S}$, etc. For example:

$$\begin{pmatrix} \underline{\Delta z}_r^{(1,1')(k,k')(a)} \\ \vdots \\ \underline{\Delta z}_{1'}^{(k,k')(a)} \\ \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_{k'}^{(a)} \end{pmatrix} = (\bar{S}_{(k,k')}^{(a)}) \cdot \begin{pmatrix} \underline{\Delta z}_r^{(1,1')(k,k')} \\ \vdots \\ \underline{\Delta z}_{1'}^{(k,k')} \\ \underline{\Delta z}_k^{(a)} \\ \underline{\Delta z}_{k'}^{(a)} \end{pmatrix} \qquad (17.25)$$

Then the condition model can be established as follows:

$\tilde{\Delta z}_r^{(1,1')(a)} = \tilde{\Delta z}_r^{(2,2')(a)}$	$\tilde{\Delta z}_v^{(2,2')(a)} = \tilde{\Delta z}_v^{(3,3')(a)}$
$\tilde{\Delta z}_i^{(1,1')(a)} = \tilde{\Delta z}_i^{(2,2')(a)}$	
$\tilde{\Delta z}_s^{(1,1')(a)} = \tilde{\Delta z}_s^{(2,2')(a)}$	$\tilde{\Delta z}_s^{(2,2')(a)} = \tilde{\Delta z}_s^{(3,3')(a)}$
	$\tilde{\Delta z}_s^{(4,4')(a)} = \tilde{\Delta z}_s^{(3,3')(a)}$
	$\tilde{\Delta z}_j^{(4,4')(a)} = \tilde{\Delta z}_j^{(3,3')(a)}$
$\tilde{\Delta z}_t^{(1,1')(a)} = \tilde{\Delta z}_t^{(4,4')(a)}$	$\tilde{\Delta z}_w^{(4,4')(a)} = \tilde{\Delta z}_w^{(4,4')(a)}$

. . . . (17.26)

As for the notation, the remark made after (17.23) also applies here.

Of course the condition model (17.26) can also be replaced by extending (17.23), but the computation then follows a chain of S-transformations different from the one suggested in (17.25).

Given coordinates must not be corrected

This situation from practice is treated according to section 7. This means that given coordinates in the (a)-system are introduced as non-stochastic constants in the adjustment procedure. In the condition model (17.23) this means that the tilde-sign is omitted in $\widetilde{\Delta z}_k^{(a)}$, $\widetilde{\Delta z}_l^{(a)}$, $\widetilde{\Delta z}_m^{(a)}$ and $\widetilde{\Delta z}_n^{(a)}$. The adjustment procedure then furnishes again pseudo least-squares estimators. To the functions of observation variates thus obtained, the law of propagation of variances must be applied, using the *complete* covariance matrix {including the "given" coordinate variates}. If this is not done, then the pseudo least-squares method only furnishes pseudo covariances. In particular, the pseudo covariances of the given coordinate variates are zero!

Adjustment of photogrammetric networks by connecting "independent models"

The method for the connection of {stochastically} independent networks, outlined in the preceding part of this section, makes it possible to execute this adjustment problem entirely by the method of standard problem I {condition equations} because the unknown parameters of the different similarity transformations are eliminated by a consistent application of S-transformations.

It is possible to construct the covariance matrices for coordinate variates in each of these systems, because each time an S-system can be defined. As long as no pseudo least-squares adjustment is executed, the precision computation and testing is always possible, and the theory of the criterion matrix and of "reliability" can always be applied. This means that predefined specifications regarding precision and error control of networks can be complied with by planning. This is almost always possible for a pseudo least-squares adjustment as well, but this necessitates additional and more complicated computations.

Therefore the method sketched can be used to investigate closer the computing system of the so-called "aerotriangulation with independent models". It should be noted here that, as remarked in section 1, the two-dimensional treatment in this publication can in principle be generalized to the spatial three-dimensional case. The systems (1)−(4) in (17.1) then refer to the "independent models", the (a) system to the given coordinates of pass points.

It is interesting to compare the method with a very successful practical photogrammetric method, the Anblock-method, whose construction for the two-dimensional and for the three-dimensional situation is entirely identical, and in which the unknown parameters of the different similarity transformations are not eliminated in the {parametric} condition model. The Anblock-method is essentially based on the idea of C. M. A. VAN DEN HOUT[1] that this parametric condition model simply would be linear in the variates it contains. This misunderstanding led to a stochastic model of variances that was so drastically simplified that the necessity of sharply defining coordinate systems, also from the stochastic point of

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[1] See notes at the end of this section.

view, was overlooked. Starting from this, in fact too strongly simplified, computing model, which makes it very difficult to interpret the results of the adjustment, a computationally very elegant computer programme was developed by VAN DEN HOUT, in collaboration with the staff of the International Institute for Aerial Survey and Earth Sciences (I.T.C.) at Delft.[1,2] Prof. ACKERMANN and his staff in Stuttgart[3] then further developed and analysed the programme.

Although the Anblock-method resulted in a considerable improvement of the practical procedure for the adjustment of photogrammetric networks, it raises many theoretical questions. A clear definition of coordinate systems is sacrificed to the efficiency of computing technique, by which seemingly S-systems as a base for variance matrices are passed unnoticed. Perhaps this explains the absence of testing procedures. Given coordinates of pass points are introduced without variances, but the adjustment leaves "residuals" in these coordinates, and likewise in the coordinates of points connecting models. It is clear that a pseudo least-squares method of adjustment is applied, so that also the variances of coordinates computed after the adjustment have the character of pseudo variances, because the given coordinate variates still seem to have variance zero. This, also, makes the definition of S-systems seemingly superfluous. This procedure is, indeed, also in terrestrial networks more the rule than the exception.[5]

The unpleasant result of such a computing technique – simple to execute but hard to interpret – is that neither criteria for "precision" nor criteria for "reliability" of networks, as indicated in this publication and in [9], can be applied.

Sometimes an artificial covariance matrix[4] is used for interpolation, but without an S-system the interpretation is hardly possible.

The computation of many terrestrial networks at Delft has shown that checking by test methods is indispensable; in almost every network a number of gross errors was found.

Bearing in mind that the coordinates per photogrammetric model as initial data are only weakly checked, whereas the Anblock-method also provides relatively poor checking possibilities,[2] the question raised by Prof. HALLERT on the "reliability" of photogrammetric methods[6] gets an urgent significance. Of course one may wonder if HALLERT was right in restricting his statements to photogrammetry!

Notes to section 17

[1] C. M. A. VAN DEN HOUT – Een exacte procedure voor een numerische vereffening van een planimetrisch sektie-blok {An exact procedure for the numerical adjustment of a planimetric section-block} – I.T.C., Delft, 16 november 1962.
C. M. A. VAN DEN HOUT – Analytical Radial Triangulation and "Anblock" – Photogrammetria 19/1962–1964, pp. 445–447.
D. ECKHARDT – The I.T.C.-Catalogue of Block Adjustment – Photogrammetria 19/1962–1964, pp. 472–478.
F. ACKERMANN – Some Results of an Investigation into the Theoretical Precision of Planimetric Block-Adjustment – Photogrammetria 19/1962–1964, pp. 505–509.
C. M. A. VAN DEN HOUT – The Anblock Method of Planimetric Block Adjustment: Mathematical Foundation and Organization of its Practical Application – Photogrammetria 21/1966, pp. 171–178.
Further papers have not been mentioned, because the ideas developed are sufficiently clear from these five publications. If the condition model is differently arranged, but the strongly simplified stochastic model maintained, many variants of "Anblock" are possible. An elegant example is:
R. ROELOFS – Une méthode de compensation planimétrique de blocs par des équations de condition – Bulletin trimestriel de la Société belge de Photogrammétrie, no. 82, dec. 1965.

[2] F. Ackermann – On the Theoretical Accuracy of Planimetric Block Triangulation – Photogrammetria 21/1966, pp. 145–170.

[3] F. Ackermann, R. Bettin, H. Ebner, H. Klein, K. Kraus, W. Wagner – Aerotriangulation mit unabhängigen Modellen. Beiträge aus dem Institut für Photogrammetrie der Universität Stuttgart – Bildmessung und Luftbildwesen 38/1970, Heft 4, pp. 197–257. See also a paper by E. Stark in Allgemeine Vermessungsnachrichten 1970, pp. 318–328.
F. Ackermann, H. Ebner, K. Heiland, H. Klein, K. Kraus – Numerische Photogrammetrie. Erfahrungen mit neuen Rechenprogrammen – Nachrichten aus dem Karten- und Vermessungswesen. Reihe I. Heft Nr. 53. Frankfurt a/M., 1971.

[4] K. Kraus – Interpolation nach kleinsten Quadraten in der Photogrammetrie – Zeitschrift für Vermessungswesen 95/1970, pp. 387–390.

[5] E. Gotthardt – Genauigkeitsuntersuchungen an schematischen trigonometrischen Netzen – Festschrift zum 70. Geburtstag von Prof. Walter Grossmann – Konrad Wittwer, Stuttgart, 1967, pp. 123–131.

[6] B. Hallert – Is photogrammetry a giant on feet of clay? – Photogrammetria 25/1969–1970, pp. 147–148. Discussion: A. J. van der Weele, B. Hallert, in: Photogrammetria 25/1969–1970, pp. 149–150; E. H. Thompson, in: Photogrammetria 26/1970, p. 163.

18 APPLICATION OF THE THEORY*)

The evolution of ideas

Since 1968, when the present theory began to take definite shape, numerous examples have been computed. These examples included networks obtained in practice as well as artificial networks of abstract regular shape.

In the beginning only point- and relative standard ellipses of coordinates of network points resulting from the adjustment of observations, with and without fictitious observations, were compared with the circular standard ellipses computed from a criterium matrix with assumed parameter values (15.a.26). The figures on pages 25–28 are examples of this type of computations. Whether or not the first mentioned standard ellips enclosed the latter one appeared to be dependent however on the S-system chosen. Furthermore it showed that a comparison of the standard ellipses not always gave a clear indication of local differences in quality in a network.

About 1970, it appeared that both drawbacks could be eliminated after J. C. P. DE KRUIF succeeded in developing efficient computer programmes for the computation of eigenvalues according to section 8. The eigenvectors computed at the same time, in addition to the above mentioned comparison of the standard ellipses in pairs, assisted in indicating weaker parts of the network. As the technique applied has not been rounded-off in a satisfactory way yet, a further elaboration will not be given in this section. Of great importance for the simplication of computational problems is the fact that a detailed study by W. BEEKMAN and J. C. P. DE KRUIF of all possible shapes and types of geodetic {two-dimensional} networks have shown that the parameter c_p from (15.a.26) can be restricted to c_1. In these investigations is, among other things, (8.23) applied as a criterium. For networks of lower or the lowest order in plane surveying it appears from investigations by J. VAN MIERLO that the introduction of the parameters Δd_i^2 {hence also c_0} from (15.a.26), in addition to c_1, is indeed essential. This means that the criterium matrix now applied by the Computing Centre of the Delft Geodetic Institute to many Dutch networks from practice resembles in broad lines the theory on which the HTW-1956 was based {cf. pp. 5–7, 24–29, 65, 93–94, 107–110}.

As with all new theories, *all* possible applications of such a theory are immediately tried out. Very often the results of the trials show such a confusing multitude of variations that the initial optimism about finding a logical principle to order these results, turns into pessimism. This was also the case with this theory. First it was attempted to describe variance matrices of coordinate variates of network points by a $H^{(rs)}$-matrix in accordance with (8.21). At the same time such a $H^{(rs)}$-matrix was introduced as a substitute for usually not known or poorly known covariance matrices for "given" coordinate variates, hence as a substitute {or computational, or pseudo} covariance matrix in the adjustment procedure according to section 7. And finally came the confrontation with the problem of adjusting

*) Written just before finalizing the present publication.

to one another the values of the parameters c_1 and Δd_i^2 {hence also c_0} by considering networks from the highest to the lowest order.

First it became clear that for networks of higher order the influence of the parameter c_1 is domineering. Only for networks of the lowest order {smallest average distance between network points} the influence of the parameters Δd_i^2 becomes of paramount importance. That implied that first the factors that could possibly explain the variations in computed c_1-values should be traced and, if possible, give a description of these factors.

Like in the speculations on "reliability of networks" in [9], it appeared that considering only independent or "free" networks could supply vital information. Hence this implied for the present that in a first analysis situations as described in sections 5 and 7 were ruled out.

Some striking results of test computations induced the author to ask J. C. P. DE KRUIF to have computed a number of schematized network forms. Computations and working up of the results were carried out in the years 1970–1972. At the same time networks from practice were computed in a more directional form. Of paramount importance was finally an analysing computation of the primary network of The Netherlands by DE KRUIF.

The results of the latter computations confirmed the supposition of the author that computing eigenvalues of submatrices from coordinate covariance matrices might give a considerable contribution to the solution of the c_1-problem. This supposition was inferred from ideas of the HTW-1956 and from the theory developed in [9].

The problem of eigenvalue computations is after all a difficult to describe increase of c_1 when the shape of the network under consideration deviates more and more from all-sided symmetry. In this procedure a value for c_1 equal to λ_{max} from (8.21) was computed, with a start value of $c_1 = 1$ cm^2/km in $H^{(rs)}$. A good explanation can be given for this behaviour of c_1 because the criterion matrix can be considered as being linked to a network of unlimited size, with "measuring" of the natural logarithm of the distance ratios and of the angles, combined in the complex variate $\Pi = \ln v + i\alpha$.

Taking now submatrices – referring to coordinates of a subset of the set of all network points – from the total coordinate covariance matrix of the network under consideration, then the sketched application of (8.21) suddenly gives a picture of λ_{max} that is much less dependent on the shape of the group of points. Arranging the λ_{max} of submatrices in order of increasing values of the number of corresponding points, then in general an increase in accordance with the theoretical considerations is observed {see note on page 55}. Of course from the theory follows that arranging should be done in order of the rank of the subsets, hence in considerations on precision in a S-system arrangement should be carried out according to $(n-2)$ if n is the number of points corresponding to the submatrix.

Considering submatrices means considering local precision, and that is starting-point for connecting networks of lower order. This is a sound and practical principle. The same principle was also the basis for the HTW-1956.

However a more difficult problem is to establish a criterium for this λ_{max}. To this end it should be verified first if some functional relationship can be formed. In doing so attention should be paid to the possibility of attuning the reconnaissance of a network to such a criterium. This implies following the reconnaissance as a growing process of the network. Consequently it follows that one is dealing with a consecutive series of independent networks increasing in size, whereby earlier independent networks from this sequence as subsets of points may give rise to submatrices of covariances. Therefore it should be tried to express in *one* λ_{max} {or c_1}-picture independent networks as well as partial networks in the sense of subsets of points. This appears to be fairly possible if one restricts oneself to independent networks about round or square in shape, or to subsets of points from not too oblong independent networks. Other types of networks need some further analysis.

In this connection two effects are very important. Firstly, per independent network {inclusive of possible partial networks} should be introduced one and the same factor $(m-b)/m$ {m = number of observations, b = number of condition equations, referring to the independent network}. This factor is well-known from studies on the so-called "Strength of Figure". Secondly the relation to be given apparently refers to λ_{max} from submatrices for points forming a partial network at the border of the independent network. The more the partial network is situated nearer the centre of the independent network, the more the λ_{max} of the corresponding submatrix decreases in value {an analogous situation presents itself in the measures of internal reliability of a network in [9]}.

On theoretical grounds a third effect is to be expected, viz. the appearance of the factor:

$$l\sigma^2$$

with:

l = average side length of the network {l in km, if c_1 is expressed in cm^2/km},
σ^2 is a function of $\sigma_{\underline{r}}^2$ = variance of direction variates $\underline{r}$ and/or of $\sigma_{\underline{\ln s}}^2$ = variance of the natural logarithm of distance-measure variates $\underline{\ln s}$ {see note 1 added to this section}.
This factor can be looked upon as being the main cause for the relative small range of c_1-values for different types of networks {see also note 1 at the end of this section}.

In fact one opts in this case for networks of about equal side lengths. Test computations have already shown that deviations from this situation give a large and unpredictable increase of the λ_{max}-value, hence also in this case a description is very difficult, if possible at all. Therefore it seems that both the choice of the form and the build-up of the network has to follow classical rules. This procedure fits in very well with the classical pattern of the build-up of a network from large to small whereby every time a network of higher order gives a regular pattern of "loops" of small subsets of points of the network, inside which a lower order network gives the next step of densification with again "loops" inside which further densification takes place. In this procedure every time the method of section 7 is applied*) whereby of great practical importance is the small range of possible c_1-values.

A speculative line of thought, based on this concept but not verified yet, may provide at the same time the link to the present development of ever continuing linking-up or fitting-in of higher order networks. Consider to this end a network of higher order and suppose the coordinate covariance matrix of points of this network is acceptable in respect of a criterion matrix according to (8.22). Subsequently consider a subset of points in this higher order

*) However, the application of the computation method according to section 7 adds an extra difficulty. As the adjustment is being done with a pseudo covariance matrix, pseudo least squares estimators are obtained in which the variances of coordinates are in general {somewhat} larger than those resulting from a real least square adjustment. An increase in value of λ_{max} is then quite possible, as numerical computations have shown. How these increase of λ_{max} fits in with the earlier mentioned functional description of λ_{max}, is not quite clear yet. More examples must be computed before it can be decided which situations should be considered as normal and which ones as abnormal.

network – forming a "loop" in between which a densification network of lower order is fitted in – as given points in accordance with section 7 for this densification network. Ascribe as coordinate covariance matrix to these given points the corresponding submatrix from the just mentioned criterium matrix. Then there might be a real chance that the in this way imposed influence of a criterium matrix cancels for the greater part the disturbing influence of a possible unsymmetrical shape of the densification network on the computation of λ_{max} according to (8.21) {whereby of course the position of the given points in the densification network might be of significance}. Then per country only the national network of the highest order should satisfy the requirements regarding symmetry in shape. Often these requirements can not be met; an example of this case is e.g. the national network of The Netherlands. To cancel the shape-effect it can be tried to link-up national networks to greater entities, as e.g. the West-European network. If the requirements regarding shape are then still not fulfilled, consideration might finally be given to a world-wide network of points fixed by satellite measurements. In the latter case, however, at a certain stage the computations with complex numbers should be replaced by spatial computations with quaternions {for which is referred to the remarks in section 1}.

In this respect it should be borne in mind that the use of a criterium matrix as a compilation of criteria for precision is mainly suited for an overall objective of a network, whereby it is not possible to formulate a limited number of sharply defined requirements for e.g. technical constructions. In the latter case the precision of the network itself plays a small role compared to the requirements for precision in pegging out axes for tunnel building or pillar building for bridges. In such cases there is no objection if the network used is oblong in shape because the network in question will seldom be considered as a densification network.

Examples

In the following examples is λ_{max} the largest eigenvalue computed from {see section 8}:

$$|G^{(rs)} - \lambda \cdot H^{(rs)}| = 0$$

and $H^{(rs)}$ computed with the initial parameter value $c_1 = 1\ \mathrm{cm}^2/\mathrm{km}$.

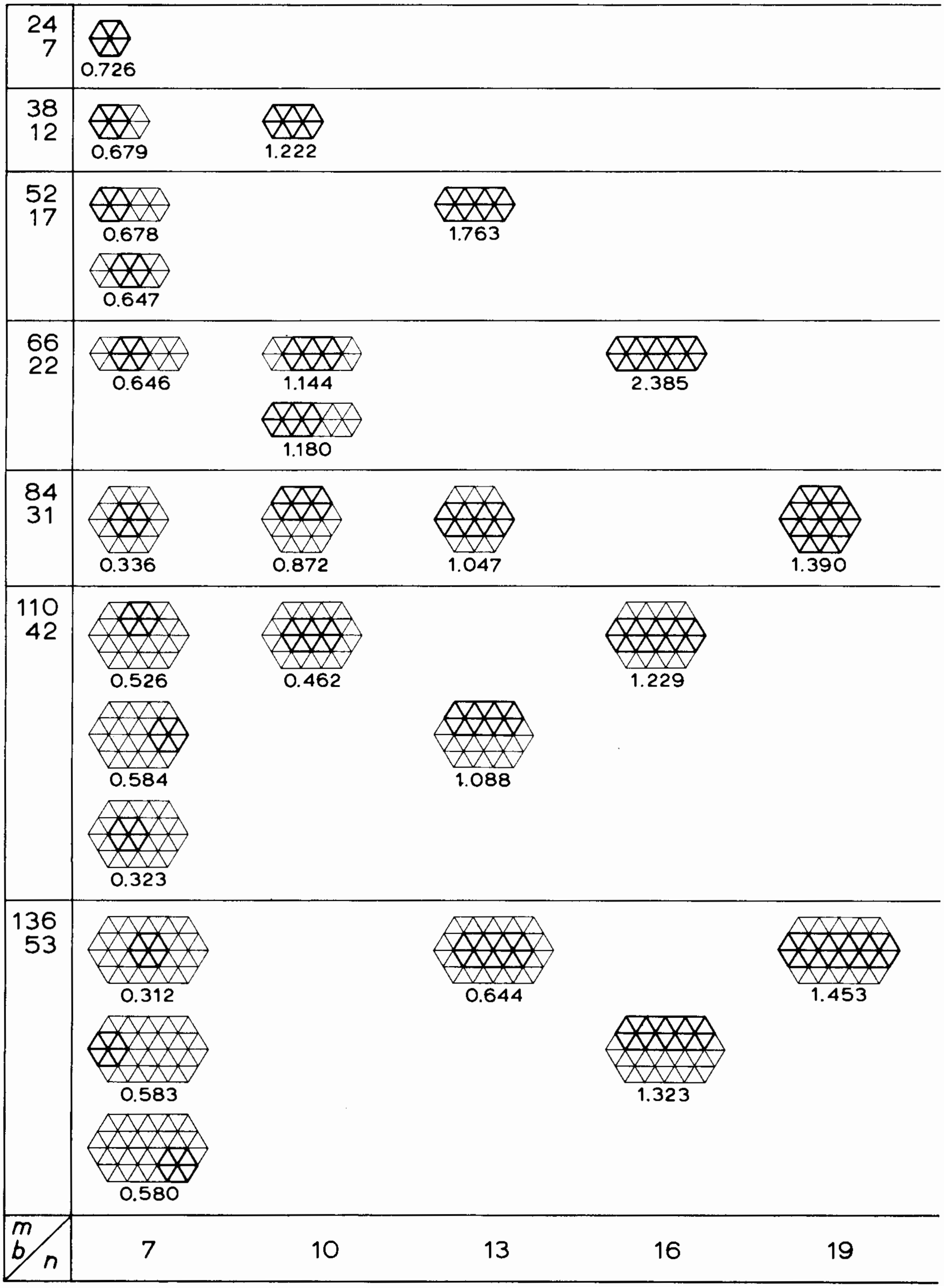
The measured elements are in the figures for the sake of brevity indicated by "direction" or "distance". The meaning in all cases is {see note 1 to section 18}:

direction: measurement of directions with local orientation;

distance: measurement of distance-measures with regional scale factor {quasi-distances}.

Points shown by ▲ in the figures are "given" points or controll points, the coordinates of which are determined in a network of higher order. The adjustment of the network on these "given" coordinates is in these examples not taken into consideration.

It is recommended to compare the examples given in this section with the examples and conclusions published in [12].

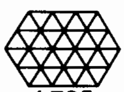

24
7
0.726
38
12
0.679
1.222
52
17
0.678
1.763
0.647
66
22
0.646
1.144
2.385
1.180
84
31
0.336
0.872
1.047
1.390
110
42
0.526
0.462
1.229
0.584
1.088
0.323
136
53
0.312
0.644
1.453
0.583
1.323
0.580
m
b
n
7
10
13
16
19

TRIANGULATION

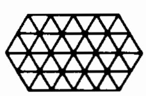
Observation variates: directions $\underline{r}$ (no correlation)
Standard deviation: $\sigma_r = 1$ dmgr $= 0.16.10^{-5}$ rad.
Side length: $l = 25$ km
Number of observations: m
Number of condition equations $= b$
Number of networkpoints $= n$

$\lambda_{\max}$
(subset of points)

$\lambda_{\max}$
(all points of network)



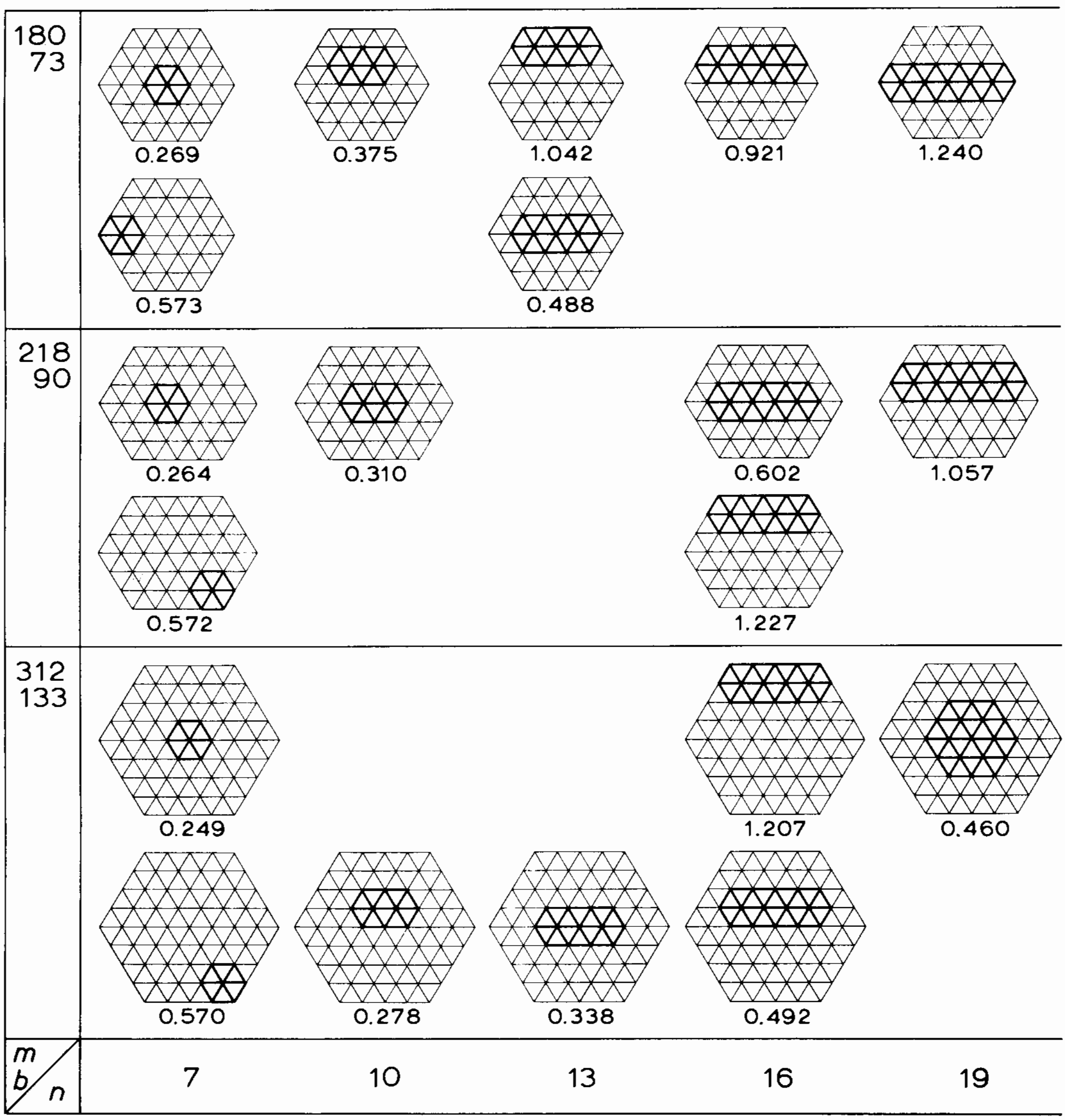
1.735



2.125

24	29	37	44	61

Fig. 18-1a


180
73
0.269
0.375
1.042
0.921
1.240
0.573
0.488
218
90
0.264
0.310
0.602
1.057
0.572
1.227
312
133
1.207
0.249
0.460
0.570
0.278
0.338
0.492
m
b
n
7
10
13
16
19

TRIANGULATION

Observation variates: directions $\underline{r}$ (no correlation)
Standard deviation: $\sigma_r = 1$ dmgr $= 0.16.10^{-5}$ rad.
Side length: $l = 25$ km
Number of observations: m
Number of condition equations $= b$
Number of networkpoints $= n$

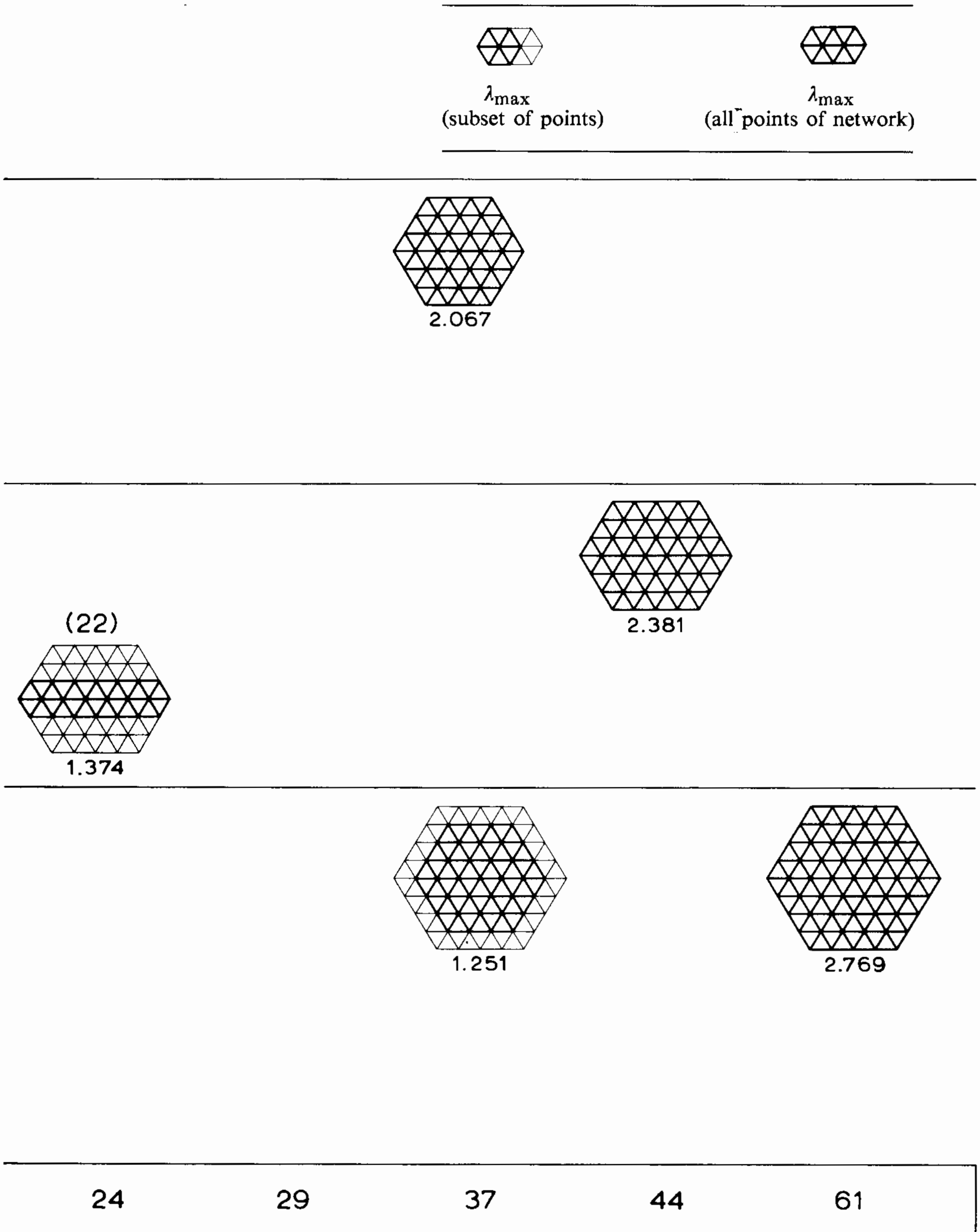
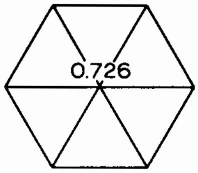
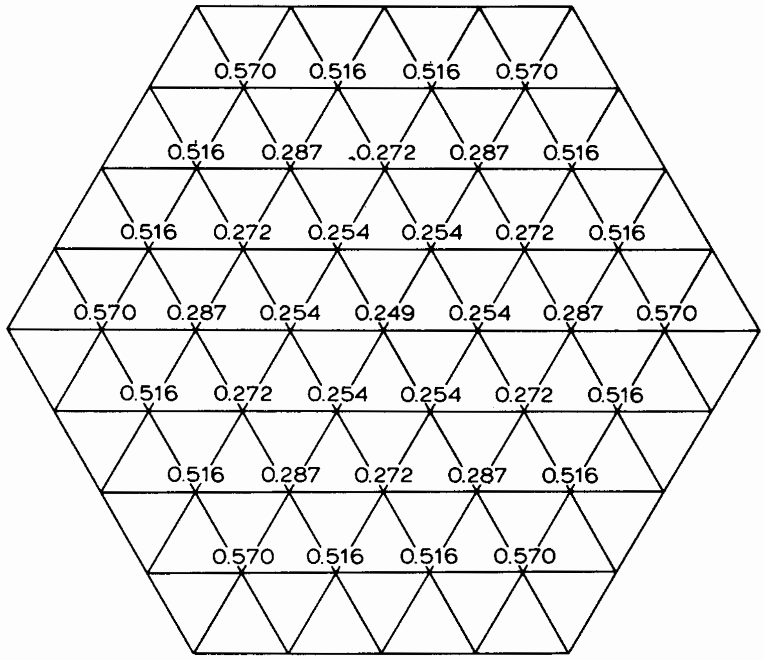


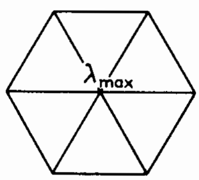
Fig. 18-1b



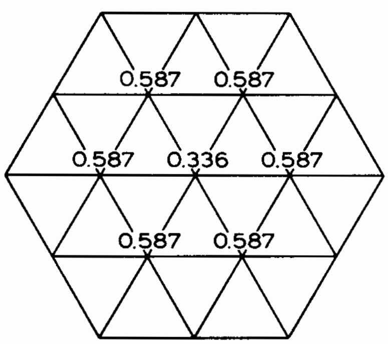


TRIANGULATION

EFFECT OF
POSITION OF
SUBSET WITHIN
NETWORK
Same data as in
fig. 18-1a, 1b



(one chosen type of subset of points)



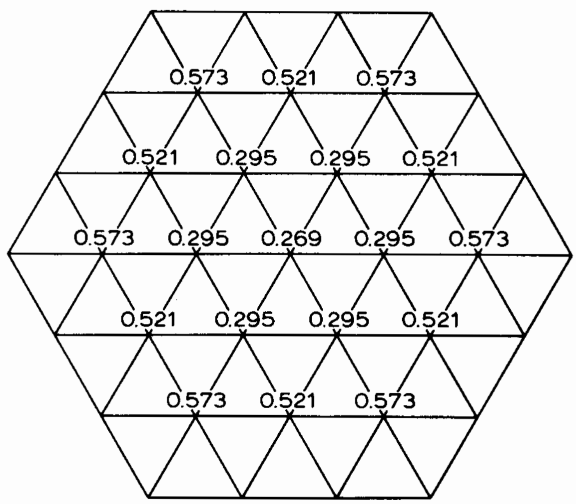


Fig. 18-2

NETHERLANDS TRIANGULATION NETWORK

Observation variates:
directions $\underset{\sim}{r}$ (no correlation)
Standard deviation: $\sigma_r = 1.1$ dmgr
Average side length: $l \approx 25$ km
Number of observations: $m = 449$
Number of condition equations:
$b = 191$
Number of network points: $n = 85$

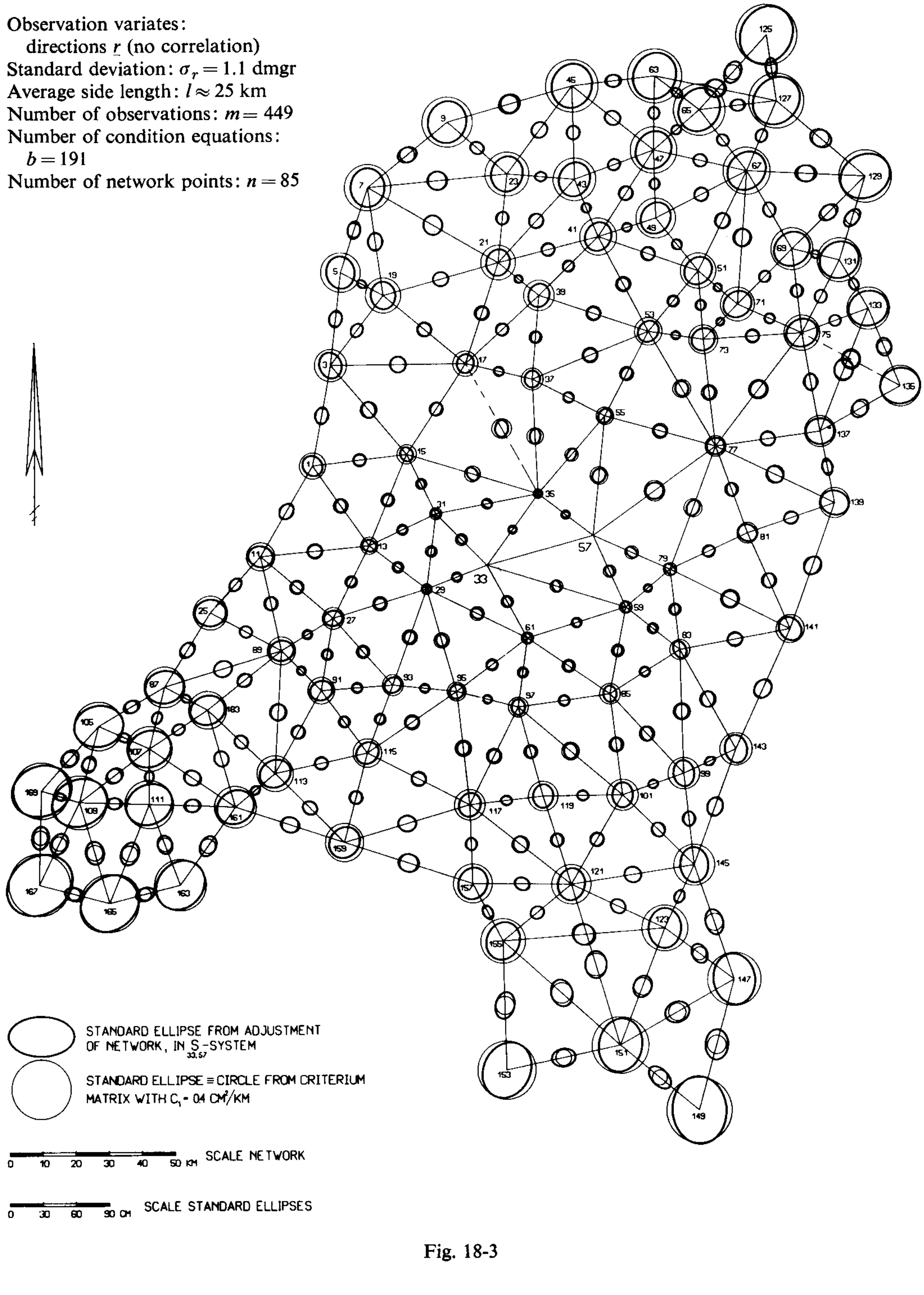
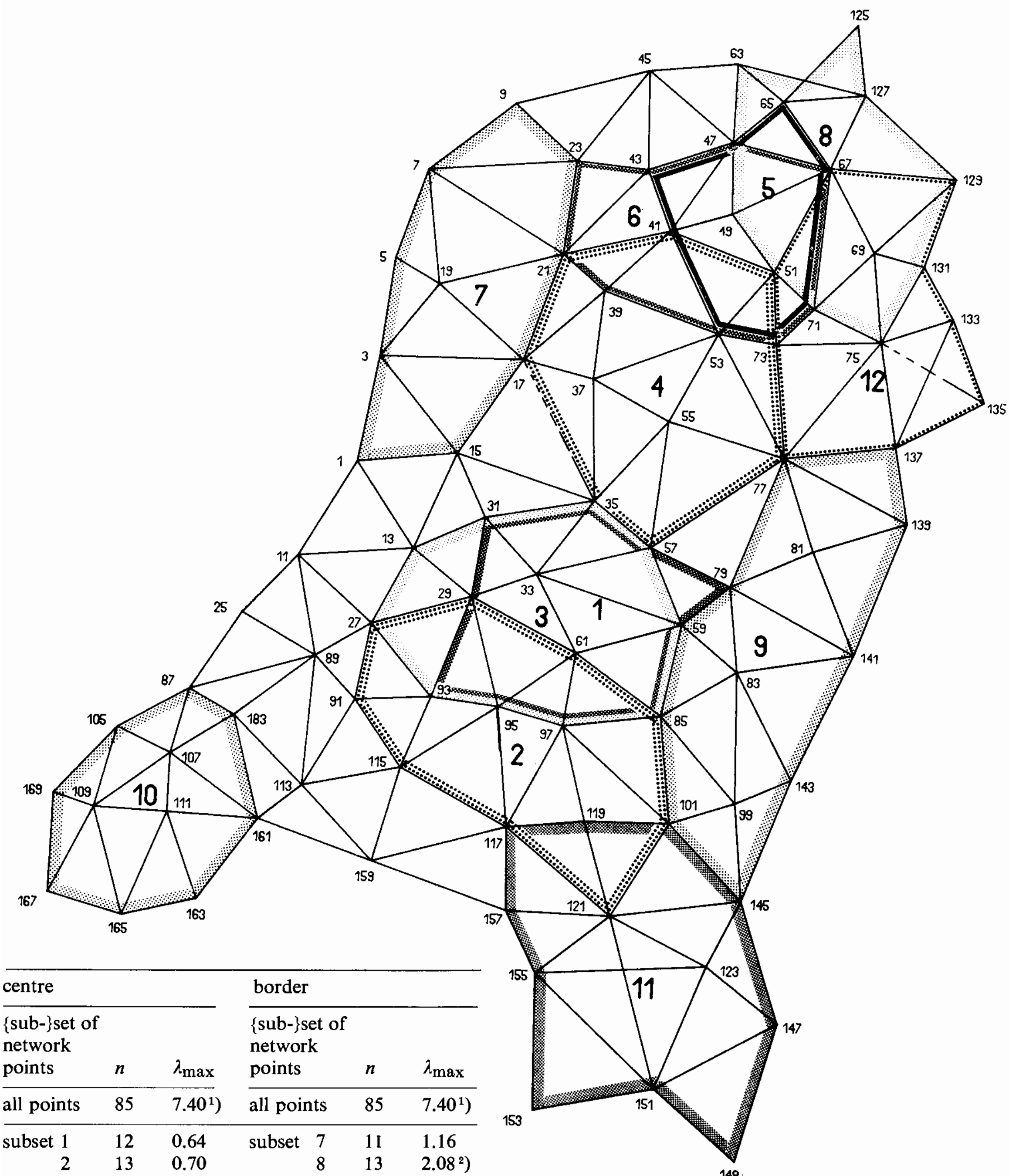


Fig. 18-3

NETHERLANDS TRIANGULATION NETWORK

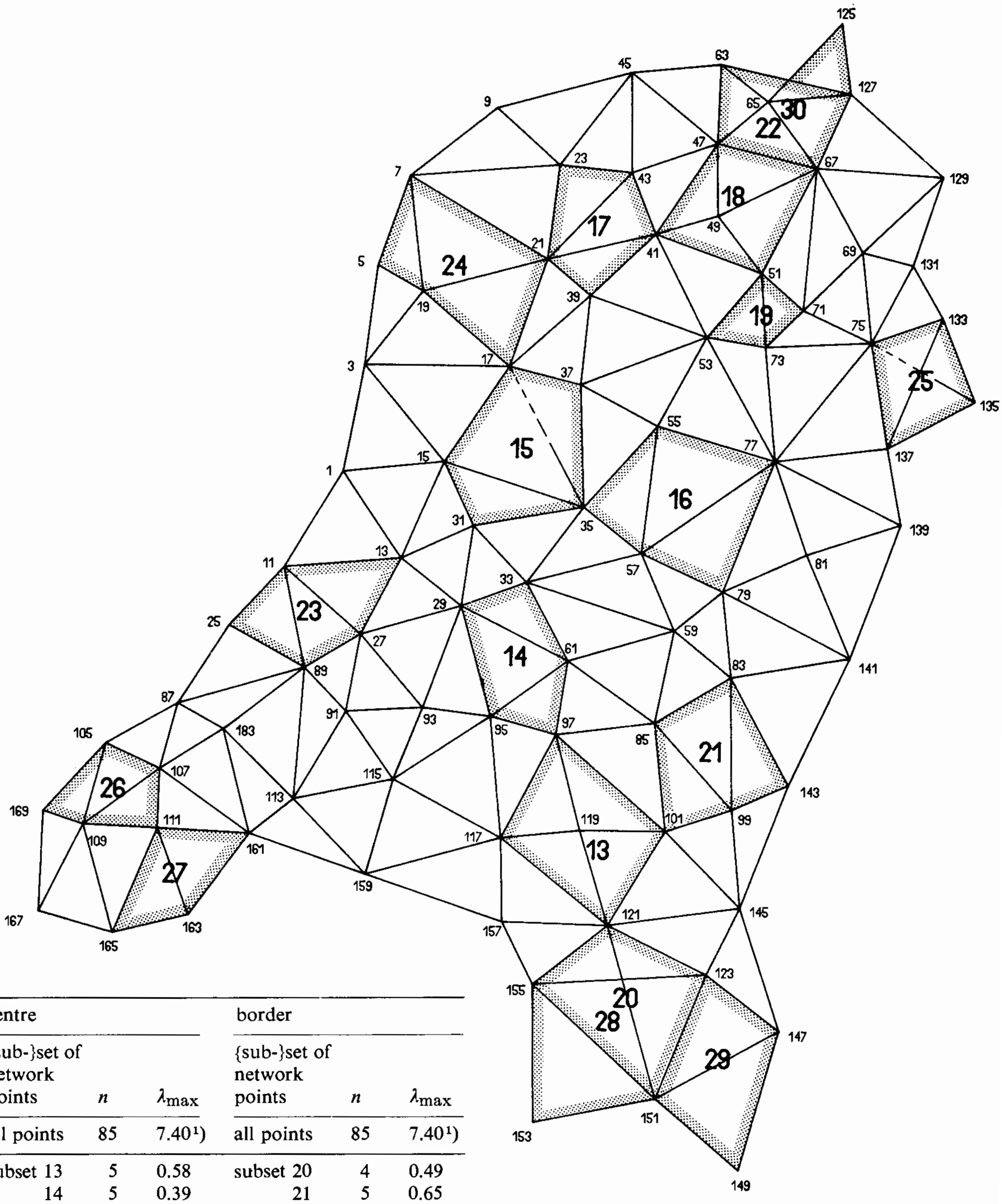


centre {sub-}set of network points	n	λ_{max}	border {sub-}set of network points	n	λ_{max}
all points	85	7.40[1])	all points	85	7.40[1])
subset 1	12	0.64	subset 7	11	1.16
2	13	0.70	8	13	2.08[2])
3	13	0.58	9	13	1.47
4	12	0.74	10	11	1.17
5	12	0.76	11	12	2.21[2])
6	13	0.67	12	12	1.94[3])

[1]) Non-symmetric shape of network
[2]) Loose border triangle
[3]) Loose triangle with one extra direction

Fig. 18-4a

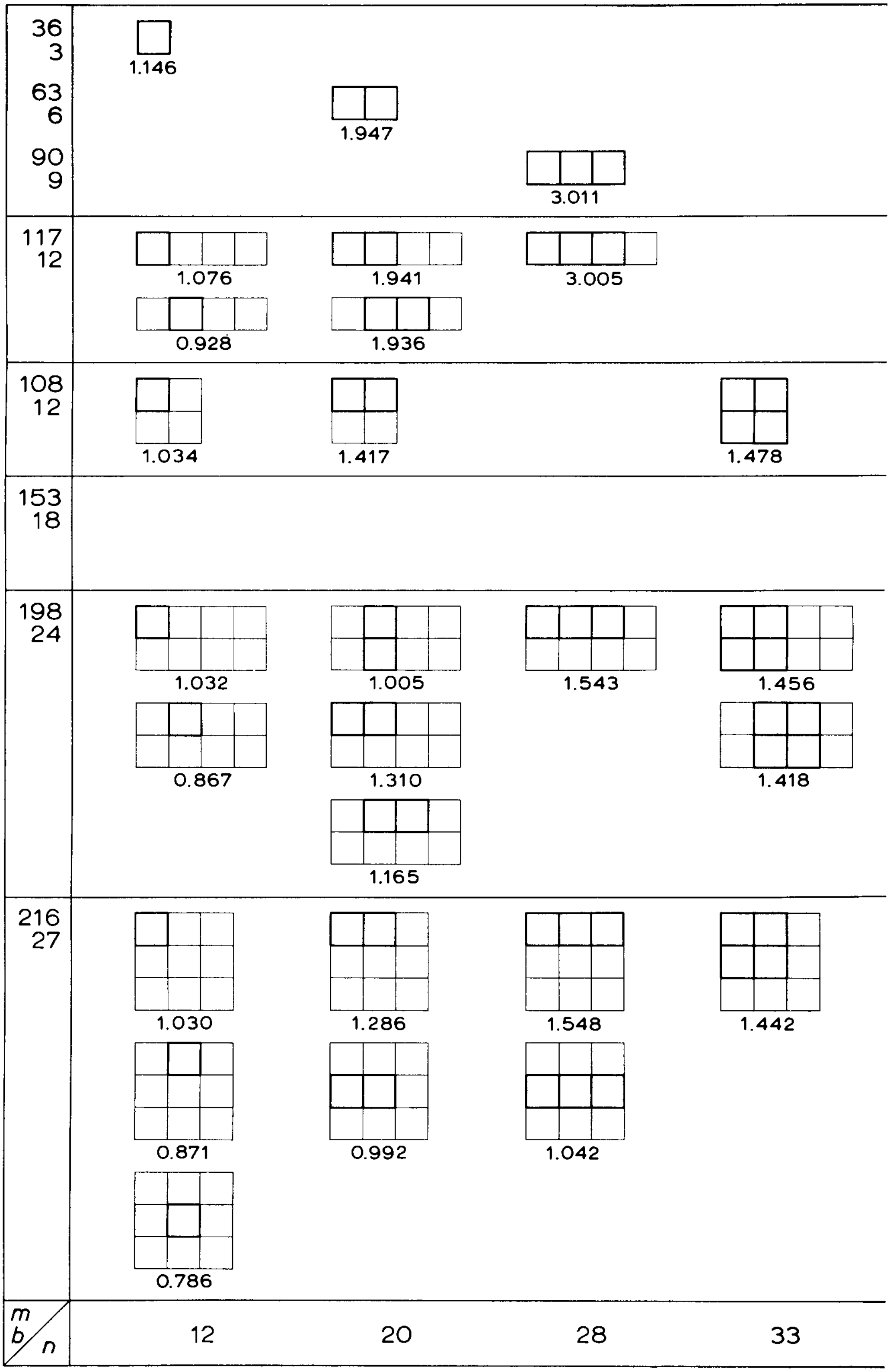
SUBSETS OF NETWORK POINTS



centre			border		
{sub-}set of network points	n	λ_{max}	{sub-}set of network points	n	λ_{max}
all points	85	7.40[1])	all points	85	7.40[1])
subset 13	5	0.58	subset 20	4	0.49
14	5	0.39	21	5	0.65
15	5	0.59	22	5	0.63
16	5	0.49	23	5	0.61
17	5	0.45	24	5	0.74
18	5	0.43	25	4	0.76[3])
19	4	0.33	26	5	0.46
			27	4	0.78
			28	5	0.80[2])
			29	4	1.36[2])
			30	6	1.70[2])

[1]) Non-symmetric shape of network
[2]) Loose border triangle
[3]) Loose triangle with one extra direction

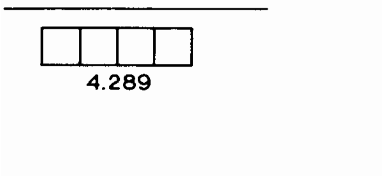
Fig. 18-4b


36
3
1.146
63
6
1.947
90
9
3.011
117
12
1.076
1.941
3.005
0.928
1.936
108
12
1.034
1.417
1.478
153
18
198
24
1.032
1.005
1.543
1.456
0.867
1.310
1.418
1.165
216
27
1.030
1.286
1.548
1.442
0.871
0.992
1.042
0.786
m
b
n
12
20
28
33

POLYGONS (Units of 3×3 sides)

Observation variates: directions $\underline{r}$, distances $\underline{s}$ (no correlation)
Standard deviations: $\sigma_r = 6.37$ dmgr $= 1.10^{-5}$ rad.
$\sigma_{\ln s} = 1.10^{-5}$ rad.
Side length: $l = 1$ km
Number of observations: m
Number of condition equations: b
Number of network points: n

λ_{max} (subset of points) | λ_{max} (all points of network)



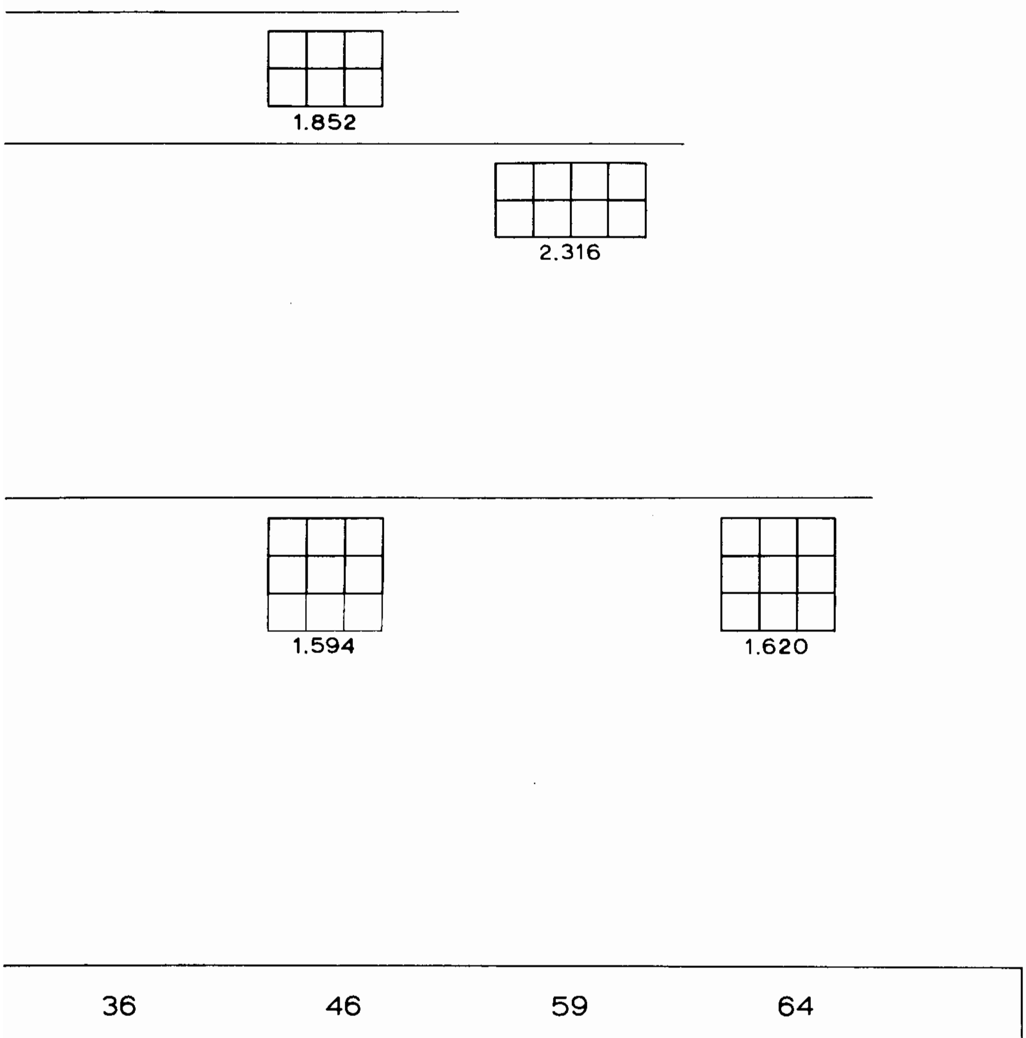


Fig. 18-5

POLYGONS (units of 1×1 sides)

EFFECT OF $(m-b)/m$

Same data as in Fig. 18-5, no subsets of network points considered

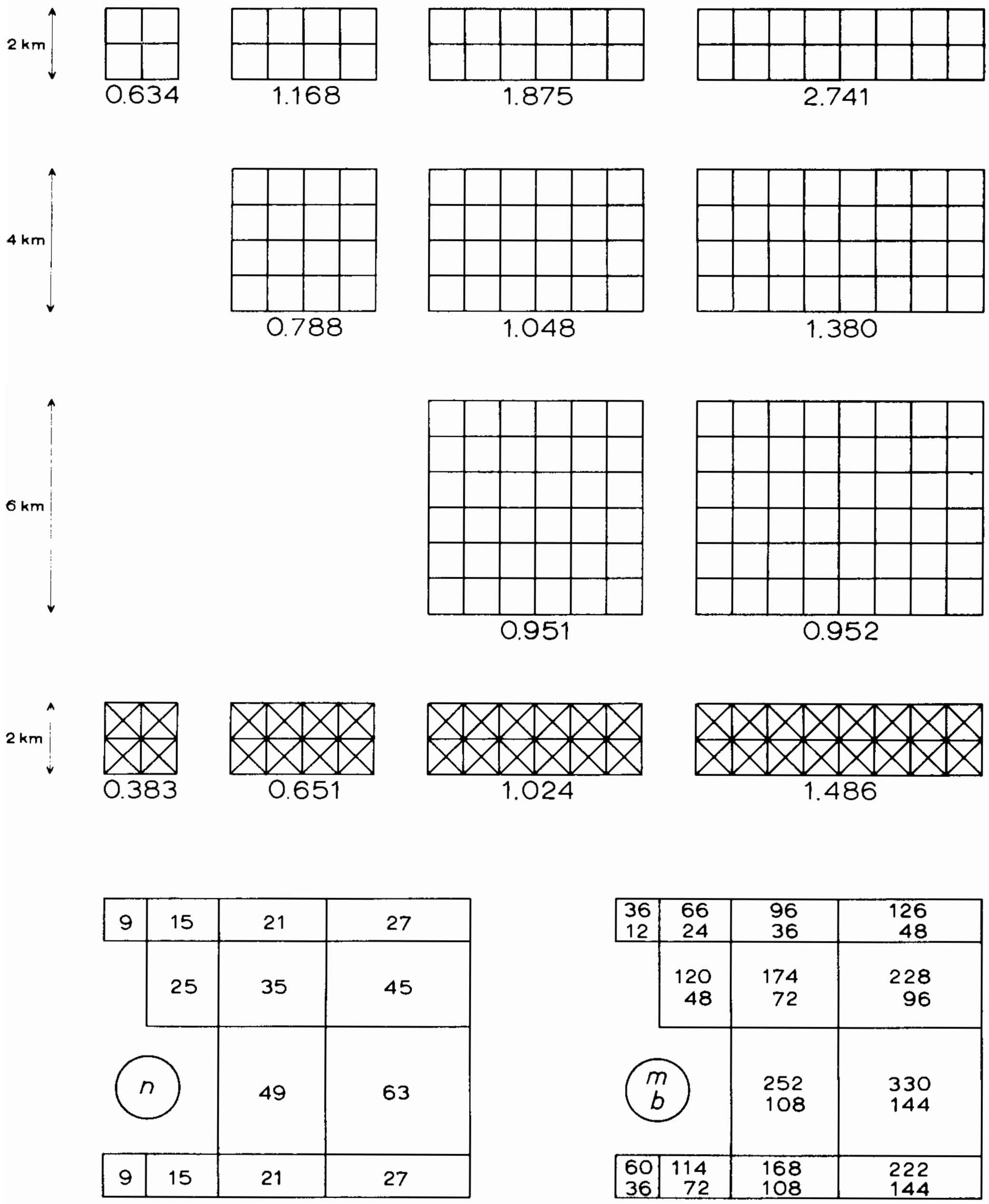


Fig. 18-6

TEST NET MEER EN BEEK

POLYGONS

Observation variates: directions $\underline{r}$, distances $\underline{s}$ (no correlation)

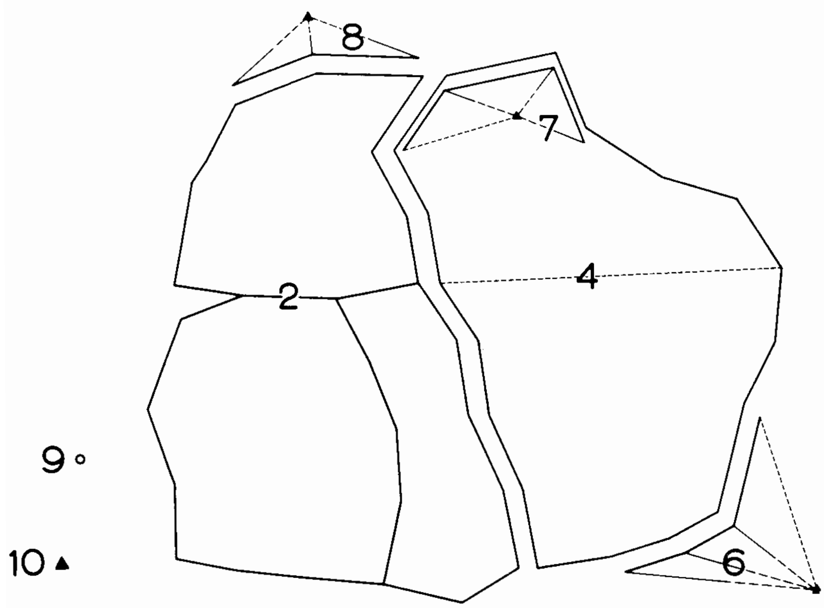
Standard deviations: $\sigma_r = 7$ dmgr
$\sigma_s = 1.5$ cm

Average side length: $l \approx 1$ km

Number of observations: m

Number of condition equations: b

Number of network points: n



{sub-}set of network points	combination of subsets	first version m / b	n	λ_{max}	fourth version m / b	n	λ_{max}
all points	2, 4, 6, 7, 8, 9, 10	128 / 15	40	24.98	146 / 24	43	4.24
subset 1	2, 4		36	4.64		38	3.88
2	2		26	4.62		26	3.33
3	2, 8, 9		28	4.77		28	3.80
4	4		18	2.61		20	2.69
5	4, 6, 7		20	23.84		22	2.75
6	6		6	16.45		6	2.03
7	7		5	2.27		5	2.30
8	8		4	2.60		4	1.73
subset 9	point 61						
10	point 81						

Fig. 18-7

TEST NET MEER EN BEEK

PRECISION

VERSION 1 FREE NETWORK

Ellipses (full drawn): standard ellipses from adjustment of network, in S-system$_{5,7}$

Circles (dashed): standard ellipses ≡ circles from criterium matrix with $c_1 = 2$ cm²/km

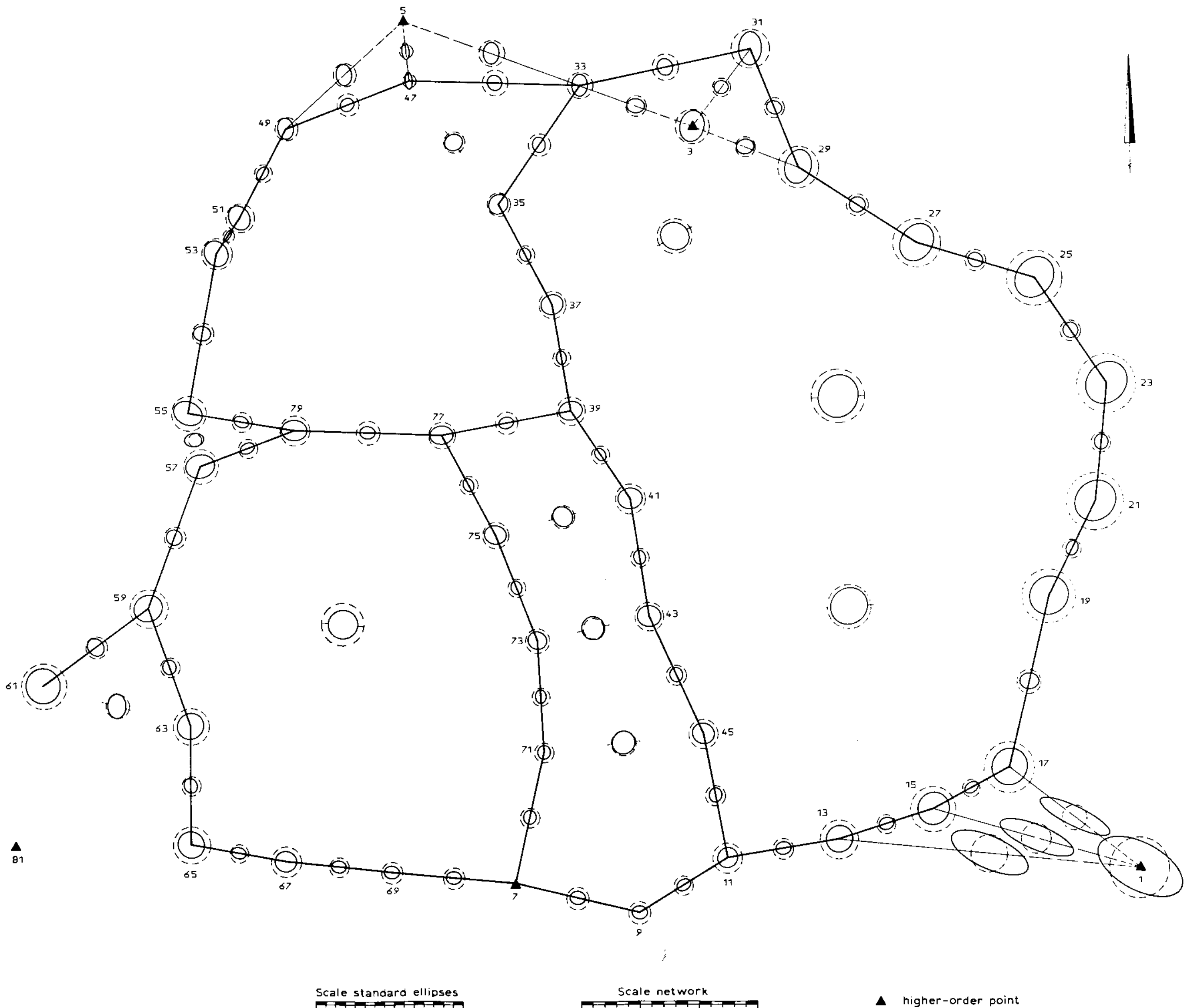


Fig. 18-8a

TEST NET MEER EN BEEK

PRECISION

VERSION 4 FREE NETWORK

Ellipses (full drawn): standard ellipses from adjustment of network, in S-system $_{5,7}$

Circles (dashed): standard ellipses ≡ circles from criterium matrix with $c_1 = 2\ \text{cm}^2/\text{km}$

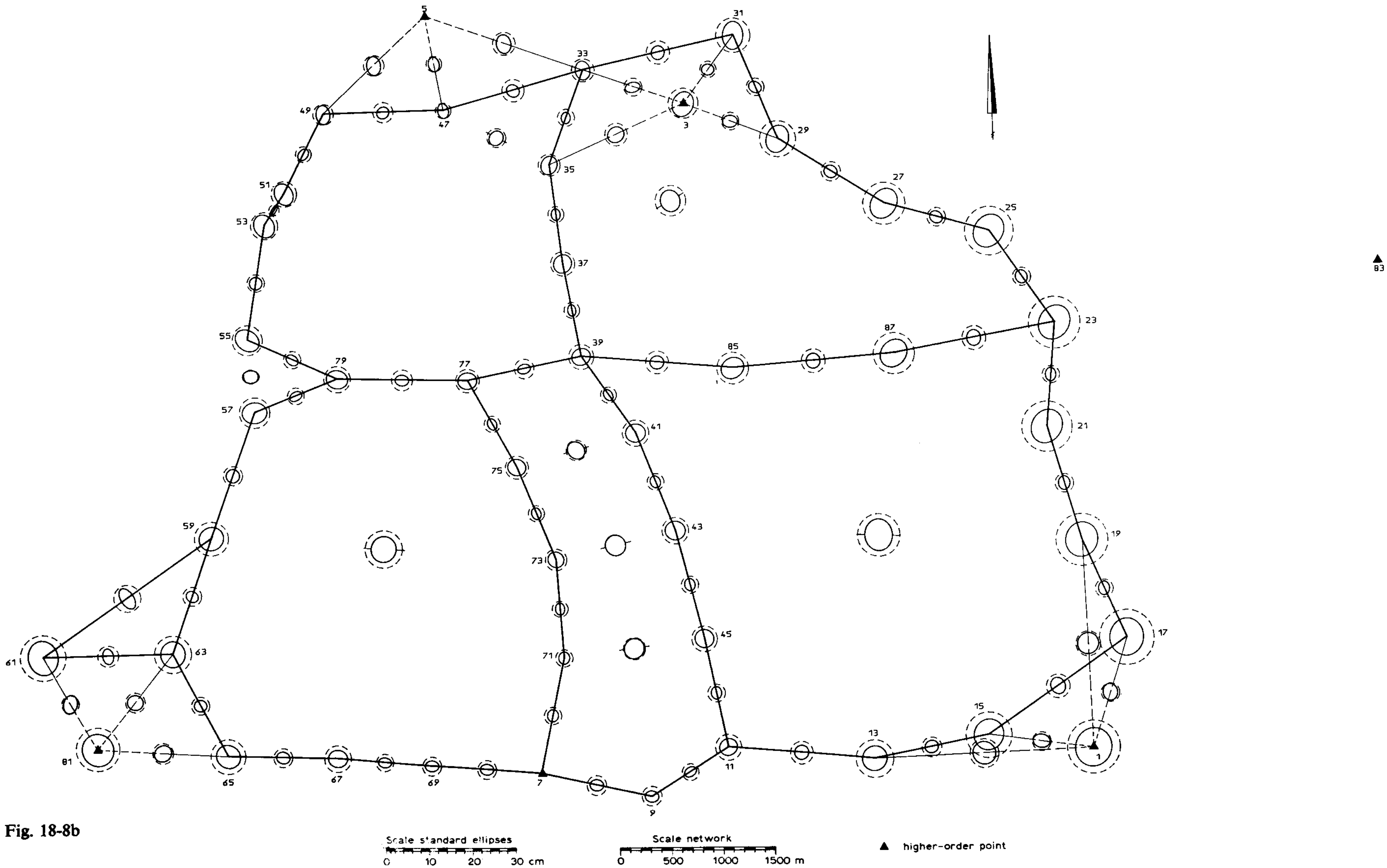


Fig. 18-8b

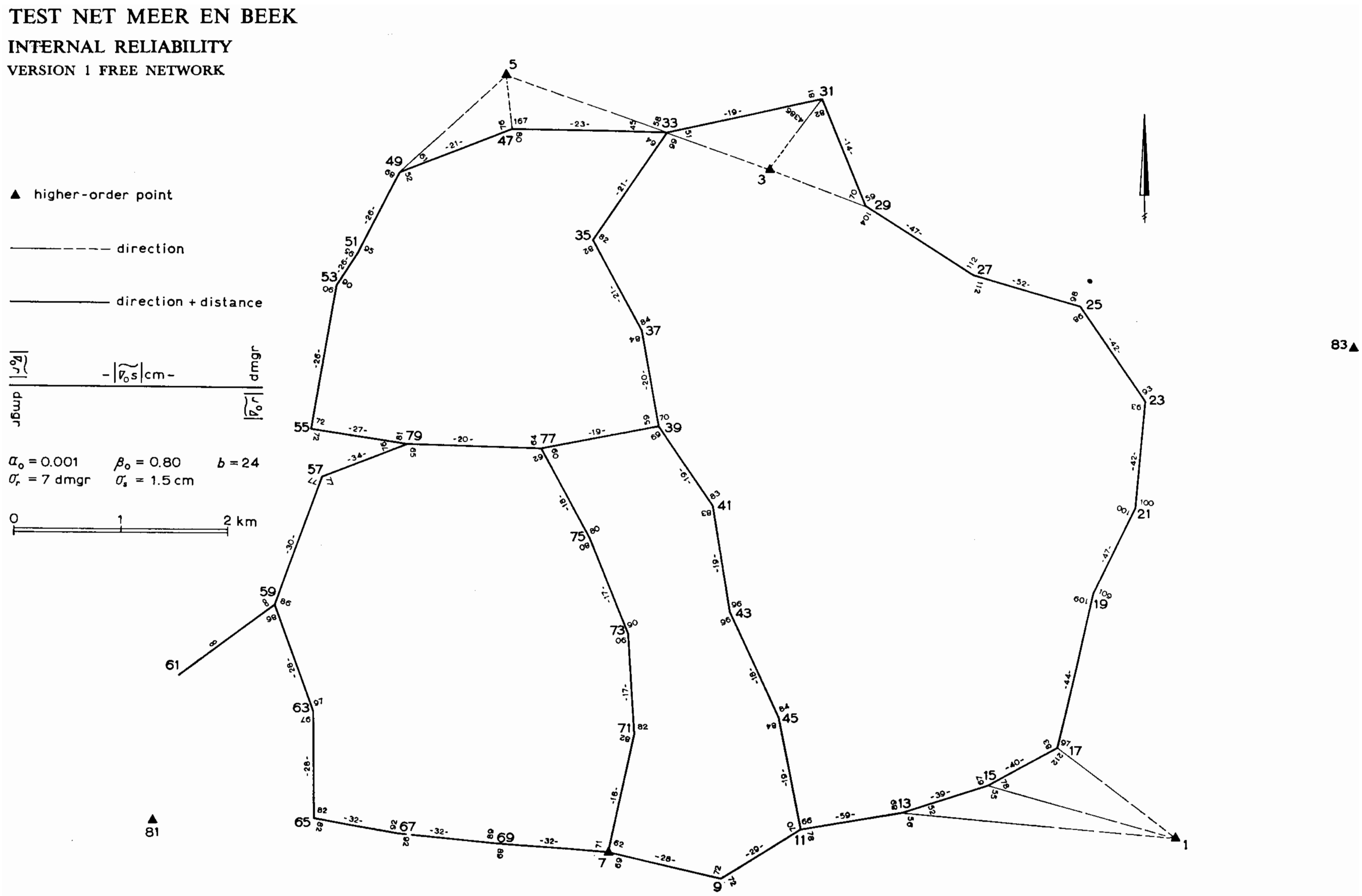

TEST NET MEER EN BEEK
INTERNAL RELIABILITY
VERSION 1 FREE NETWORK
▲ higher-order point
direction
direction + distance
$-|\widetilde{\nabla_0 s}|$ cm-
$|\widetilde{\nabla_0 r}|$ dmgr
$\alpha_0 = 0.001$
$\sigma_r = 7$ dmgr
$\beta_0 = 0.80$
$\sigma_s = 1.5$ cm
$b = 24$
0
1
2 km

Fig. 18-9a

TEST NET MEER EN BEEK

INTERNAL RELIABILITY

VERSION 4 FREE NETWORK

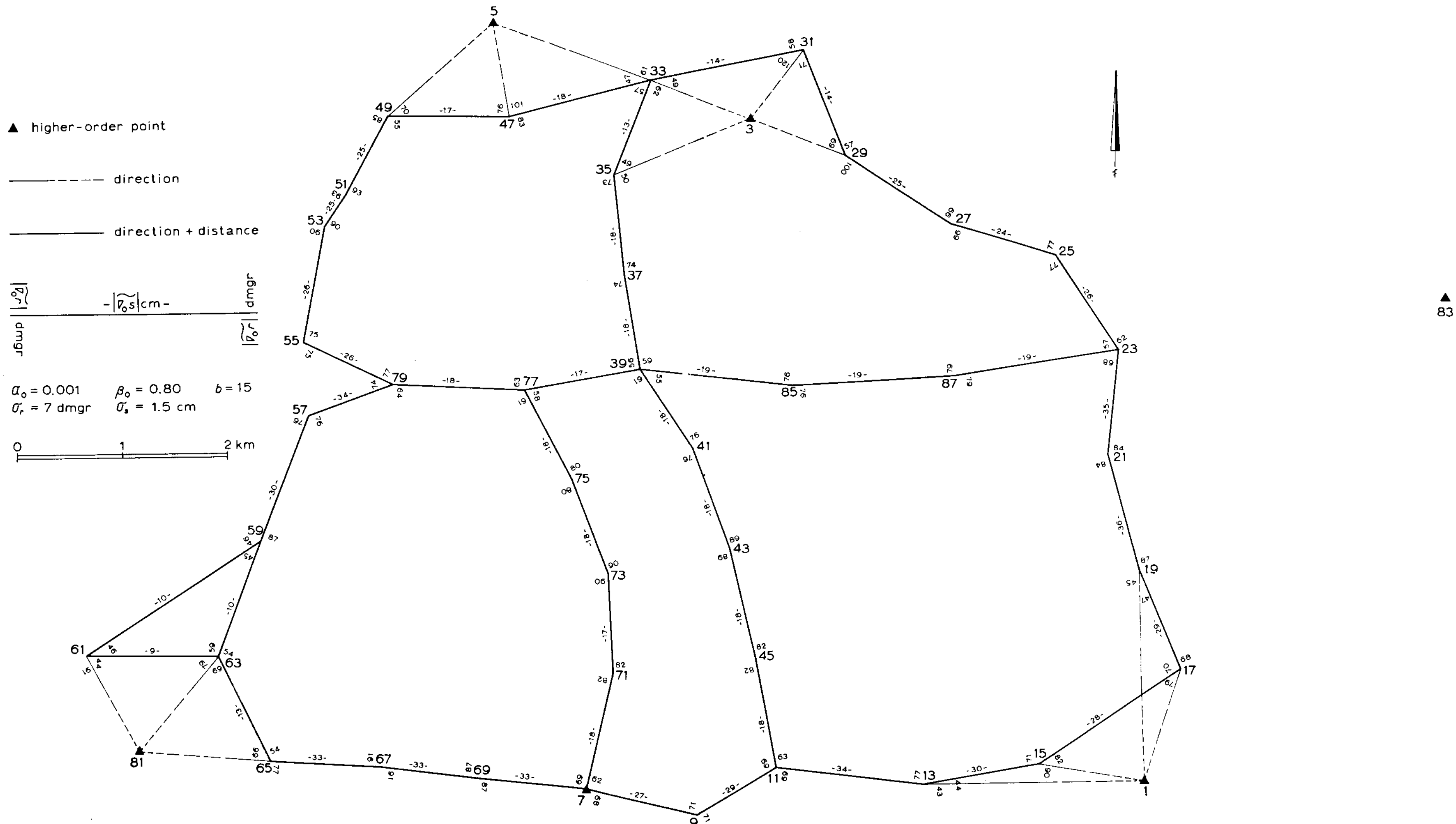


Fig. 18-9b

A. *Schematic triangulation networks*

Side length $l = 25$ km, $\sigma_r = 1$ dmgr.

Test computation set up for tracing relations from λ_{max} in national triangulation networks.[2]

Figs. 18-1a, 18-1b: λ_{max} corresponding to sets or subsets of network points; effects of shape, size and position {border or central position within the network of subsets of points}.

Fig. 18-2: Effect of position for one type of subset of points.

B. *Netherlands triangulation network*

Average side length $l \approx 25$ km, $\sigma_r = 1.1$ dmgr.

Computation and analysis by J. C. P. DE KRUIF.

Fig. 18-3: point- and relative standard ellipses following from adjustment of network on Hayford ellipsoid,[3] standard circles from criterion matrix with parameter value $c_1 = 0.4$ cm^2/km, both in S-system.

33,57

Figs. 18-4: λ_{max} corresponding to set or subsets of network points.[4]

C. *Schematic polygon networks*

Side length $l = 1$ km, $\sigma_r = 6.4$ dmgr, $\sigma_s = 1$ cm.

Test computation set up for tracing relations for λ_{max} in densification networks, like those being applied in The Netherlands for the determination of pass points in photogrammetry.

Fig. 18-5: λ_{max} corresponding to sets or subsets of network points; effect of shape, size and position. Networks built up from units of 3×3 sides.[5]

Fig. 18-6: λ_{max} corresponding to sets of network points; effects of shape and size, effects of increasing the number of condition equations. Networks built up from units of 1×1 sides.[6]

D. *Polygon network*

Average side length $l \approx 1$ km, $\sigma_r = 7$ dmgr, $\sigma_s = 1.5$ cm.

In 1970 E. F. MEERDINK and W. BEEKMAN devised for studying reliability and precision a test net that contained various situations from practice for network build-up. BEEKMAN contrived several versions of network build-up with the object of improving both reliability and precision and maintaining at the same time the practical and economical feasibility. In 1972 and 1973 this analysis was complemented by J. E. J. VAN ANGELEN, H. DE HEUS, J. C. P. DE KRUIF and J. VAN MIERLO.[7] In this example some data, borrowed from the first and the fourth versions of the test net, serve a further analysis within the framework of the present publication.

Fig. 18-7: λ_{max} corresponding to set or subsets of network points.

Fig. 18-8a: point- and relative standard ellipses following from adjustment of network, standard circles from criterium matrix with parameter value $c_1 = 2$ cm^2/km, both in S-system. First version of network.

5,7

---

[2] See notes 2–8 at the end of this section.

Fig. 18-8b: same as Fig. 18-8a. Fourth version of network.
Fig. 18-9a: in view of a certain interconnection of reliability and precision of networks {see pp. 12, 13} a summary is given of bounds concerning observation variates under conventional alternative hypotheses as measures for internal reliability of the network. First version of network.
Fig. 18-9b: same as Fig. 18-9a. Fourth version of network.
First conclusion: The fourth version of the network is considerably better in respect of precision as well as reliability.

Analysis of λ_{max} in free and border partial networks

Fig. 18-10 shows, for border networks, points on logarithmic coordinate paper corresponding to the value-pairs:

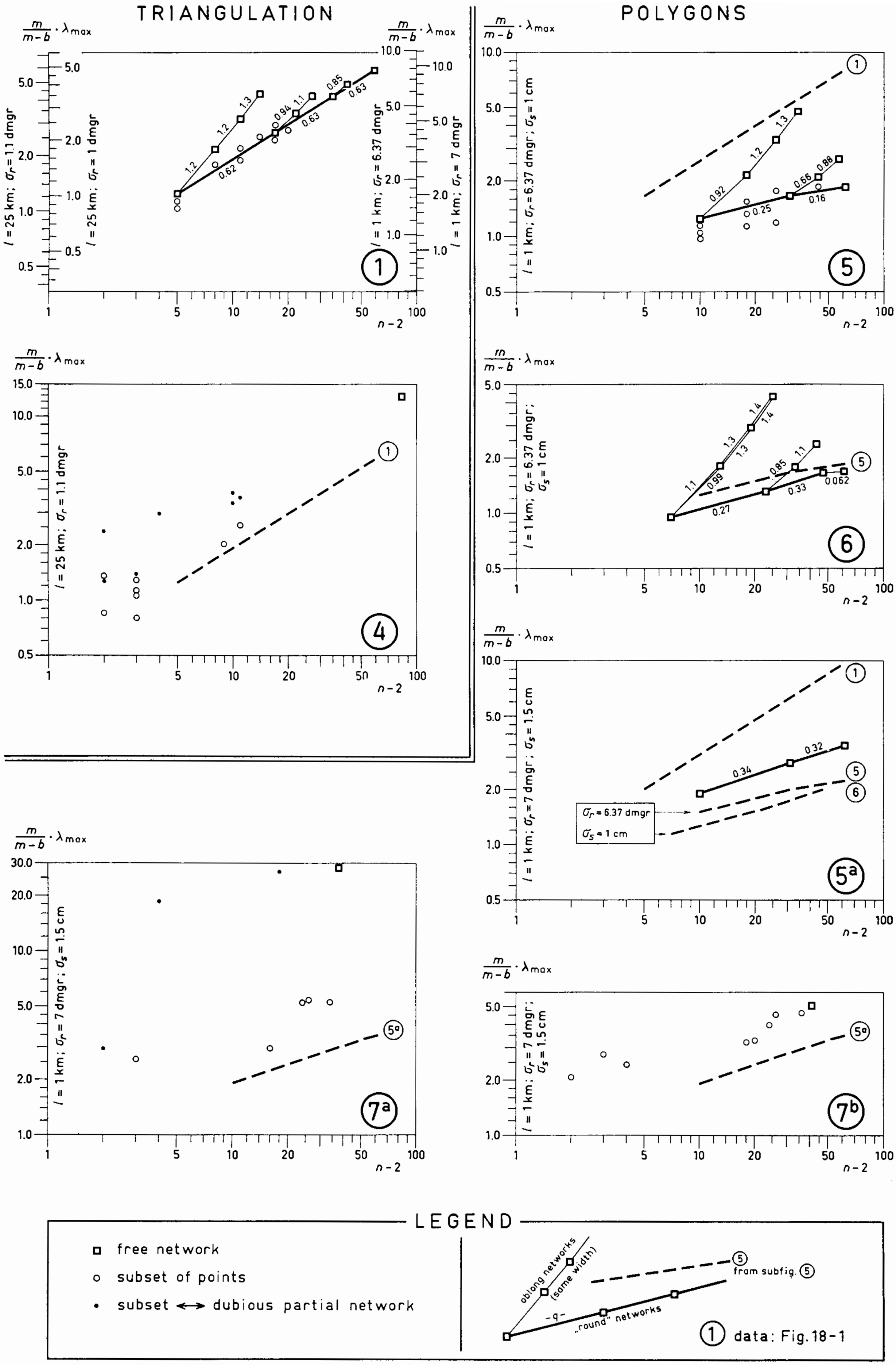
$$\left(\frac{m}{m-b}\cdot\lambda_{max}, \quad n-2\right)$$

Data from Fig. 18-1 are worked up into subfig. 1, in which 4 scales for

$$\frac{m}{m-b}\cdot\lambda_{max}$$

are indicated corresponding to $l\sigma^2\{\sigma^2=\sigma_r^2\}$.
Data from Fig. 18-4 are worked up into subfig. 4; in this and other subfigures are for comparison included data from other subfigures, indicated by a dashed line. Partial networks are shown as "dubious", in accordance with notes 2 and 3, Fig. 18-4.
Data from Fig. 18-5 are worked up into subfig. 5.
Data from Fig. 18-6 are worked up into subfig. 6.
If σ_r^2 and σ_s^2 do not change proportional, it is not possible to show directly the effect of the change of $l\sigma^2$. The computation of λ_{max} for the three "round" free networks of Fig. 18-5 is therefore once more executed for other values of σ_r and σ_s. The result is shown in subfig. 5a.
Data from Fig. 18-7 are worked up into subfig. 7a {version 1} and subfig. 7b {version 4}. Partial networks are shown as "dubious", in accordance with the comparison of standard ellipses {adjustment of network} and standard circles {criterium matrix with $c_1=2$ cm^2/km}, Fig. 18-8a.

For schematic networks {Fig. 18-10, sugfigures 1, 5, 6, 5a} it appears to be a tree-like structure in which the "trunk" connects points corresponding to "round" free networks and a "branch" connects points of oblong free networks of the same width; the juncture of branch and trunk is a "node". Points corresponding to not too oblong border partial networks fluctuate around the trunk; the deviations give an impression of the approximation of the description of the data resulting from the functional relationship suggested. These deviations give at the same time an indication whether the fluctuations of networks from



TRIANGULATION
POLYGONS
m/(m−b) · λmax
n − 2
l = 25 km; σr = 1.1 dmgr
l = 25 km; σr = 1 dmgr
l = 1 km; σr = 6.37 dmgr
l = 1 km; σr = 7 dmgr
l = 1 km; σr = 6.37 dmgr; σs = 1 cm
l = 1 km; σr = 7 dmgr; σs = 1.5 cm
σr = 6.37 dmgr
σs = 1 cm
1
4
5
5a
6
7a
7b
LEGEND
free network
subset of points
subset ⟷ dubious partial network
oblong networks (same width)
"round" networks
from subfig. 5
1 data: Fig. 18-1

Fig. 18-10

practice can possibly be accepted {see subfigures 4, 7a and 7b} apart from the fact that it can clearly be demonstrated that the trunk line moves slightly parallel in the direction of increasing λ_{max} on account of the always somewhat irregular shape of networks from practice. In subfigures 4, 7a and 7b a clear indication of dubious border partial networks is evident; this opens up the possibility to apply the way of depiction {mapping} shown in Fig. 18-10 to the reconnaissance of networks.

With a view to the reconnaissance of networks the functional relationship of λ_{max} for not too oblong partial networks is of much importance; the trunk line therefore is the most important part of the functional relationship. With regard to the above mentioned fluctuations the trunk line {pertaining to a certain type of network} can be considered as a straight line; applying somewhat stronger approximation this is possibly also true for branch lines. However, this assumption means that the lines must be given a certain thickness, so one works in fact with "bands".

In this way one arrives at the following functional relationships in which for oblong networks the node fills a part corresponding to the width of the networks:

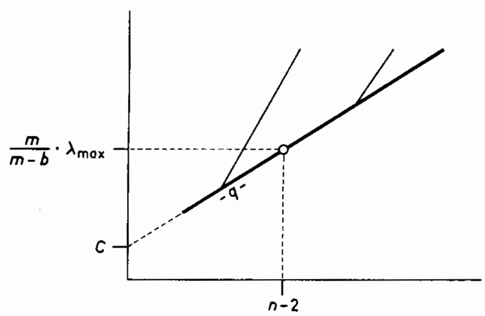


Fig. 18-11a

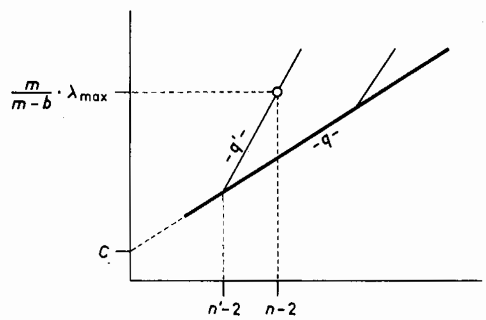


Fig. 18-11b

"Round" free networks and not too oblong border partial networks			
$\lambda_{max} \approx C \cdot \frac{m-b}{m} \cdot (n-2)^q$	$C \sim \sigma^2$	Fig. 18-11a	(18.1)
Oblong free {and from them oblong border partial} networks			
$\lambda_{max} \approx C \cdot \frac{m-b}{m} \cdot (n'-2)^q \cdot (n-n')^{q'}$	$C \sim \sigma^2$	Fig. 18-11b	(18.2)
C and q depend on type of network. q' depends on type and width of network? Lines → bands means λ_{max} → interval for λ_{max}.			
$C_{\text{network from practice}} > C_{\text{schematized network}}$		Fig. 18-10	

As stated before, formula (18.1) is of paramount importance for the reconnaissance of networks. At the same time this formula makes possible the formulation of *criteria for precision of networks.*

The meaning of formula (18.2) is still doubtful. Firstly, the computed examples give no clear indication about the behaviour of branch lines {see the course of q' in Fig. 18-10} and secondly, a reasonably unique determination of n' in networks from practice {whose build-up is always somewhat irregular} encounters difficulties.

To both formulae applies that more test computations are needed to make the picture of Fig. 18-10 more complete.

Formulae (18.1) and (18.2), combined with the method of depiction shown in Fig. 18-10, make possible an analysis of the precision of networks.

In this field of the analysis of precision of networks much literature has been published, some of which are mentioned in [12]. In particular mention should be made of the work of P. MEISSL,[8] which is based on ideas related to the theory of S-transformations.

Note 1 to section 18. The factor $l\sigma^2$

Fig. 18-12 shows the application of the law of propagation of variances on the coordinates of points of a network prior to adjustment, computed in a $\underset{r,s}{\text{S}}$-system.

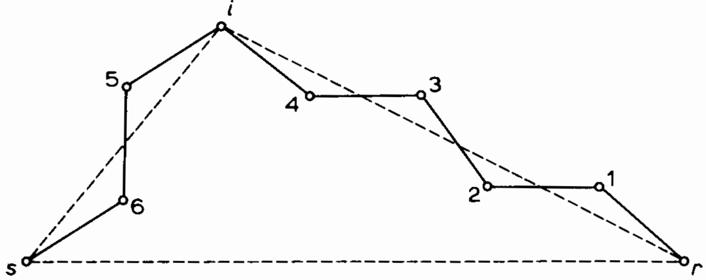


Fig. 18-12

From (2.12) follows, again omitting the superscript 0 indicating approximate values:

$$\underline{\Delta z_i^{(rs)}} = -\frac{z_{ri}z_{is}}{z_{rs}}\cdot\Delta\Pi_{ris}$$

We have then, according to section 14:

$$\overline{z_i^{(rs)}, z_i^{(rs)T}} = \left(\frac{l_{ri}l_{is}}{l_{rs}}\right)^2\cdot\overline{\Pi_{ris}, \Pi_{ris}{}^T}$$

$$\boxed{\begin{aligned}\tfrac{1}{2}\cdot\mathrm{Re}\{\overline{z_i^{(rs)}, z_i^{(rs)T}}\} &= \tfrac{1}{2}(\overline{y_i^{(rs)}, y_i^{(rs)}}+\overline{x_i^{(rs)}, x_i^{(rs)}})\\ &= \tfrac{1}{2}\left(\frac{l_{ri}l_{is}}{l_{rs}}\right)^2\cdot\mathrm{Re}\{\overline{\Pi_{ris}, \Pi_{ris}{}^T}\}\end{aligned}}$$

The latter result can be compared with the comparable value obtained from a criterium matrix with only the parameter $c_1 \neq 0$. From (15.a.19) with (15.a.17) it follows then:

$\frac{1}{2}\cdot\mathrm{Re}\{\overline{z_i^{(rs)}, z_i^{(rs)T}}\}_C = 2c_1 \frac{l_{ri}l_{is}}{l_{rs}}\cdot[\cos\alpha]$	$\{C$ denotes criterion matrix$\}$
with: $[\cos\alpha] = \cos\alpha_{isr}+\cos\alpha_{sri}+\cos\alpha_{ris}$, $1 \leqq [\cos\alpha] \leqq 1.5$	

From this an estimate for c_1 can be obtained, which is, however, now dependent on the base r, s:

$$c_1 \approx \frac{1}{4}\frac{l_{ri}l_{is}}{l_{rs}}\cdot[\cos\alpha]^{-1}\cdot\mathrm{Re}\{\Pi_{ris}, \Pi_{ris}{}^T\}$$

Supposing now that the sides of the network are about equal in length, whereas:

$$l_{ri}l_{is} = \frac{1}{4}\{(l_{ri}+l_{is})^2-(l_{ri}-l_{is})^2\} =$$
$$= \left(\frac{l_{ri}+l_{is}}{2}\right)^2\left\{1-\left(\frac{l_{ri}-l_{is}}{l_{ri}+l_{is}}\right)^2\right\}$$

then for c_1 can be written:

$$\boxed{c_1 \approx \left(\frac{l_{ri}+l_{is}}{4l_{rs}}\right)^2\left\{1-\left(\frac{l_{ri}-l_{is}}{l_{ri}+l_{is}}\right)^2\right\}\cdot[\cos\alpha]^{-1}\cdot\frac{l_{rs}}{l}\cdot l\cdot\mathrm{Re}\{\overline{\Pi_{ris},\Pi_{ris}{}^T}\}}$$

with: l = average side length in network

First the computation of Π_{ris} in this network is carried out. In this computation every time the shortest trajectories between points are chosen, so that in Fig. 18-12 the trajectories r, 1, 2, 3, 4, i and i, 5, 6, s are about straight line traverses.

Then it follows from net- and central conditions*):

$$N_{(r),i,4,3,2,1}: \; z_{ir}\cdot\underline{\Delta\Pi}_{ri4}+z_{4r}\cdot\underline{\Delta\Pi}_{i43}+z_{3r}\cdot\underline{\Delta\Pi}_{432}+z_{2r}+\underline{\Delta\Pi}_{321}+z_{1r}\cdot\underline{\Delta\Pi}_{21r}=0$$

$$N_{(s),i,5,6} \;: \; z_{is}\cdot\underline{\Delta\Pi}_{si5}+z_{5s}\cdot\underline{\Delta\Pi}_{i56}+z_{6s}\cdot\underline{\Delta\Pi}_{56s}=0$$

$$W_i \;: \; \underline{\Delta\Pi}_{ris} = \underline{\Delta\Pi}_{ri4}+\underline{\Delta\Pi}_{4i5}-\underline{\Delta\Pi}_{si5}$$

The above mentioned choice of trajectories gives:

$$\frac{z_{4r}}{z_{ir}} \approx \frac{4}{5}, \quad \text{etc.}$$

In addition, we have:

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*) See note on page 120.

$$\underline{\Delta\Pi}_{i43} = -\underline{\Delta\Pi}_{34i} \,, \qquad \text{etc.}$$

Substitution in W_i gives then:

$$\boxed{\underline{\Delta\Pi}_{ris} \approx \tfrac{1}{5}\cdot\underline{\Delta\Pi}_{r12}+\tfrac{2}{5}\cdot\underline{\Delta\Pi}_{123}+\tfrac{3}{5}\cdot\underline{\Delta\Pi}_{234}+\tfrac{4}{5}\cdot\underline{\Delta\Pi}_{34i}+\underline{\Delta\Pi}_{4i5}+\tfrac{2}{3}\cdot\underline{\Delta\Pi}_{i56}+\tfrac{1}{3}\cdot\underline{\Delta\Pi}_{56s}}$$

In a network various types of measurement are possible, i.e.:

a. *Measurement of distance-measures and directions with local {per point} scale factor and orientation, see* [2] *and* [4]:

$$\underline{\Pi}_{123} = \underline{\ln v}_{123} + i\underline{\alpha}_{123}$$

$$\underline{\ln v}_{123} = \underline{\ln s}_{23} - \underline{\ln s}_{21} \,, \qquad s_{kj} \neq s_{jk}$$

$$\underline{\alpha}_{123} = \underline{r}_{23} - \underline{r}_{21} \,, \qquad r_{kj} \neq r_{jk} + \pi$$

Supposing non-correlated observations, with:

$$\sigma^2_{\ln s_{jk}} = \sigma^2_{\ln s} \;;\; \sigma^2_{r_{jk}} = \sigma^2_r$$

then we have:

$$\overline{\Pi_{123}, \Pi_{123}{}^T} = 2(\sigma^2_{\ln s} + \sigma^2_r), \qquad \text{etc.}$$

or, in more general form:

$$5 \rightarrow \bar{n}, \qquad 3 \rightarrow \bar{\bar{n}}$$

$$\overline{\Pi_{ris}, \Pi_{ris}{}^T} \approx \left\{ \frac{1^2+\ldots+(\bar{n}-1)^2}{\bar{n}^2} + 1 + \frac{1^2+\ldots+(\bar{\bar{n}}-1)^2}{\bar{\bar{n}}^2} \right\} \cdot 2(\sigma^2_{\ln s} + \sigma^2_r)$$

or, with:

$$1^2+\ldots+(\bar{n}-1)^2 = \tfrac{1}{6}(\bar{n}-1)\bar{n}(2\bar{n}-1) = \bar{n}^2\left(\tfrac{1}{3}\bar{n} - \tfrac{1}{2} + \frac{1}{6\bar{n}}\right)$$

$$\boxed{\overline{\Pi_{ris}, \Pi_{ris}{}^T} \approx (\bar{n}+\bar{\bar{n}})\cdot\left(\tfrac{1}{3} + \frac{1}{6\bar{n}\bar{\bar{n}}}\right)\cdot 2(\sigma^2_{\ln s} + \sigma^2_r)}$$

b. *Measurement of quasi-distances and -bearings, or distance-measures and directions with regional {per network} scale factor and orientation*:

$$\underline{\Pi}_{123} = \underline{\Lambda}_{23} - \underline{\Lambda}_{21}$$

$$\underline{\Lambda}_{23} = \underline{\ln s}_{23} + \mathrm{i}\underline{r}_{23}, \qquad s_{kj} = s_{jk}, \qquad r_{kj} = r_{jk} + \pi$$

$$\underline{\Delta\Lambda}_{23} = \underline{\Delta\Lambda}_{32}, \qquad \text{etc.}$$

$$\underline{\Delta\Pi}_{ris} \approx -\tfrac{1}{5}(\underline{\Delta\Lambda}_{ri} + \underline{\Delta\Lambda}_{12} + \underline{\Delta\Lambda}_{23} + \underline{\Delta\Lambda}_{34} + \underline{\Delta\Lambda}_{4i}) - \tfrac{1}{3}(\underline{\Delta\Lambda}_{i5} + \underline{\Delta\Lambda}_{56} + \underline{\Delta\Lambda}_{63})$$

Supposing again non-correlated observations, with:

$$\sigma^2_{\ln s_{jk}} = \sigma^2_{\ln s} \; ; \qquad \sigma^2_{r_{jk}} = \sigma^2_r$$

The we have, again with:

$$5 \to \bar{n}, \qquad 3 \to \bar{\bar{n}}$$

$$\overline{\Pi_{ris}, \Pi_{ris}}^T \approx \left(\frac{\bar{n}}{\bar{n}^2} + \frac{\bar{\bar{n}}}{\bar{\bar{n}}^2}\right)(\sigma^2_{\ln s} + \sigma^2_r)$$

or:

$$\boxed{\overline{\Pi_{ris}, \Pi_{ris}}^T \approx (\bar{n} + \bar{\bar{n}}) \cdot \frac{1}{\bar{n}\bar{\bar{n}}}(\sigma^2_{\ln s} + \sigma^2_r)}$$

c. *A customary mixture of cases a and b* {*"polygons"*}: $\mathrm{Re}\{\Pi\}$ *according to b*, $\mathrm{Im}\{\Pi\}$ *according to a*

Then again we have:

$$\boxed{\overline{\Pi_{ris}, \Pi_{ris}}^T \approx (\bar{n} + \bar{\bar{n}}) \left\{\frac{1}{\bar{n}\bar{\bar{n}}} \cdot \sigma^2_{\ln s} + \left(\tfrac{1}{3} + \frac{1}{2\bar{n}\bar{\bar{n}}}\right) \cdot 2\sigma^2_r\right\}}$$

d. *Measurement only of directions with local orientation* {*"triangulation"*}, *with* $\ln v$ *computed applying the law of sines in a triangle*:

$$\underline{\Delta\Pi}_{123} = \underline{\Delta \ln v}_{123} + \mathrm{i} \cdot \underline{\Delta\alpha}_{123}$$

$$\underline{\Delta \ln v}_{123} = \underline{\Delta \ln v}_{127} + \underline{\Delta \ln v}_{728} + \underline{\Delta \ln v}_{823}$$

Hence applying the law of sines, see Fig. 18-13:

$$\begin{aligned} \underline{\Delta \ln v}_{123} = {} & (-\cot\alpha_{271} \cdot \underline{\Delta\alpha}_{271} + \cot\alpha_{712} \cdot \underline{\Delta\alpha}_{712}) + \\ & + (-\cot\alpha_{287} \cdot \underline{\Delta\alpha}_{287} + \cot\alpha_{872} \cdot \underline{\Delta\alpha}_{872}) + \\ & + (-\cot\alpha_{238} \cdot \underline{\Delta\alpha}_{238} + \cot\alpha_{382} \cdot \underline{\Delta\alpha}_{382}) \end{aligned}$$

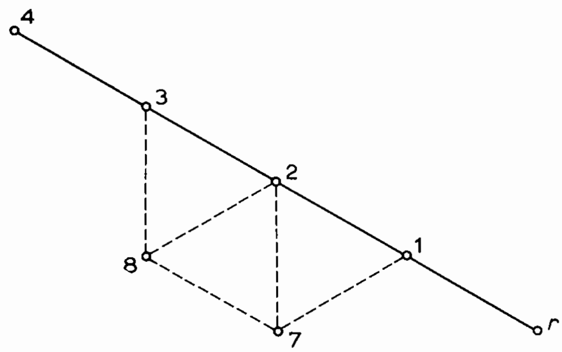

Fig. 18-13

$$\underline{\Delta\alpha}_{123} = \underline{\Delta r}_{23} - \underline{\Delta r}_{21}, \qquad \text{etc.}$$

Suppose now:

$$\cot\alpha \approx \frac{1}{\sqrt{3}}$$

then we have:

$$\underline{\Delta\Pi}_{123} \approx \frac{1}{\sqrt{3}}(\underline{\Delta r}_{12} - \underline{\Delta r}_{17} - \underline{\Delta r}_{71} + 2\cdot\underline{\Delta r}_{72} - \underline{\Delta r}_{78} - \underline{\Delta r}_{87} +$$
$$+ 2\cdot\underline{\Delta r}_{82} - \underline{\Delta r}_{83} - \underline{\Delta r}_{38} + \underline{\Delta r}_{32}) + \mathrm{i}(-\underline{\Delta r}_{21} + \underline{\Delta r}_{23})$$

Supposing again non-correlated observations, with:

$$\sigma^2_{r_{jk}} = \sigma^2_r$$

then we have:

$$\overline{\Pi_{123}, \Pi_{123}{}^T} \approx \left(\frac{12}{3} + 2\right)\sigma_r^2 = 6\sigma_r^2$$

Ignoring {unjustly} the correlation between $\underline{\Pi}_{r12}$, $\underline{\Pi}_{123}$, ..., $\underline{\Pi}_{56s}$, then analogous to case *a*, we have:

$$\boxed{\overline{\Pi_{ris}, \Pi_{ris}{}^T} \approx (\bar{n} + \bar{\bar{n}})\cdot\left(\tfrac{1}{3} + \frac{1}{2\bar{n}\bar{\bar{n}}}\right)\cdot 6\sigma_r^2}$$

e. *Measurement only of quasi-distances, hence of distance-measures with regional scale factor {"trilateration"}, with α computed applying the cosine law in a triangle*:

$$\underline{\Delta\Pi}_{123} = \underline{\Delta \ln v}_{123} + \mathrm{i}\cdot\underline{\Delta\alpha}_{123}$$

$$\underline{\Delta\alpha}_{123} = \underline{\Delta\alpha}_{127} + \underline{\Delta\alpha}_{728} + \underline{\Delta\alpha}_{823}$$

Hence via the law of cosines, see Fig. 18-13:

$$\begin{aligned}\underline{\Delta\alpha}_{123} &= (\cot\alpha_{271}\cdot\underline{\Delta \ln v}_{271}-\cot\alpha_{712}\cdot\Delta\ln v_{712})+\\ &+(\cot\alpha_{287}\cdot\underline{\Delta \ln v}_{287}-\cot\alpha_{872}\cdot\Delta\ln v_{872})+\\ &+(\cot\alpha_{238}\cdot\Delta\ln v_{238}-\cot\alpha_{382}\cdot\Delta\ln v_{382})\end{aligned}$$

$$\underline{\Delta \ln v}_{123} = \underline{\Delta \ln s}_{23}-\underline{\Delta \ln s}_{21}, \qquad \text{etc.}; \qquad s_{kj}=s_{jk}$$

Supposing again:

$$\cot\alpha \approx \frac{1}{\sqrt{3}}$$

then we have:

$$\begin{aligned}\underline{\Delta\Pi}_{123} &\approx (-\underline{\Delta \ln s}_{12}+\underline{\Delta \ln s}_{23})+\frac{\mathrm{i}}{\sqrt{3}}(-\underline{\Delta \ln s}_{12}+2\cdot\underline{\Delta \ln s}_{17}-2\cdot\underline{\Delta \ln s}_{72}+\\ &+2\cdot\underline{\Delta \ln s}_{78}-2\cdot\underline{\Delta \ln s}_{82}+2\cdot\underline{\Delta \ln s}_{83}-\underline{\Delta \ln s}_{23})\end{aligned}$$

Supposing again non-correlated observations, with:

$$\sigma^2_{\ln s_{jk}} = \sigma^2_{\ln s}$$

then we have:

$$\overline{\Pi_{123},\, \Pi_{123}{}^T} \approx (2\cdot\tfrac{4}{3}+5\cdot\tfrac{2}{3})\cdot\sigma^2_{\ln s} = 6\sigma^2_{\ln s}$$

In trilateration the derivation of $\overline{\Pi_{ris}, \Pi_{ris}{}^T}$ is much more difficult because in this case correlation can certainly not be ignored, as in case d has been done. Moreover the networks in question are, compared with triangulation networks, built up according to a different pattern with the object of obtaining a reasonable number of condition equations. This makes the supposed order of magnitude of cot α also less real. Therefore a further elaboration of formulae will not be given for the present, although in analogy with the other cases we will continue, as reasonable suggestion, with:

$$\boxed{\overline{\Pi_{ris}, \Pi_{ris}{}^T} \approx (\bar{n}+\bar{\bar{n}})\cdot(\text{factor})\cdot 6\sigma^2_{\ln s}}$$

In accordance with the supposition made in estimating the value of c_1 as described above, the following substitution can be carried through:

$$\bar{n} \approx \frac{l_{ri}}{l}, \qquad \bar{\bar{n}} \approx \frac{l_{is}}{l}$$

With this, the estimates of c_1 can be summarized as follows:

$c_1 \approx \frac{1}{2}\left(\frac{l_{ri}+l_{is}}{2l_{rs}}\right)^3 \cdot \left\{1-\left(\frac{l_{ri}-l_{is}}{l_{ri}+l_{is}}\right)^2\right\} \cdot [\cos\alpha]^{-1} \cdot \left(\frac{l_{rs}}{l}\right)^2 \cdot l \cdot \sigma^2$	
case a	$\sigma^2 \approx \left(\frac{1}{3}+\frac{1}{6}\frac{l^2}{l_{ri}l_{is}}\right) \cdot 2(\sigma^2_{\ln s}+\sigma^2_r)$
case b	$\sigma^2 \approx \frac{l^2}{l_{ri}l_{is}}(\sigma^2_{\ln s}+\sigma^2_r)$
case c	$\sigma^2 \approx \frac{l^2}{l_{ri}l_{is}}\sigma^2_{\ln s}+\left(\frac{1}{3}+\frac{1}{6}\frac{l^2}{l_{ri}l_{is}}\right) \cdot 2\sigma^2_r$
case d	$\sigma^2 \approx \left(\frac{1}{3}+\frac{1}{6}\frac{l^2}{l_{ri}l_{is}}\right) \cdot 6\sigma^2_r$
case e	$\sigma^2 \approx (\text{factor}) \cdot 6\sigma^2_{\ln s}$

. . . . . (18.3)

In these formulae it appears that the term:

$$\frac{1}{6}\frac{l^2}{l_{ri}l_{is}}$$

is relatively small or negligible, while from the substitution {± constant side length}:

$$\frac{l^2}{l_{ri}l_{is}} \cdot \sigma^2_{\ln s} \approx \frac{l^2}{l_{ri}l_{is}} \cdot \frac{\sigma^2_s}{l^2} = \left(\frac{\sigma_s}{\sqrt{l_{ri}l_{is}}}\right)^2$$

it appears that for bigger networks the influence of $\sigma^2_{\ln s}$ becomes less.

From the here developed estimate for c_1 {hence in this form dependent on the S-system chosen, as opposed to the estimate via the eigenvalue problem}, it might be possible to trace some effects that show up when estimating via the eigenvalue problem.

Choosing base points r, s situated relatively far apart the dimensionless factors:

$$\left(\frac{l_{ri}+l_{is}}{2l_{rs}}\right)^3 \cdot [\cos\alpha]^{-1}, \qquad \text{resp.} \qquad \left(\frac{l_{rs}}{l}\right)^2$$

might indicate the influence of the shape, resp. the number of points, of the network. The influence of an adjustment does not show up and is moreover difficult to give. But this influence is described adequately by the factor $(m-b)/m$ in the estimate from the eigenvalue problem.

Of importance is the factor:

$$l\sigma^2$$

This factor suggests that, *per case a–e*, when scaling up or scaling down networks and maintaining the ratio $\sigma^2_{\ln s}:\sigma^2_r$, the value of c_1 varies proportional to $l\sigma^2$. This has indeed been confirmed numerically in estimating the value of c_1 from the eigenvalue problem. Relations of c_1-estimates *in between* the cases *a–e* cannot be shown in this simple way because for this purpose the formulae developed are too coarse.

However these formulae do indicate that the estimates of c_1 via $\lambda_{\max}$ from the eigenvalue problem will not vary very much as has been shown by numerical computations. This is closely bound up with the course of the order of magnitude of standard deviations of observation variates in networks of different order. As an illustration may serve the following table of standard deviations:

$l\cdot\sigma^2_{\ln s} = l\cdot\sigma^2_r = 1$			
l [km]	$\sigma_{\ln s}$ [10^{-5} rad] σ_r	σ_r [10^{-4} gr]	σ_s [cm]
0.1	3.2	20	0.32
1	1.0	6.4	1.0
40	0.16	1.0	6.4
1000	0.032	0.2	32

. . . . (18.4)

The σ_s for $l \approx 0.1$ km does not correspond with the precision-specifications of the instruments used in this case. That this is in practice not experienced as annoying might be due to the in case c mentioned decrease of the influence of $\sigma^2_{\ln s}$ by the factor $l^2/l_{ri}l_{is}$. Computations in test nets have confirmed this conclusion but naturally the theory behind it will be more complicated.

Notes 2–8 to section 18

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