GEMMA FRISIUS, TYCHO BRAHE AND SNELLIUS AND THEIR TRIANGULATIONS

PRINTED IN THE NETHERLANDS BY W. D. MEINEMA N.V., DELFT

PUBLICATION OF THE NETHERLANDS GEODETIC COMMISSION

GEMMA FRISIUS, TYCHO BRAHE AND SNELLIUS AND THEIR TRIANGULATIONS

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1968

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CONTENTS

GEMMA FRISIUS, TYCHO BRAHE AND SNELLIUS AND THEIR TRIANGULATIONS

1 Introduction

In the past few years several papers in the Dutch language have been published on the triangulation of the Dutchman SNELLIUS (WILLEBRORD SNEL VAN ROYEN, 1580–1626) [1], [2], [3], [4]. It seems to be justified to bring these papers in a somewhat different form and in the English language to the attention of a greater number of readers and to have preceded the description of SNELLIUS' work by an examination of triangulations or ideas about triangulations published before 1615.

In the German translation of G. PERRIER: *Petite histoire de la géodésie* [5] the translator GIGAS, writing on SNELLIUS' triangulation, says, that, as in so many cases, an important discovery was made rather simultaneously and independent of each other by different people. He refers also to the report of the Baltic Geodetic Commission on the year 1930 [6], on page 45 of which report the then president, the Dane NØRLUND, makes some remarks on the history of geodetic and cartographic activities in Denmark. In the years 1578–1579 the famous Danish astronomer TYCHO BRAHE (1546–1601) carried out a triangulation, which, according to his intention, was to be the basis of a map of the whole kingdom of Denmark.

Neither GIGAS in [5] nor NØRLUND in [6], however, makes mention of a very remarkable publication of a triangulation, inserted already since 1533 in the second and following prints of *Cosmographia Petri Apiani* by the Dutch geographer GEMMA FRISIUS (1508–1555). Already in 1889 VAN DER PLAATS in [7] refers to this publication as does VAN ORTROY in a very elaborate and excellently documented book concerning GEMMA [8] and SCHMIDT in his well-known *Geschichte der geodätischen Instrumente* [9]. Recently it was mentioned by KOOPMANS in the Dutch periodical Geodesia [10].

Further on in this paper it will be shown that TYCHO BRAHE knew GEMMA's scientific work and that SNELLIUS visited TYCHO in Prague in 1600 or 1601. The exchange of scientific ideas at that occasion makes it doubtful to me whether GIGAS is right in his supposing independence of their inventions of the art of triangulation.

In my opinion it is obvious that TYCHO BRAHE borrowed his rather primitive triangulation from GEMMA's ideas and that SNELLIUS could realize his famous meridian chain thanks to TYCHO's work and GEMMA's publication which must have been known to him [11].

GEMMA FRISIUS (1508–1555)

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2 His life in Dokkum and his studies in Louvain

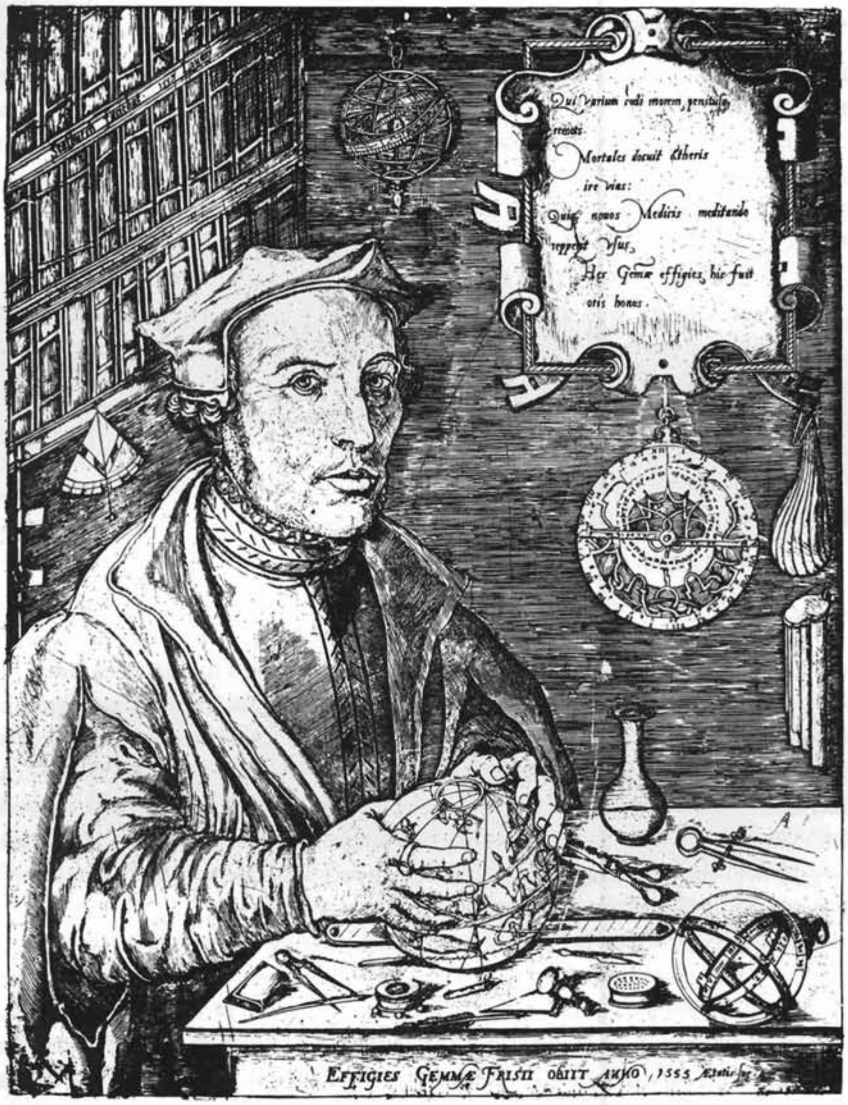
REINIER (REGNERI, RAINERUS) GEMMA (GEMME, JEMME) FRISIUS was born on December 8th, 1508 in Dokkum in the present Dutch province Friesland. The meaning of his name might be GEMMA, son of REINIER, born in Friesland [12]. His parents were well-to-do. According to his son CORNELIS, GEMMA used crutches until his sixth year, his feet being deformed since birth. On the occasion of the name day of Saint Bonifacius he went to the church in Dokkum which is consecrated to this Saint. "After having offered up his alms, he got up in the sight of all the people present and suffered no more from his infirmity" though his health remained weak during all his life [13].

Probably till 1515, "the year of his miraculous recovery", he lived in Dokkum. After the early death of his parents he was educated by friends in Groningen where he began studying literature. Then he was sent to the university of Leuven (Louvain). The date of his registration is unknown as the relevant register No. III (1485–1528) is lost [14]. As GEMMA graduated in the *Faculté des Arts* on March 19th, 1528 at the end of a two year's (October–October) course, his registration was certainly not later than 1525. It was an obligatory course preceding the study of medicine. It was not before August 1st, 1536 that he became *licentiate in medicines* [15]. As this was a six year's course, illness may have retarded a regular study. It is known indeed that he suffered from "sudoris anglicus" (sweating fever) [16].

3 Marriage and doctor's career

GEMMA married in Louvain on June 2, 1534. The family name of his wife BARBARA is unknown. On February 28th, 1535 a son – CORNELIS – was born who became later on a famous physician and who was appointed professor in the Louvain university in 1569. According to several authors he succumbed to the plague on October 13th, 1578 [17]. The other children – one of them died already before December 12th, 1539 – did not play an important part in the science of those days.

According to VAN ORTROY it is likely that GEMMA was appointed professor in Louvain between 1537 and 1539 [18] though he obtained his doctors degree in medicines not before August 30th, 1541. There is, however, not a single explicit evidence for this appointment [19].



Engraving J. van Stalburch, 1557. Reproduced by courtesy of the "Rijksprentenkabinet", Amsterdam

Fig. 1

In his profession he proved to be a man of strong character, rather than a man with rich scientific gifts. He considers his profession a holy profession in which he is as much engaged as in the care for the public affair. His science is not only meant for the greats of the earth, the life of the poor being for God of as much value as that of the mightiest potentates. This attitude towards life made him a good physician, helpful for the poor whom he treated gratis. The rich, however, had to pay largely for his services [20].

It made also that he was not an eye-servant and that he dared freely speak his mind. So in a letter to his benefactor DANTISCUS [21] he wrote that the number of physicians in the university of Louvain was larger than their reputation or the number of their auditors [22].

4 Devotion to mathematics and geography; his death

Though GEMMA was a physician he was mostly devoted to mathematics, astronomy and geography. In 1543 he started a course in mathematics and astronomy. As a talented man he was highly appreciated by a great many auditors among whom his friend MERCATOR (GERARD KREMER, 1512–1594). According to several authors the emperor CHARLES V would have discussed with him many times matters of scientific interest [23]. One of them even states that he was distinguished with the order of the Golden Fleece. He enjoyed also fame abroad. TYCHO BRAHE e.g. says that he belongs to the prominent mathematicians [24]. As a child of his age GEMMA was a supporter of the opinion that the motion of the heavenly bodies, especially the motion of the moon, influenced the periodical occurence of fevers, the progress of illness and the life of people in general. This way of thinking is not as queer as it looks like, in view of the fact that astrology was taught in Louvain until 1568.

GEMMA died, 46 years old, in Louvain on May 25th, 1555. According to his son CORNELIS he died of stones in the kidneys, a disease from which he had suffered for seven years at least. He is buried in the church of the Dominicans [25].

5 Publications

I have already remarked before that in Van ORTROY's study GEMMA does not appear as a very original scientist. He had, however, an excellent feeling for application possibilities and for the solution of practical problems. He had with the greatest care several instruments built which were not invented by himself. Moreover he himself was an able constructor who made several celestial globes, earth globes, astronomical rings (*annulus astronomicus*), cross-staffs (*baculus Jacob*), astrolabes and quadrants. He was also an excellent geographer and cartographer; his world map from 1540 is an introduction to the famous cartographic work by e.g. MERCATOR, ORTELIUS and BLAEU.

His most important books – they all treat mathematics, astronomy and geography and not medical art – were reprinted several times: his *Arithmeticae Practicae methodus facilis* 73 times and his *De principiis Astronomiae et Cosmographiae* 11 times. His *Cosmographicus liber Petri Apiani* counts 30 prints between 1529 and 1609 (16 in Latin, 8 in Dutch, 5 in French and 1 in Spanish) [26]. As the title suggests it is his version of APIANUS' book *Cosmographia liber* which was published in 1524. GEMMA's first version (in Latin) is from 1529. He was then only 20 years old. Even the second print (Antwerp 1533 and also in Latin) [27] is extended with an utmost important appendix of 16 pages in which the principles of triangulation are treated completely.

The Latin title of this Appendix is *Libellus de locorum describendorum ratione*. It is copied in all the other 28 prints of the book. I borrowed the Dutch translation from a photographic copy of the last (Dutch) edition from 1609 [28].

It runs: *Een boecxken seer nut ende profijtelijck allen geographiens leerende hoemen eenighe plaetsen beschrijven ende het verschil oft distantie derselver meten sal welck tevoren noyt ghesien en is gheweest. Ghemaeckt bij Gemmam Frisium Mathematicien ende Licentiaet inder Medecijnen.*

The English translation could run:

A booklet very useful and profitable for all geographers, teaching how to measure and to compute the distance between two places, which was never seen before. Made by GEMMA FRISIUS, mathematician and Licentiate in Medicines.

In 1889 it is VAN DER PLAATS who writes already appreciatively about the booklet [7], according to VAN ORTROY the principles of triangulation in it *sont absolument conformes à ceux de la planimétrie ou de la topographie moderne* [29] and DE VOCHT declares that "its importance can hardly be gauged: for it revealed the final definite way of representing any country with its towns . . . by means of a series of triangles with one common basis which could be measured with preciseness so that it led to accurate distances and became the beginning of actual geography; subsequent times have only been able to add to it more facilities in the checking and the registering of the various elements" [30].

6 Description of a triangulation

The booklet has 7 chapters. In the first and most important chapter GEMMA gives first a definition of what we call nowadays a magnetic bearing. Then he treats the principles of triangulation. They are illustrated with some drawings but there is not a single formula in the text [31].

For measuring a whole "province" with all its towns and villages an instrument must be made consisting of a circle which is divided into four quadrants. Each quadrant must be divided into 90 degrees. In the centre of the circle is fastened the end of a sight rule. The other end with a sighting device can be moved along the circumference.

This very primitive goniometer is set up at a station, e.g. a tower *A* which lies in the area that must be measured. The plane of the circle must be horizontal and the line that connects the centre with the zero point of the graduation must be pointed at the magnetic north, which is done with a "mariner's compass". The instrument is now oriented. The compass is taken away and with the sighting device one can read magnetic bearings on the horizontal circle of the apparatus, e.g. the bearing to a tower *B* or to another detail in the terrain. The bearings can be plotted with a protractor. By every radius the name of the relative tower is mentioned, e.g. *B* or *C*.

"Now somebody might ask me: what is the purpose of this method; for, even if I have a great number of bearings, they are of no use if I have not the distances to the several details in the terrain" [32].

In order to give an answer to this question "travel to another town (e.g. *B*) and act there in the same way with the bearings to the surrounding places which you can see there" [32]. On the map with the bearings in *A* the point *B* is chosen on the line *AB* at an in principle arbitrary distance from *A*. The line to the magnetic north is drawn parallel to that in *A* and the bearings in *B* are plotted in an analogous way as those in *A*. The intersection point of the radii in *A* and *B* to e.g. *C* represents *C* at the assumed scale.

In this way "you must go from tower to tower", attending to it that each detail in the terrain to be plotted has two bearings. If the point to be plotted lies on or almost on the connecting line of the points from which the bearings are measured, a third bearing is necessary in order to fix the point.

GEMMA describes an example of his method on the pages 105 and 106 of his booklet. On the tower of Antwerp he "measures" the following bearings: Gent 80° west of the north, Lier 30° south of the east, Mechelen "almost" 8° west of the south, Leuven 4° east of the south, Brussels 25° west of the south, Middelburg 30° north of the west and Bergen op Zoom 20° west of the north. Then he travels in theory to Brussels where he "measures" the following bearings: Leuven nearly 14° south of the east, Mechelen and Lier on one line 47° north of the east, Gent 29° west of the north, Middelburg 33° west of the north and Bergen op Zoom 9° east of the north. It is true, he says on page 106, that these last two towers cannot be seen from Brussels "but I give them as an example and I don't wish that somebody would think that I mention here real bearings".

7 Measurement of a base line and determination of the length of a side of a triangulation network

GEMMA'S elucidating sketch map on page 105 is reproduced in a somewhat different form as fig. 2.

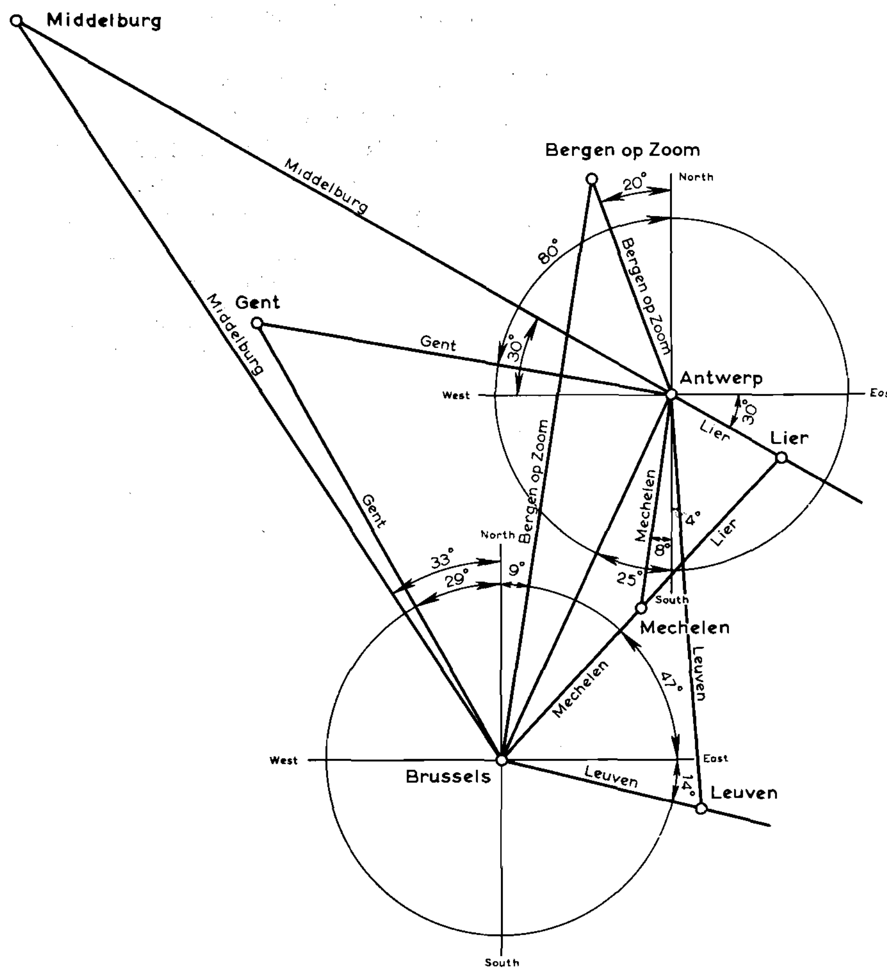


Fig. 2

He remarks that (the scale of) the map is larger when Brussels and Antwerp are chosen farther from each other. The mutual proportions, however, remain unaltered. In order to obtain a map on a known scale it is necessary that the distance between two towers on the map is known in the terrain. It can be determined "by walking over this distance" [33].

Two more accurate methods are described in the third and the fourth chapter (pages 107 and 108). For the first method "a large field is necessary whereupon you can go hither and thither; it does not matter if it is not quite flat".

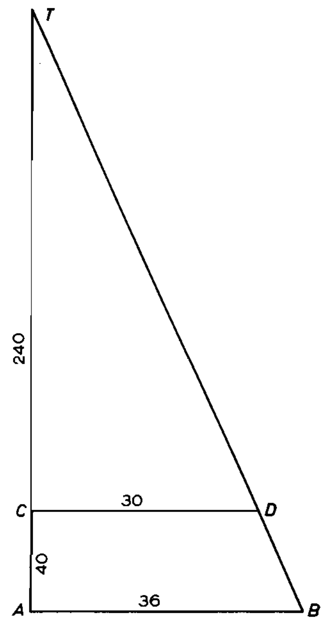


Fig. 3

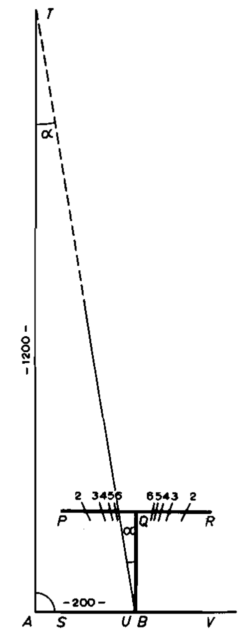


Fig. 4

In fig. 3 AB is a base line in such a terrain; it has – by way of an example – a length of 36 units. $CD = 30$ units is a line parallel to AB at a distance of 40 units. T is the tower. In an extremely long-winded manner GEMMA says how $AT = 240$ units must be computed from these three data. "If somebody wants the mathematical proof, let he come round to me".

For the second method, described on page 108, a "mathematical instrument" must be used, the so called *scala altimetra* or *scala geometrica.* Quite rightly GEMMA states that the instrument is more accurate when it is larger.

In the shape of fig. 4 it is a *scala geometrica* because it is used here for the measurement of angles α in the horizontal plane. It consists of a cross-sight vane ($QU \perp SUV$ with which right angles can be set out. It is fastened on a staff "with a length of five or six feet". Parallel to SUV a calibration for $\cot \alpha$ is made which can be read with a sighting device .In fig. 4 the point $\cot \alpha = 6$ lies on BT. AT is therefore 6 AB. AB is set out with the cross-sight vane.

In chapter V GEMMA treats the measurement of the angles of the trigonometrical network, instead of the bearings of the sides. After the angles have been drawn on the map the distances between arbitrary points can be scaled-off on an arbitrary scale. If one distance is known in the terrain all other distances can be computed by a proportion. On page 111 he states that the distances can also be computed "with the tables of sine, but I omitted this intentionally as it is too difficult for the common man".

At the end of the booklet we find that, without deformation, the spherical earth cannot be represented on a flat map, even not "if Ptolemy would come back". "In a province of about 50 or 100 miles the error is of no importance. If, however, Europe would be measured in this way, the earth must be considered as a sphere. As this knowledge is no common property I shall not enter into that".

8 Speculations on his "Libellus"

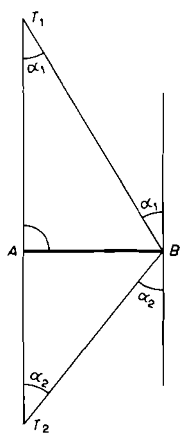


Fig. 5

The reader will agree that in 16 pages of this remarkable book from 1533 GEMMA treats the principles of triangulation completely and describes clearly the measurement of the angles of the network. It is even striking that he recommends a complete circle for the angular measurement, since in his time – I shall presently come back to that – the cross-staff was the mostly used goniometer. A quadrant was used exceptionally. The measurement of a base line in a suitable terrain is also quite modern. The text implies that A in the figures 3 and 4 is a point in the terrain and not a tower. Whether it forms part of the trigonometrical network is not clear. GEMMA only states that T is a tower. If A forms no part of the network it seems most likely that he would have determined the distance between the towers T_1 and T_2 as is shown in fig. 5. The top part of this figure agrees with fig. 4; the lower part gives an analogous construction. In that case A must be marked out between T_1 and T_2. The large field "whereupon you can go hither and thither" would plead for this idea. If so, an important part of the honour for the "invention" of base extension which is now given to SNELLIUS, would be due to GEMMA.

9 Application in practice

It is doubtful whether the excellent theory on triangulation on the preceding pages was applied in practice by GEMMA. According to DE VOCHT [34] his weak health will presumably have prevented this. Also VAN ORTROY has no indications for it. SCHMIDT, however, says [35] that GEMMA executed measurements for a triangulation with a cross-staff and KOOPMANS mentions that a triangulation by GEMMA was the basis of a map of Lorraine [36]. Unfortunately this map is lost. He gives even the "accuracy" of the measurement: 1 to 2000 for the base line and 2′ for the angles. It is not clear what should be understood by "accuracy". As will be shown later on the standard deviation in the angles of SNELLIUS' triangulation from 1615 is almost 4′; that in TYCHO's triangulation from 1578–1579 is much larger (almost 6′). The amount of 2′ given by KOOPMANS must therefore be much too low. The source of the accuracy of 1 to 2000 in the length measurement is also not clear.

10 Tycho Brahe must have known Gemma's work

It is important to state that TYCHO BRAHE had connections with members of GEMMA's family. For, in his "Description of his instruments and scientific work" [37] he mentions that the *radius astronomicus* (cross-staff) and *annulus astronomicus* (astronomical ring) which he uses "are not constructed by myself but by WALTER ARSCENIUS, a grandson of the eminent mathematician GEMMA FRISIUS who at one time lived in Louvain in Belgium". Calling him "grandson" in this quotation is a very disputable point. According to VAN ORTROY [38] WALTERUS (GAUTHIER) ARSCENIUS (ARSENS, AERTSSENS, VAN AERTSSENS), a well known instrument maker, was GEMMA'S nephew. He founds that on information given by GEMMA himself on an astrolabe being made *per nepotem nostrum Gualterum Arsenium* [39]. It is unknown, however, whether WALTERUS and his brothers REIGNIER (REINIER) and REMI were related to GEMMA through his wife's family or through his sister's marriage. Possibly the signalized difference in relationship in [37] and [38] might be carried back to the translation of the Latin word *nepos* which can mean grandson as well as nephew. From the quotation "eminent mathematician GEMMA FRISIUS" it is, anyhow, clear that TYCHO must have known GEMMA's work. Moreover, it was written in the language (Latin) which was accessible to him. If he had not known it from his own investigation – which is improbable – ARSCENIUS would have drawn his attention to it. Therefore TYCHO's triangulation over The Sound in Denmark could probably be carried out because he knew the principles of triangulation which were published 45 years earlier in GEMMA's remarkable *Libellus*.

TYCHO BRAHE (1546–1601)

11 His youth and his settlement on Hven

TYCHO (TYGE) BRAHE descends from an old noble family. He was born December 14th, 1546 as the second child (first son) from the marriage of OTTO BRAHE with BEATE BILLE, on the family estate Knudstrup, about 30 km east of Hälsingborg, at that time belonging to the kingdom of Denmark [40]. He was educated on the estate Tostrup of his uncle JØRGEN BRAHE. When he was still very young he learned, besides the usual subjects, also Latin that he could write and speak fluently.

In 1559 he went to the university of Copenhagen where he had the greatest interest in astronomy and astrology. This interest was stimulated through the total sun eclipse that could be seen in Portugal on August 21st, 1560. He stayed in Copenhagen for three years. Then uncle JØRGEN sent him to the university of Leipsic with the very talented ANDERS SØRENSEN VEDEL who acted as his mentor and who was his senior by only four years. In Leipsic he makes his first astronomical observations with the only instrument at his disposal: a pair of compasses of which the turning point had to be held in the eye. Later on he bought a cross-staff which was made according to the directions of GEMMA FRISIUS [41].

In 1565 TYCHO and VEDEL went back to Denmark where a war had broken out between Sweden and Denmark and where uncle JØRGEN died in consequence of a successful attempt to save king FREDERIK II from drowning.

With the exception of his uncle STEEN BILLE no one of his family and his acquaintances had any sympathy for TYGE and did not speak disapprovingly of his "absurd tendency to make observations". He was therefore glad to leave Denmark for the second time in 1566.

After a short stay in Wittenberg he goes to Rostock where he is matriculated in the university on September 24th, 1566 and where he loses part of his nose in a duel on Decem-

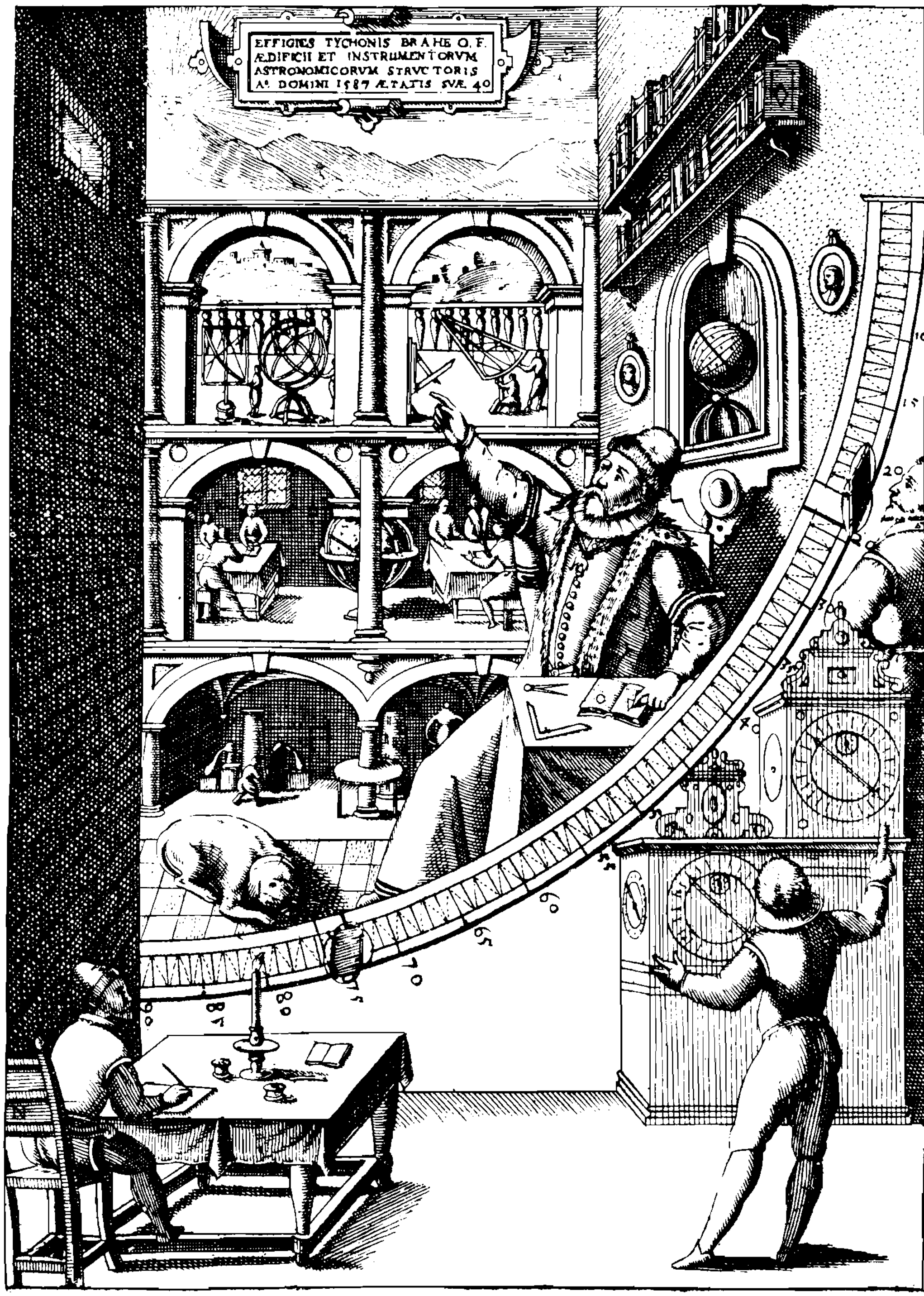

EFFIGIES TYCHONIS BRAHE O. F.
ÆDIFICII ET INSTRUMENTORVM
ASTRONOMICORVM STRVCTORIS
A°. DOMINI 1587 ÆTATIS SVÆ 40

Fig. 6

ber 27th, 1566. As a "prosthesis" the missing part was replaced by a mixture of gold and silver. It could be fastened with salve which he had always with him [42]. We owe this information to WILLEM JANSZOON BLAEU (1571–1638), the famous Dutch instrument-maker, cartographer and maker of globes. He lived with TYCHO on the isle of Hven from 1594 till 1596.

From Rostock he goes to Wittenberg and in 1568 to Basel. Here his first quadrant is made "of dry oak, so large (radius 18 feet) and so heavy, that 20 workmen could hardly place it in position".

In 1570 he goes back to Denmark because of illness of his father who died in 1571. He experiments there for some time with chemistry and alchemy but is "called back" to astronomy by the appearance of "the new star in Cassiopeia". Almost every day during the following 18 months he measures its distance to the nine brightest stars of the constellation. From these observations the (constant) declination 61°47′ of the nova could be computed. A publication of his observations appears in 1573. He did not allow to mention his name in it "because many people would find it foolish that a nobleman engaged himself with science" [43].

After the publication of his book he did intend to go on journey. Illness and his relationship with CHRISTINE prevented this. According to some sources she was the daughter of a peasant in Knudstrup, according to others a maid-servant or the daughter of a clergyman. At any rate she was not a woman of his social position. This relation estranges him still more from his family [44]. Apparently he did not marry her but they had several children. Their relation lasted till his death, 28 years later.

Early in 1575 he starts upon his intended journey. It leads him to several places in Germany and even to Venice. He is back home in December of the same year. His intention to settle in Basel could fortunately be foiled because the king's attention was drawn to him and FREDERIK II quite rightly was of the opinion that the scientist TYCHO had to be kept for Denmark. He offered him the loan of the isle of Hven in The Sound where he could cultivate his astronomical science and also an allowance of 500 thaler (about 140 pounds). On February 22nd, 1576 TYCHO paid his first visit to the isle and on May 23rd, 1576 the king signed the deed in which the loan of the isle and its proceeds were assigned to TYCHO [45]. The isle belongs already since 1658 to the territory of Sweden.

12 Scientific career

About in the centre of the isle "160 feet (about 40 m) above sealevel" TYCHO built his house and his observatory. Though the first stone was laid on August 8th, 1576 it was not quite ready until 1580. It was called Uraniborg [46] after the muse of astronomy, Urania. The principal instrument in the observatory was the great mural quadrant with a radius of almost 5 cubits [47]. With TYCHO's portrait this quadrant is pictured as fig. 6.

The exact length of the cubit was unknown till 1943. In that year NØRLUND published in his *Danmarks Kortlaegning* (Cartography of Denmark) a very important paper on TYCHO's geodetic work [48]. In this paper he derives – I come presently to the details – that his unit of length, the *passus geometricus*, was 1.552 m [49]. As 1 *passus* = 6 feet and 1.5 feet = 1 cubit [50], the length of the (Tychonian) foot is 0.2587 m and the length of the cubit 0.388 m. It brings the radius of the mural quadrant at about 1.94 m. Ten minutes of arc on the limb of this instrument represent about 5.6 mm. According to TYCHO 10″ or 5″ could be read easily on it [51].

The years 1576–1596 were the happiest and most productive of his life because he could work free from financial worries. His allowance of 500 thaler was raised and as a consequence of his appointment as canon of Roskilde in 1579 his income increased once more. True, the population of Hven complained of his ill treatment and his arbitrariness but TYCHO will have been neither better nor worse than most noblemen of his time [52].

In his castle he had – as was customary in his time – a jester, a dwarf, who, during the meals, sat at his feet and received now and then a piece of food out of TYCHO's hand [53].

On his isle TYCHO not only occupied himself with astronomy but also with preparing medicines which were available free of charge. Many people therefore crossed The Sound to obtain those medicines. He casted also horoscopes, e.g. those of the princes CHRISTIAAN (born April 12th, 1577) and ULRICH, sons of king FREDERIK II. The latter horoscope, 300 pages, written by TYCHO himself and bound in green velvet, is still kept in the royal library in Copenhagen [54]. TYCHO did, however, not sympathize very much with astrology; he practised it only on the king's desire. He himself wished to promote astronomy; "for only by this science and with good instruments the truth could be found".

King FREDERIK died on April 4th, 1588. His name will always be connected with that of TYCHO whom he gave an opportunity to follow his scientific career. TYCHO was aware of this support and he was grateful for it. His thankfulness found expression in the text on the celestial globe on which he plotted the stars (approximately 1000) of which he had determined the co-ordinates. It was about 6 feet (1.55 m) in diameter [55].

Unfortunately it was lost by fire. In golden letters the text said that it was made in 1584 "four years before king Frederik, of glorious memory, departed this life, he who generously and graciously supported me and my studies and followed them with royal favour as long as he lived" [56].

The king was succeeded by his eldest son CHRISTIAAN who was then only 11 years old. Till his majority the reign was executed by a regency of four which paid the debts of 6000 thaler (about 1700 pounds) which TYCHO had made "in honour of his country" [57].

The new king, CHRISTIAAN IV, come to the throne on August 17th, 1596, was a thrifty man and TYCHO met with this savingness. Part of his allowance was stopped, also probably because several of his influential friends had died. It must be said, however, that TYCHO himself contributed to a high degree to the diminution of his influence because of his obstinacy and the negligence of his duties as canon of Roskilde.

When the peasants on Hven perceived that TYCHO had fallen into disgrace they did attack him as well by sending a letter to the king in which they complained of his tyranny. These complaints were examined on the spot on April 4th, 1597. First of all the clergyman of Hven was discharged for his omission not to have admonished and punished TYCHO for not having taken part in the sacraments for 18 years [58]. It can not be said, however, that TYCHO was not religious. The contrary can be understood from several of his works in which he shows himself a supporter of the geocentric world system: The earth is the centre of the universe and it must be a physical absurdity that it should move. Moreover it is in conflict with the wording of the Holy Scripture. The enormous velocity with which the eighth sphere in his system moves around the earth is a token of God's great wisdom and omnipotence [59].

13 Settlement in Prague; his death

Immediately after Easter 1597 TYCHO and his family leave Hven for good. After a short stay in Copenhagen and several other cities he settles in Prague where he enters upon the duties of the emperor RUDOLF II and where he has the castle Benatky at his disposal. He has KEPLER (1571–1630) as his co-worker who, later on, could build up his famous laws of planetary motion, because he had at his disposal the thousands of accurate observations made by TYCHO.

In 1600 or 1601 the Dutch mathematician SNELLIUS visited TYCHO in Prague. It is possible that on that occasion they spoke about TYCHO's triangulation. If this is correct – but it can not be proved – SNELLIUS' performances in this field are not quite independent of TYCHO's.

After a short illness TYCHO died on October 13th, 1601. On November 4th he was buried with great pomp in Teyn church in Prague. On the grave, in the nave of the church, the children erected later on a fine monument which still exists. His wife, who died in 1604, was also buried there.

It seems that the buildings on Hven were of a poor construction as already in 1599 several buildings were very ruinous. In 1623 60,000 bricks were broken from the castle in order to be used for other objects and it seems that in 1645 nothing was left of it [60].

14 Instruments

In the preceding pages I expatiated intentionally on TYCHO's life in order to be able to project his work to the time in which he lived. A time which lies so far behind us that it seems almost unreal.

The instruments which he used – the quadrant with a radius of about 4.5 m and so heavy that 20 workmen could hardly place it in position (see § 11) – seem equally unreal. As I told already before TYCHO obtained very good results with these great and heavy instruments, at least with the astronomical instruments which had a permanent setting up. Bearings to or angles between terrestrial reference points, however, could not be determined with this great accuracy because they had to be measured with smaller, transportable instruments. For the measures of these instruments we are entirely dependent on TYCHO's description in [37] as all his instruments are lost. The radius e.g. of the goniometer in fig. 7 is 4 cubits (about 1.55 m). It is described on the pages 80–83 of that book and it was used for the measurement of angles up to about 30°. It had to be supported, it was levelled with the aid of some plumb lines and it dates from about 1572. "It is not as excellent as those that I invented and had constructed in later years with much trouble and at great cost" [61]. Its radius is not very much smaller than that of the mural quadrant in fig. 6 ($r = 1.94$ m).

The limbs of all TYCHO's instruments had an interval of 10′. In order to read more accurately within this interval, so called transversals were used. The principle of the transversals emanates from LEVI BEN GERSON (1288–1344), a Jewish philosopher who lived in the south of France. Especially TYCHO saw the usefulness of these transversals and he used them on all his instruments, even on his first cross-staff from 1564 [62].

For a quadrant with a radius r and an interval of 10′ these transversals are represented

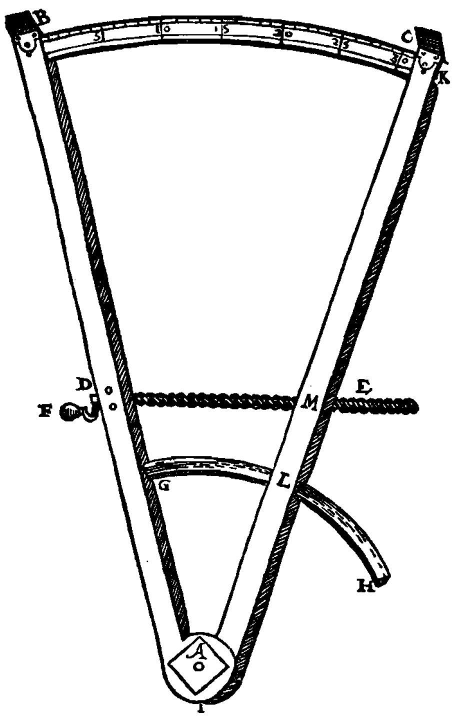


Fig. 7

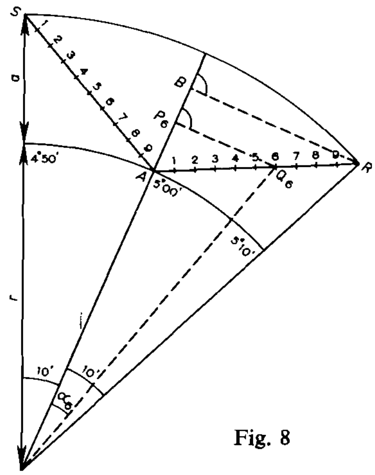


Fig. 8

i	α_i	i	α_i
0	0'00.0"	5	5'03.1"
1	1 01.1	6	6 03.0
2	2 02.0	7	7 02.6
3	3 02.6	8	8 02.0
4	4 03.0	9	9 01.1

Table 1

by the lines AR and AS in the very deformed fig. 8. The lines are clearly visible on the mural quadrant in fig. 6. R and S lie on a circle with a radius $r+a$. AR and AS are divided into ten equal parts. As

$AB = (r+a)\cos 10' - r$ and
$BR = (r+a)\sin 10'$ we have for the point $i (i = 0, \ldots, 9)$ on the transversal AR:
$AP_i = \{(r+a)\cos 10' - r\} i : 10$ and
$P_iQ_i = i(r+a)\sin 10' : 10$

whence

$$\tan \alpha_i = \frac{i(r+a)\sin 10' : 10}{r + \{(r+a)\cos 10' - r\} i : 10} =$$

$$\frac{i(r+a)\sin 10'}{10r + \{(r+a)\cos 10' - r\} i} =$$

$$i\left(1 + \frac{a}{r}\right)\sin 10' : \left[10 + \left\{\left(1 + \frac{a}{r}\right)\cos 10' - 1\right\} i\right].$$

For all TYCHO's instruments $\frac{a}{r} = \frac{1}{48}$ so that

$$\tan \alpha_i = \frac{49\, i \sin 10'}{480 + (49 \cos 10' - 48) i}$$

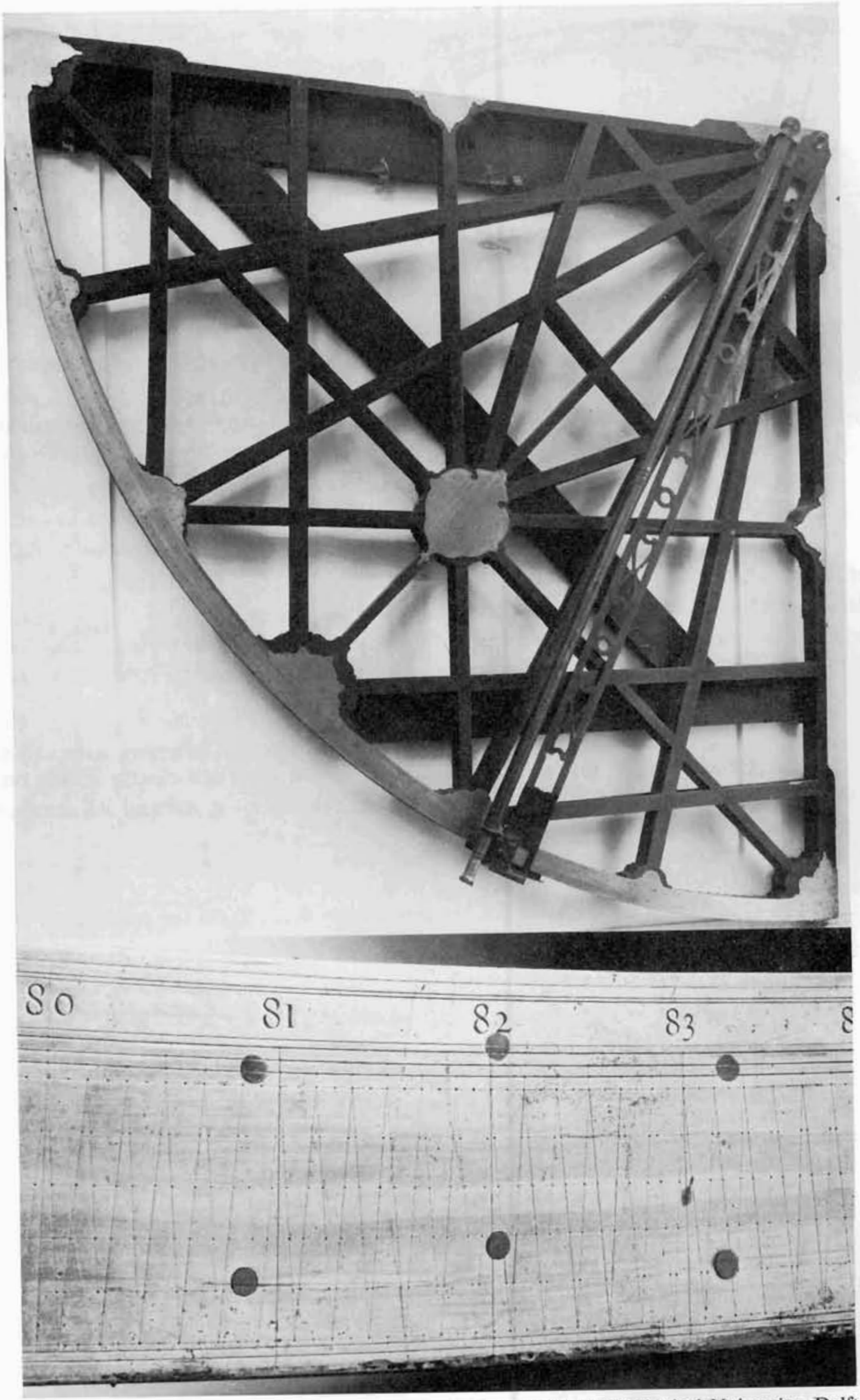


Fig. 9 Photo: W. Rietveld, Technological University, Delft

The amounts α_i are arranged in table 1. It shows that the reading device with transversals always gives too high a reading on the rising transversal *AR* of an instrument that is calibrated to the right. Upon the falling transversal *SA* the readings are too low. TYCHO also computed the errors in a somewhat long-winded argumentation [63]. His results, however, are good. "The greatest difference to be added or subtracted is a little over 3″, a quantity so small, that a keen vision is in no way able to distinguish it in any instrument". In fig. 9 I have represented a still existing quadrant whereupon such a transversal division has been made. It has a radius of about 2.20 m and it has been made on SNELLIUS' commission by the instrument maker WILLEM JANSZOON BLAEU whose name I mentioned already in § 11.

After SNELLIUS' death in 1626 it was bought by his successor GOLIUS who sold it in 1632 for 125 guilders (about 12 pounds) to Leiden University. It was the first instrument of the astronomical observatory in Leiden, founded in 1632. It is still present there [64]. Underneath the instrument one sees a full-size reproduction of a part of its limb.

15 Cross-staff

As I remarked already before in TYCHO's time the cross-staff was the mostly used instrument for the measurement of angles. Its original Latin name, given by LEVI BEN GERSON, was *baculus Jacob* (*Jacob-staff*; in French: *Crosse* de *Saint Jacques*; German: *Jacobsstab*) but GEMMA FRISIUS and TYCHO BRAHE called it *radius geometricus* when it was used for the

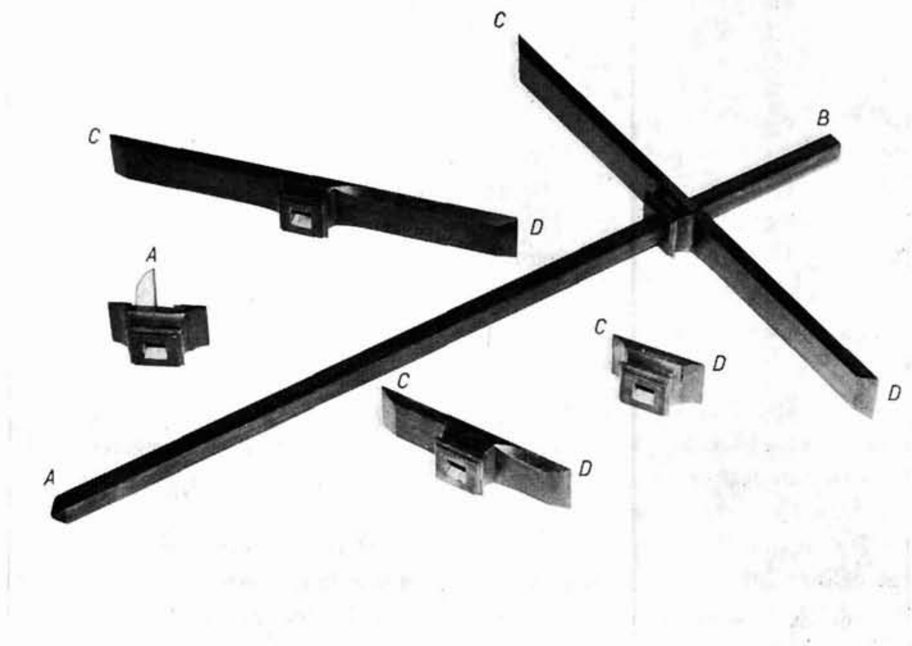


Fig. 10 Photo: W. Rietveld, Technological University, Delft

measurement of horizontal angles. The principle of the instrument dates back to ARCHIMEDES and HIPPARCHUS (third and second century B.C.) who applied it in the dioptra for the measurement of the sun's diameter [65].

A reproduction of a cross-staff is given in fig. 10. It is a photograph of a faithful copy of an original in the Shipping Museum in Rotterdam. The calibration on it, however, has been omitted. The copy belongs to "Snellius", a society of geodetic students at the Delft Technological University.

The wooden instrument consists of a staff *AB*, square in section. It is provided with a metric scale or a scale of cotangent. The zero point *A* of the calibration has sometimes – as in fig. 10 – a sighting device. For the instrument in fig. 10, apparently used for the fixation of a ship's position, *AB* is only 75 cm. For LEVI BEN GERSON's instrument *AB* was about 1 m and for REGIOMONTANUS' (1436–1476) cross-staff between 2 and 3 metres [66].

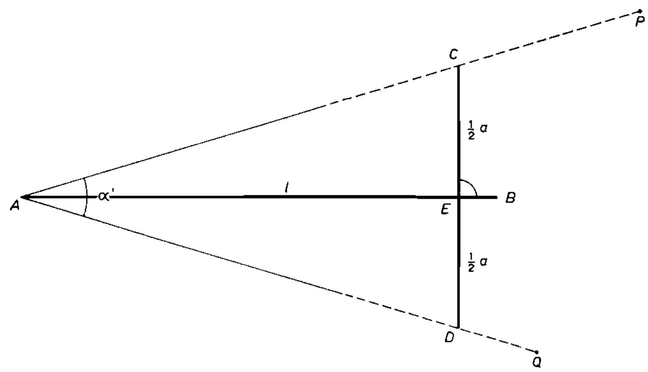


Fig. 11

Perpendicular to *AB* cross-bars *CD* of different lengths *a* can be shifted along *AB*. Some of these bars are shown in fig. 10. After the instrument has been levelled with plumb-lines or by sighting on the horizon the horizontal angle α' in *A* between the reference points *P* and *Q* in fig. 11 can be measured. This is done by shifting *CD* along *AB* in such a way that *PC* and *QD* intersect in *A*, the point where the eye is held. We have then

$$\tan \tfrac{1}{2}\alpha' = \frac{a}{2AE} = \frac{a}{2l}$$

As *a* is known and *l* can be read on *AB*, α' can be computed.

In order to avoid a computation one can also make a calibration in degrees on *AB*. For a length $a = 0.50$ m the calibration line $\alpha' = 20°$ must be marked at a distance:

$l = 0.25 \cot 10° = 1.418$ m from *A*.

For $CD = a = 0.30$ m this distance is 0.851 m. It is therefore possible to make four different calibrations on the four lateral faces of *AB* corresponding with four different lengths *a* of *CD*. It is obvious that the most accurate determination of α' will be obtained with a long *l* and – therefore – a long *a*. For $\tan \tfrac{1}{2}\alpha' = a/2l$ can be computed more accurately from the quotient of two great numbers than from the quotient of two small numbers.

When α' is directly read on AB the preference for a great a demonstrates itself by a greater interval e.g. between $\alpha' = 20°$ and $\alpha' = 20°10'$. For $a = 0.50$ m this interval is about 12 mm, for $a = 0.30$ m about 7 mm. In the first interval interpolation is more accurate than in the latter. Interpolation in an interval was mostly done with the aid of transversals which were drawn on AB. They run so to say round the whole staff and give it the appearance of the sheep-hook which should have been used by the patriarch JACOB. For this reason LEVI BEN GERSON called it *baculus Jacob* [67].

When α' is large, e.g. 150°, cot $\frac{1}{2}\alpha'$ changes but little. Readings on AB close to A are therefore inaccurate. Moreover in this case the points C and D of the staff are too close to the eye. This gives difficulties for the eye to accomodate. In order to prevent that the users would measure too large an angle with too short a length CD the first part of AB was not calibrated. According to LEVI BEN GERSON angles larger than about 45° should not be measured with the cross-staff [68].

16 Systematic errors in readings on the cross-staff

In § 15 I assumed that during the observation the observer's eye was in the sighting device A of fig. 10 or in the zeropoint A (cot $\frac{1}{2}\alpha' = 0$) of fig 11. This is not quite correct as the point where the intersection of the lines PC and QD is observed lies in O, at a small distance e to the left of A (see fig. 12).

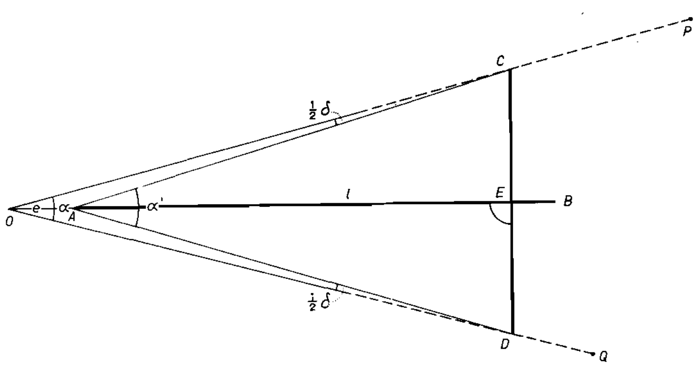


Fig. 12

The observed angle α between PC and QD is therefore smaller than the angle α' read on the instrument. The error $\alpha' - \alpha = \delta$ can be computed as follows:

$$\frac{e}{\sin\frac{1}{2}\delta} = \frac{AC}{\sin\frac{1}{2}\alpha} = \frac{AC}{\sin\frac{1}{2}(\alpha'-\delta)} = \frac{l}{\cos\frac{1}{2}\alpha' \sin\frac{1}{2}(\alpha'-\delta)} =$$

$$= \frac{l}{\cos\frac{1}{2}\alpha'(\sin\frac{1}{2}\alpha' \cos\frac{1}{2}\delta - \cos\frac{1}{2}\alpha' \sin\frac{1}{2}\delta)}$$

As $\frac{1}{2}\delta$ is small cos $\frac{1}{2}\delta \approx 1$ and sin $\frac{1}{2}\delta \approx \delta/2\varrho$ so that

$$\frac{2\varrho e}{\delta} = \frac{2\varrho l}{\cos\frac{1}{2}\alpha'\,(2\varrho\sin\frac{1}{2}\alpha' - \delta\cos\frac{1}{2}\alpha')}$$

or

$$l\delta + e\delta\cos^2\tfrac{1}{2}\alpha' = 2\varrho e\sin\tfrac{1}{2}\alpha'\cos\tfrac{1}{2}\alpha' = \varrho e\sin\alpha'$$

whence

$$\delta \approx \frac{\varrho e\sin\alpha'}{l + e\cos^2\frac{1}{2}\alpha'} \approx \frac{\varrho e\sin\alpha'}{l\left(1 + \frac{e}{l}\cos^2\frac{1}{2}\alpha'\right)}$$

$$\approx \frac{\varrho e\sin\alpha'}{l}\left(1 - \frac{e}{l}\cos^2\tfrac{1}{2}\alpha'\right) \quad \ldots\ldots \quad (1)$$

From this formula one sees that δ is small for small values of α' and for large l's. The distance e is a source of personal errors. When the staff is not too thick (1.5 cm) it can easily be pressed in the inner corner of the eye, the heart-line of AB in the direction of the pupil. In order to avoid injury of the eye the staff must be rounded off in A. According to SCHMIDT [69] the distance e can then be reduced to about 3–6 mm. LEVI BEN GERSON makes mention of an amount of about 1 cm [70]. At any rate it is so small that in (1) $(e/l)\cos^2\frac{1}{2}\alpha'$ can be neglected with respect to 1. The error is then:

$$\delta \approx \frac{\varrho e\sin\alpha'}{l} \quad \ldots\ldots \quad (2)$$

For $e/l = 0.005$ (e.g. $e = 6$ mm and $l = 1.20$ m) and for δ expressed in minutes of arc:

$$\delta = 17.19\sin\alpha'.$$

For small values of α', δ is approximately directly proportional to α'. For $\alpha' = 5°$ e.g. $\delta = 1.5'$; for $\alpha' = 10°$, $\delta = 3.0'$; for $\alpha' = 20°$, $\delta = 5.9'$ and for $\alpha' = 30°$, $\delta = 8.6'$. $\delta = 17.2'$ for $\alpha' = 90°$ is of course not real as in that case $CD = 2l$, much too long to be used in practice. A smaller l in (2), however, introduces a larger δ. It is therefore plausible – I come presently to the practical results – that also for $\alpha' > 30°$, δ is approximately directly proportional to α'.

17 Elimination of systematic errors in readings on the cross-staff

In order to diminish the influence of δ as much as possible the constructors of cross-staffs shortened AB on the eye side with the small piece e. The zero point of the staff falls then in the eye. Already LEVI BEN GERSON made this change and also GEMMA FRISIUS applied it in his instruments. However, as e varies for every observer, the improvement is not quite effective. It can be said, however, that for long cross-staffs the error made is negligible.

δ can also be rendered harmless in another way. This is shown in fig. 13.

Instead of one cross-bar CD two cross-bars $C_1D_1 = a_1$ and $C_2D_2 = a_2$ are used. O is the eye in which the observed rays PC_1C_2 and QD_1D_2 intersect. As

$$\tan\tfrac{1}{2}\alpha = \frac{C_1F}{C_2F} = \frac{a_1 - a_2}{2E_2E_1} = \frac{a_1 - a_2}{2(l_1 - l_2)}$$

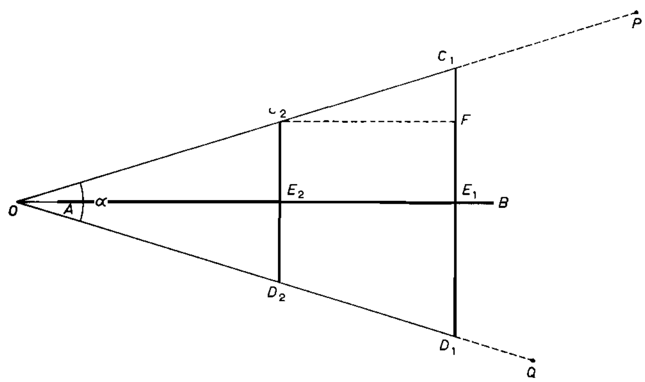


Fig. 13

α can be computed from the lengths a_1 and a_2 and from the difference of the readings $l_1 = AE_1$ and $l_2 = AE_2$.

The objection to this method is that it is very difficult to observe simultaneously whether P, C_1 and C_2 and Q, D_1 and D_2 lie on a straight line. Moreover, every observation requires a computation which is the more inaccurate as a_1 and a_2 (l_1 and l_2) differ less. It seems that this working method was used at sea with short cross-staffs. For long staffs it was not applied very often.

18 Example of readings on a cross-staff

In table 2 is given an example which shows that, when only one cross-bar is used, the influence of e manifests itself approximately in the way expressed by formula (2). I have mentioned there (column 2) 19 angles α' between 0° and 30°, measured principally with a cross-staff by TYCHO BRAHE and his assistants for the trigonometrical network which will be discussed presently and which is represented in fig. 15. The numbers in column 1 correspond with those in fig. 15. Column 3 gives the result of the computation of the angles α between the sighting points from the co-ordinates in the system of the Danish Geodetic Institute. I borrowed these data from NØRLUND [71]. One sees that all the differences 4 = 2−3 are positive: the angle α' read on the instrument is larger than the "real" angle α. According to (2) these differences δ are reduced to differences per 15° (column 5). They have a mean of about 12′ per 15° (24′ per 30°). The amount is not very reliable. A better result, however, could hardly be expected. For an

nº fig.15 1	α'(Tycho B.) 2	α (Geod. I.) 3	$\delta = \alpha' - \alpha$ 4	δ per 15° 5	α' - 12′ per 15° 6	v 3-6 7
2	23°45′	23°29.4′	+15.6′	+ 9.9′	23°27′	+2′
3	11 40	11 34.6	+ 5.4	+ 6.9	11 31	+ 4
6	4 30	4 19.8	+10.2	+34.0	4 26	− 6
7	17 35	17 32.7	+ 2.3	+ 2.0	17 21	+12
9	23 46	23 19.9	+26.1	+16.5	23 27	−7
10	4 44	4 40.7	+ 3.3	+10.5	4 40	+ 1
11	6 08	5 58.9	+ 9.1	+22.3	6 03	− 4
14	10 50	10 39.5	+10.5	+14.5	10 41	− 2
22	28 30	28 14.9	+15.1	+ 7.9	28 08	+ 7
26	1 28	1 27.2	+ 0.8	+ 8.2	1 27	0
36	9 52	9 44.4	+ 7.6	+11.6	9 44	0
40	20 57	20 40.8	+16.2	+11.6	20 41	0
42	20 24	19 57.3	+26.7	+20.0	20 08	−11
44	4 22	4 17.1	+ 4.9	+16.8	4 18	− 1
45	17 03	16 52.8	+10.2	+ 9.0	16 50	+ 3
47	23 50	23 28.1	+21.9	+13.8	23 31	− 3
48	6 13	6 11.3	+ 1.7	+ 4.1	6 08	+ 3
50	10 58	10 56.0	+ 2.0	+ 2.7	10 50	+ 6
54ecc	25 30 *	25 24.9	+ 5.1	+ 3.0	25 10 *	+15
				+12		

* See fig. 16

Table 2

angle $\alpha' \approx 20°$ e.g. will have been measured sometimes with a short cross-bar (small e/l, small δ), sometimes with a long one (large e/l, large δ). $\delta = +24'$ per 30°, corresponding with

$$\frac{e}{l} = \frac{24' \times 2}{3438'} \approx 0.014$$

is therefore at the most a mean. In its unreliability the errors in the observations are included as well as the centering errors, the influence of the errors caused by the often bad calibrations on the primitive instruments, eventual identification errors and mistakes of reading, and the reconstruction of several sighting points which exist no more. Column 6 gives the angles $\alpha' - 12'$ per 15° and column 7 the differences v between these angles and the values α.

It is interesting that NØRLUND arrives at the same amount for δ ($\delta \approx 80'$ per 100°) from a graphical adjustment of all angles in TYCHO's trigonometrical net. The speculations I gave in § 16, however, fail. From the differences v he computed a standard deviation $M \approx \pm 17'$ [72]. This rather large amount, much larger than can be derived from the 19 v's in table 2, is caused by some very large v's. They will be discussed in § 24.

As there is no alternative I used in my further computations the same angles α as those found by NØRLUND. There is still another reason to do this as there is no certainty that all the angles of the net were measured with the cross-staff. A smaller part of them will probably have been measured with the instrument represented in fig. 7 (§ 14).

19 Systematic errors in readings on a quadrant

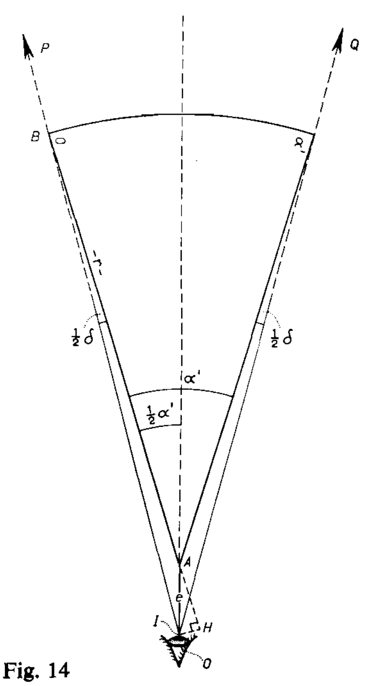


Fig. 14

As already said before it had a radius of about 1.55 m and, according to TYCHO "it was three inches (≈ 5 cm) wide and two inches (≈ 3 cm) thick" [73]. A is the centre of the limb BC. The leg AB is fixed; it has a sighting device B in the zero point of the graduation. The leg AC with a sighting device in C can be moved with a handle F. The observer's eye is as near as possible in I, at a distance $AI = e$ from A. This eccentricity gives rise to an error δ in the observation. It can be computed in an analogous way as the error in the observation with the cross-staff.

As (see fig. 14) $IH = e \sin \tfrac{1}{2}\alpha'$ and $AH = e \cos \tfrac{1}{2}\alpha'$

$$\tan \tfrac{1}{2}\delta = \frac{e \sin \tfrac{1}{2}\alpha'}{r + e \cos \tfrac{1}{2}\alpha'} \approx \frac{e}{r} \sin \tfrac{1}{2}\alpha' \left(1 - \frac{e}{r} \cos \tfrac{1}{2}\alpha'\right) \approx$$

$$\frac{e}{r} \sin \tfrac{1}{2}\alpha' - \left(\frac{e}{r}\right)^2 \sin \tfrac{1}{2}\alpha' \cos \tfrac{1}{2}\alpha'$$

or

$$\delta' \approx 3438 \left\{\frac{2e}{r} \sin \tfrac{1}{2}a' - \left(\frac{e}{r}\right)^2 \sin \alpha'\right\}$$

As one sees and as TYCHO BRAHE remarks on page 335 of his *Opera Omnia* II, δ is dependent on α' and on the proportion e/r.

Unfortunately this proportion is unknown as on the said page BRAHE makes only mention of the way how he computed δ; he gives no figures, however. In order to have an impression of δ, I scaled-off e/r from fig. 7 which seems not to be drawn too badly out of proportion. I found

$$\frac{e}{r} \approx 0.05$$

so that

$$\delta' \approx 3438\,(0.1 \sin \tfrac{1}{2}\alpha' - 0.0025 \sin \alpha') \qquad (3)$$

For $\alpha' = 0°$, $10°$, $20°$ and $30°$ respectively one finds $\delta = 0$, $\delta = 28.5'$, $\delta = 56.8'$ and $\delta = 84.7'$.

It is obvious that in practice the errors δ will be larger than these amounts, the eye being in O and not in I during the observation. BRAHE makes the same remark.

δ is approximately directly proportional to α'. Because of the large e, the amounts δ are much larger than for a cross-staff. It is obvious that it was necessary to correct the observations with these amounts. That this should not be forgotten "I had constructed a table and recorded it on the reverse side of the instrument in order that it should always be at hand" [74]. With the instrument, the table is lost, so the correctness of the δ's mentioned can not be proved. I don't know whether these negative corrections have always been given to the observations. Moreover it is unknown which angles must be corrected with these amounts. Owing to this, the uncertainty in the angles, already found in § 18, becomes still larger. One can ask oneself therefore whether an adjustment of the trigonometrical net has any sense. I have answered this question in the affirmative because, in spite of the unreliability of the observations, a good insight into the construction of the net and into its internal accuracy can then be obtained.

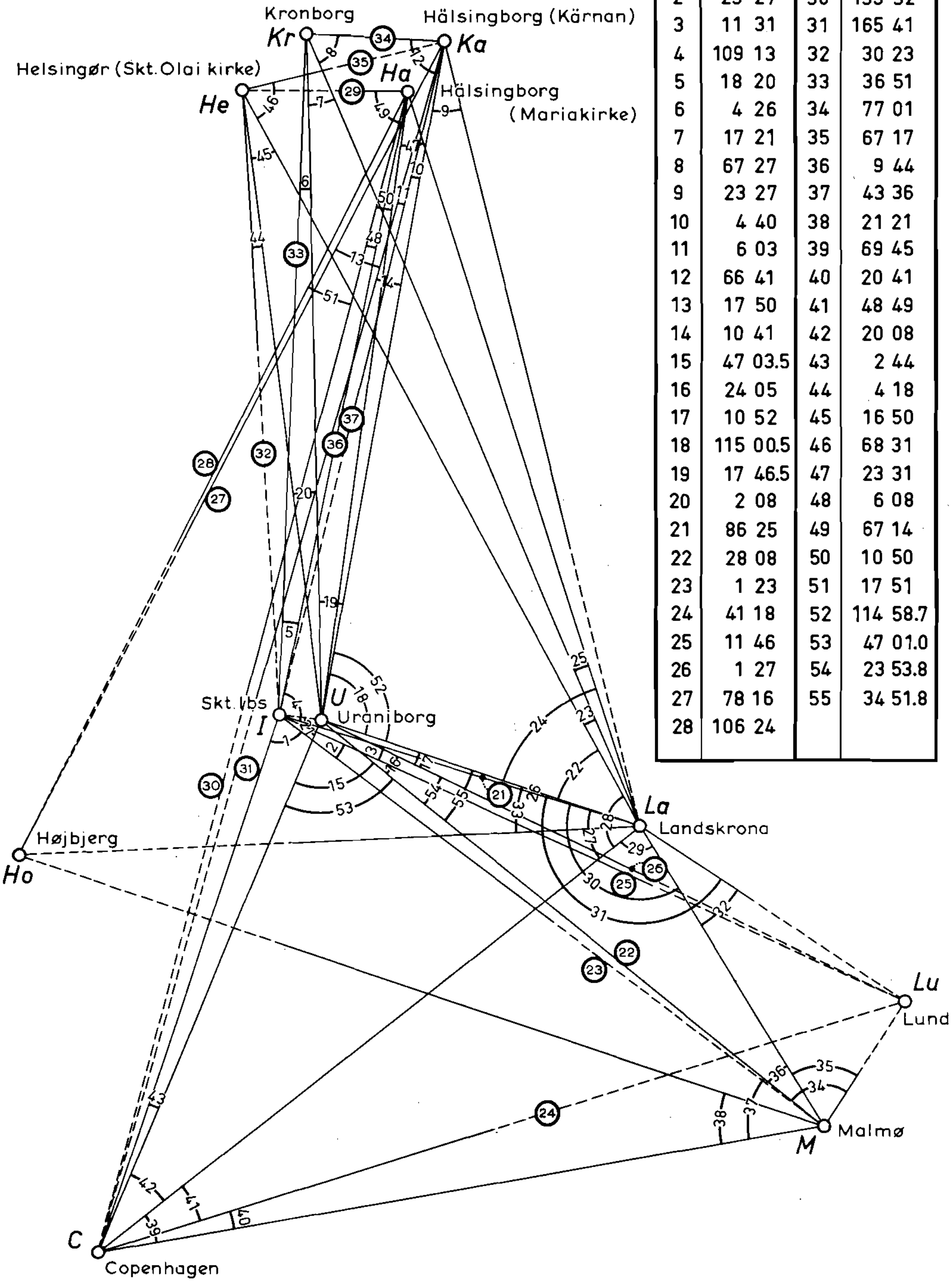
20 General view of his triangulation network, measurement of the base line and determination of the unit of length

In § 18 I said that the very deformed fig. 15 is a representation of the trigonometrical network, as it was measured by TYCHO BRAHE. It had to be the basis of a map of Denmark. As a matter of fact this is not quite correct, because some angles have not been measured directly but have been derived from his observations.

First of all it must be said that in Uraniborg, about the centre of the network, no angles were measured but astronomical azimuths to Copenhagen (26.6 km), Malmø (38.6 km), Lund (38.5 km), Landskrona (9.3 km), Hälsingborg Kärnan (15.6 km), Kronborg (15.3 km), Helsingør Skt. Olai kirke (15.1 km) and Skt. Ibs gamle kirke (1.3 km).

In order to give an impression of the dimensions of the network I mentioned in brackets the distances in km to the several angular points. The azimuths, which I borrowed from NØRLUND [76], are mentioned in two series in table 3.

The azimuth to Copenhagen e.g., counted from the north and in a clockwise direction, is 197° 18.5′, the azimuth to Malmø 150° 15′, etc. From their differences the angles 15 = 47° 03.5′ up to and including 21 have been computed. They are used as measured angles for the adjustment of the net.



1	45°34'	29	57°24'
2	23 27	30	135 32
3	11 31	31	165 41
4	109 13	32	30 23
5	18 20	33	36 51
6	4 26	34	77 01
7	17 21	35	67 17
8	67 27	36	9 44
9	23 27	37	43 36
10	4 40	38	21 21
11	6 03	39	69 45
12	66 41	40	20 41
13	17 50	41	48 49
14	10 41	42	20 08
15	47 03.5	43	2 44
16	24 05	44	4 18
17	10 52	45	16 50
18	115 00.5	46	68 31
19	17 46.5	47	23 31
20	2 08	48	6 08
21	86 25	49	67 14
22	28 08	50	10 50
23	1 23	51	17 51
24	41 18	52	114 58.7
25	11 46	53	47 01.0
26	1 27	54	23 53.8
27	78 16	55	34 51.8
28	106 24		

Fig. 15

The angles 52, 53, 54 and 55 are derived from the angles 108° 51′, 50° 20′, 25° 10′ and 43° 21′ in fig. 16. They are measured on the station Hven South, an observation hill south-east of Uraniborg and they are corrected with the amount 80′ per 100° already mentioned.

Azimuths in Uraniborg to	series *a*	series *b*	differen-ces	an-gles
Copenhagen, Frue Kirke		197°18.5′	47°03.5′	15
Malmø, Skt. Petri Kirke		150 15	24 05	16
Lund, Domkirke [75]		126 10	10 52	17
Landskrona, Skt. Joh. Bapt.		115 18	115 00.5	18
Hälsingborg, Kärnan		0 17.5	17 46.5	19
Kronborg, s.e. tower		342 31	2 08	20
Helsingør, Skt. Olai Kirke		340 23		
Hven, Skt. Ibs (Gamle Kirke)	283°25′		86 25	21
Copenhagen, Frue Kirke	197 00			

Table 3

As in not one of the points of the net outside the isle has been pointed at Hven South and as it is recommendable, however, to use these angles for its consolidation, I have regarded Hven South as an auxiliary point of Uraniborg. Each of the angles mentioned must get a correction for reduction to the centre in order to obtain the amounts of the corresponding angles 52–55 in Uraniborg. These corrections can be computed as the angle 26° 56′ in Hven South between Kärnan and Uraniborg is known as well as the distance Uraniborg–Hven South. This distance can be found from the length Uraniborg–Skt. Ibs, the base line of the net (830 *passus geometricus*) and the angles 17° 46′ and 133° 37′ which are also marked in the figure. The result, 1302.5 *passus*, can be verified with the distance 1280 *passus* which Tycho found by a direct measurement. The distances from Uraniborg to Kärnan, Landskrona, Malmø, Lund and Copenhagen which are also necessary for the computation of the correction of the reduction to centre, are borrowed from the co-ordinates of the points (see table 4). They are copied from Nørlund [76]. On account of the detailed considerations concerning Snellius' triangulation (§§ 28–50) I mentioned these co-ordinates in an analogous way as used in the Netherlands (positive X'-axis to the east, positive Y'-axis to the north).

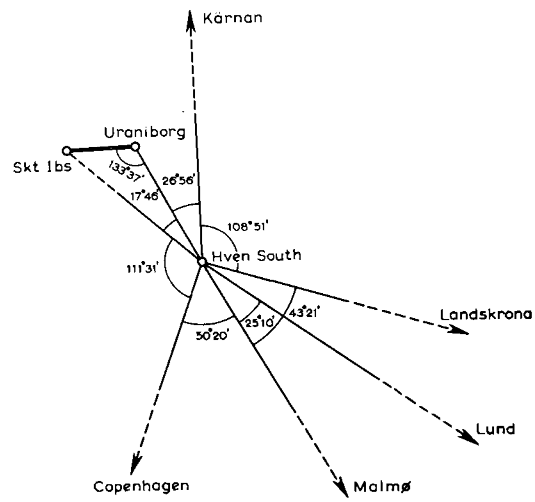


Fig. 16

Points	Co-ordinates	
	X'	Y'
Uraniborg, Obs. Centre	+145097.71	+103506.47
Hälsingborg, Kärnan	+144628.03	+119133.43
Landskrona, Skt. Joh. Bapt. [77]	+153638.—	+ 99858.—
Malmø, Skt. Petri Kirke	+165505.25	+ 70707.39
Lund, Domkirke, southern tower	+177026.60	+ 81988.90
Copenhagen, Frue Kirke	+138130.75	+ 77816.56
Hven, Skt. Ibs (Gamle Kirke)	+143833.97	+103754.77

Table 4

In § 12 I said already that until 1943 the exact length of Tycho Brahe's standard measure, the *passus geometricus*, was unknown. Nørlund derived it then from the co-ordinates which, after the reconstruction of the terminal points of the base line Uraniborg and Skt.

Ibs, were given to these points (see table 4). On the pages 35 and 36 of his book he gives an elaborate description of this reconstruction by means of still existing foundations of the buildings. I don't know whether the accuracy of the reconstruction justifies such a great accuracy of the co-ordinates.

From these co-ordinates one finds 1287.90 m, corresponding with 830 *passus*. The length of 1 *passus* is therefore 1.552 m. The distances 1302.5 *passus* and 1280 *passus* which were found for the distances Uraniborg–Hven South are therefore 2021.5 m and 1986.5 m respectively. The difference is 35 m, about 2 percent, a very large amount for a geodesist of the 20th century.

For the computation of the corrections for the reduction to the centre Uraniborg I used the mean 2004.0 m. As the point Hven South could also be reconstructed and fixed in co-ordinates ($X' = +146155.59$, $Y' = +101800.28$) it was also possible to use the distance Hven South – Uraniborg from the co-ordinates of the terminal points (2007.4 m). I did not do that in order to make myself as much as possible independent of any uncertainties in the reconstruction of the points on the isle. In my opinion it gives a better insight into the internal accuracy of TYCHO's measurement.

The attentive reader will have noticed that the sum of the angles around the point Hven South in fig. 16, is not 360° but 358°45′. A provisional computation, however, does suspect – be it with all reserve – that the angle $110°17.5' - 86' \approx 108°51'$ in that figure has an error of 1°. In my opinion it must be $111°17.5' - 87' \approx 109°50'$. Not only the error of 1°15′ in the sum of the angles is reduced then to "only" 16′, but also the computed angle in Uraniborg $52 = 109°50' + 5°08.7' = 114°58.7'$ agrees much better with the angle 114°51′17″ which can be found from the difference of the gridbearings in Uraniborg to Landkrona (112°07′57″) and Kärnan (358°16′40″). The accordance with the angle $18 = 115°00.5'$ in table 3 is also very satisfactory.

The angles 53 up to and including 55 have the following values:

$$53 = 50°20' - 3°19.0' = 47°01.0'$$
$$54 = 25°10' - 1°16.2' = 23°53.8'$$
$$55 = 43°21' - 8°29.2' = 34°51.8'$$

In the sketch and in the table of fig. 15 they are mentioned as observations in Uraniborg.

21 Speculations on the measurement of the base line

The triangulation network has 11 angular points. In two of them no measurements have been carried out, namely on the cathedral of Lund (Sweden) and on the hill which TYCHO BRAHE calls Vedbecksbierg (hill near Vedbaek). It lies on the Danish island Sjaelland, about south-west of the village of Sandbjerg and it has a height of approximately 82 metres above sealevel. In fig. 17, a reproduction of the Danish topographical map, it is called Højbjerg. I gave it the same name on the sketch in fig. 15. The point is not known in the co-ordinate system of the Danish Geodetic Institute. From TYCHO's name Vedbecksbierg one might conclude that from Malmø, Landskrona, Hälsingborg (Kärnan) and Hälsingborg (Maria kirke) was pointed at a salient point on the top of the hill, e.g. a big tree.

A local investigation made at my request by Mr. and Mrs. SKAT RØRDAM–GERLING from Virum (Denmark) has given no trace of a building whatsoever on that hill and also the

Fig. 17

detailed topographical work J. P. Trap: *Danmark* does not make mention of such a building. The top of the Højbjerg has a plateau of about 100 × 30 m, overgrown with very high trees. The slopes of the hill are also overgrown with trees and bushes.

It is peculiar that neither Trap nor his predecessor in the historical-topographical field Erik Pontoppidan (professor and bishop, 1698–1764) use the name Vedbeaksbjerg in their publications. They both call it Højbjerg. Notwithstanding diligent and highly appreciated investigations by the municipal library of Søllerød in the past few months, it could not be proved that Højbjerg-Vedbaeksbjerg are two different names for one and the same hill. Even the name Vedbaeksbjerg could not be found.

The identity, however, cannot be doubted (see my conclusions in § 25). Moreover, there is no alternative: there is no other important hill in the neighbourhood. It seems therefore plausible that TYCHO BRAHE, insufficiently aquainted with the local name, called the hill Vedbaeksbjerg for his own use.

In § 20 I gave already some distances in the trigonometrical network. It is obvious that because of the very insufficient construction of its northern part the mutual position of the points *He*, *Kr*, *Ka* and *Ha* will be very bad, the distance between *He* and *Kr* being only about 591 m and that between *Ka* and *Ha* 249 m. The distances *KrKa* and *HeHa* across The Sound are about 4.8 and 5.2 km respectively.

If TYCHO BRAHE would have computed his triangulation – but he did not do that – he would have been obliged to compute e.g. the side Uraniborg–Kronborg of his network from his base $UI = 1287.90$ m and the two angles $Kr = 6 = 4°26'$ and $I = (4+5+21+15+16+17+26)-180° = 117°25.5'$ mentioned in fig. 18.

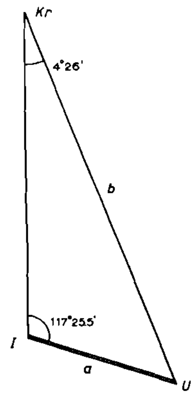


Fig. 18

He would have found then $b = 14789$ m, an amount which differs about 493 m (3.2 percent) from the "real" value 15281.7 m [78].

This difference can be illustrated by the large standard deviation in the distance.

As

$$b = \frac{a \sin I}{\sin Kr}, \quad \frac{\partial b}{\partial a} = \frac{\sin I}{\sin Kr}, \quad \frac{\partial b}{\partial I} = \frac{a \cos I}{\sin Kr}$$

and

$$\frac{\partial b}{\partial Kr} = \frac{-a \sin I \cos Kr}{\sin Kr} : \sin Kr = \frac{-b \cos Kr}{\sin Kr},$$

$$m_b^2 = \frac{1}{\sin^2 Kr}\{\sin^2 I m_a^2 + (a \cos I)^2 m_I^2 + (b \cos Kr)^2 m_{Kr}^2\} \qquad (4)$$

If the standard deviation in the measurement of an anlge of the triangulation network is m_α then

$$m_{Kr}^2 = m_\alpha^2 \quad \text{and} \quad m_I^2 = 7 m_\alpha^2.$$

For $Kr = 4°26'$, $I = 117°25\frac{1}{2}'$, $m_a = 0$ and $m_\alpha = 5.9' = 0.001716$ rad (see § 24) (4) runs

$$m_b^2 = \frac{1}{(0.07730)^2}(7.389 + 652.226) = 110396$$

so that $m_b = \pm$ 332 m.

As $(a \cos I)^2\, m_I^2 = 7.389$ is very small with respect to

$$(b \cos Kr)^2 m_{Kr}^2 = 652.226,$$

one can compute roughly

$$m_b \approx b \cot Kr\, m_{Kr} \approx 327 \text{ m},$$

in spite of the factor 7 in the first amount.

It is obvious that this inadmissable amount is caused by the very small top angle (4°26') in Kronborg.

The standard deviation in the side Uraniborg–Copenhagen (27003 m) can be computed in an analogous way in triangle *UIC* (top angle 2°44'). One finds even $m_{UC} = 971$ m.

Triangle *UILa* (top angle 1° 27′) is totally unfit for the computation of the side *ULa*.

The short base *IU* to which the long sides of the triangulation net are directly connected is of course caused by much too small top angles and consequently much too large cotangents. As the isle of Hven is too small for a considerable improvement of the said standard deviations the base should have been chosen on the continent, e.g. and if possible, between Malmø and Lund, a distance of about 16 km. The length of this side should have been determined as sketched in fig. 19. In this figure *AB* is the base *c*. The angles necessary for the computation of *PQ* are indicated with arcs of circles. 35 years later and for the first time in history of geodesy this method was applied by SNELLIUS. He demonstrated with it an excellent practical feeling for this important problem. Two centuries after SNELLIUS (in 1820) SCHWERD proved that SNELLIUS' practical insight was affirmed by mathematical considerations [79].

CD can be computed from

$$F = CD^2 = a_1{}^2 + a_2{}^2 - 2a_1a_2\cos(\beta_1+\beta_2) =$$
$$= c^2\left\{\frac{\sin^2\alpha_1}{\sin^2(\alpha_1+\beta_1)} + \frac{\sin^2\alpha_2}{\sin^2(\alpha_2+\beta_2)} - \frac{2\sin\alpha_1\sin\alpha_2\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\beta_1)\sin(\alpha_2+\beta_2)}\right\} \qquad (5)$$

and the standard deviation m_{CD} from

$$\left(\frac{\partial F}{\partial CD}\right)^2 m_{CD}{}^2 = \left(\frac{\partial F}{\partial c}\right)^2 m_c{}^2 + \left(\frac{\partial F}{\partial \alpha_1}\right)^2 m_{\alpha_1}{}^2 + \left(\frac{\partial F}{\partial \alpha_2}\right)^2 m_{\alpha_2}{}^2 +$$
$$+ \left(\frac{\partial F}{\partial \beta_1}\right)^2 m_{\beta_1}{}^2 + \left(\frac{\partial F}{\partial \beta_2}\right)^2 m_{\beta_2}{}^2$$

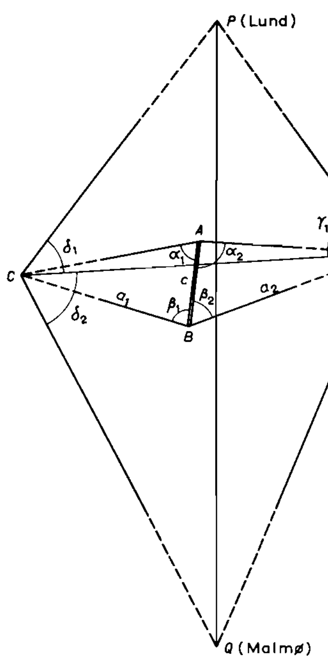


Fig. 19

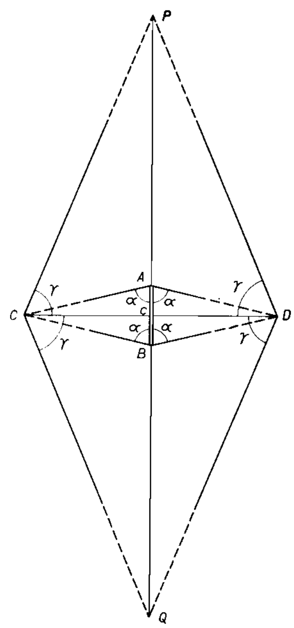


Fig. 20

Worked out this formula runs

$$m_{CD}^2 = \left(\frac{CD}{c}\right)^2 m_c^2 + \left(\frac{c^2}{2CD}\right)^2 \{A^2 m_{\alpha_1}^2 + B^2 m_{\alpha_2}^2 + C^2 m_{\beta_1}^2 + D^2 m_{\beta_2}^2\} \quad . \quad . \quad . \quad . \quad (6)$$

with

$$A = \frac{\sin(\alpha_1+\beta_1)\sin 2\alpha_1 - 2\sin^2\alpha_1\cos(\alpha_1+\beta_1)}{\sin^3(\alpha_1+\beta_1)} - \frac{2\sin\alpha_2\cos(\beta_1+\beta_2)\sin\beta_1}{\sin(\alpha_2+\beta_2)\sin^2(\alpha_1+\beta_1)}$$

$$B = \frac{\sin(\alpha_2+\beta_2)\sin 2\alpha_2 - 2\sin^2\alpha_2\cos(\alpha_2+\beta_2)}{\sin^3(\alpha_2+\beta_2)} - \frac{2\sin\alpha_1\cos(\beta_1+\beta_2)\sin\beta_2}{\sin(\alpha_1+\beta_1)\sin^2(\alpha_2+\beta_2)}$$

$$C = \frac{2\sin\alpha_1\sin\alpha_2\cos(\alpha_1-\beta_2)}{\sin(\alpha_2+\beta_2)\sin^2(\alpha_1+\beta_1)} - \frac{2\sin^2\alpha_1\cos(\alpha_1+\beta_1)}{\sin^3(\alpha_1+\beta_1)} \quad \text{and}$$

$$D = \frac{2\sin\alpha_1\sin\alpha_2\cos(\alpha_2-\beta_1)}{\sin(\alpha_1+\beta_1)\sin^2(\alpha_2+\beta_2)} - \frac{2\sin^2\alpha_2\cos(\alpha_2+\beta_2)}{\sin^3(\alpha_2+\beta_2)}$$

In an analogous way PQ can be computed with (5) and m_{PQ} with (6).

In the standardized fig. 20, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha$, $CD = c \tan\alpha$ and $m_{\alpha_1} = m_{\alpha_2} = m_{\beta_1} = m_{\beta_2} = m_\alpha$. In this case (6) runs:

$$m_{CD}^2 = \tan^2\alpha m_c^2 + \left\{\frac{c(1-\cos 2\alpha)m_\alpha}{\sin^2 2\alpha}\right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

With this formula and with the analogous formula for m_{PQ}^2 I computed to which amount the standard deviation in a side PQ = 16 km could have been reduced if TYCHO BRAHE would have known this method of base extension.

I chose AB = 1600 m, but little longer than UI on Hven and even shorter than the distance Uraniborg–Hven South. α was chosen

$$\alpha \quad = \arctan\frac{3000 \text{ m}}{800 \text{ m}} = 75°04',$$

$$\gamma \quad = \arctan\frac{8000 \text{ m}}{3000 \text{ m}} = 69°27',$$

$$m_\alpha = m_\gamma = 5.9' = 0.00172 \text{ rad (see end § 24) and}$$

$$m_c^2 = 50 \text{ m}^2,$$

corresponding with the fixation of the mean c = 1600 m from two "measurements" 1595 m and 1605 m.

One finds $m_{CD} \approx \pm$ 33.7 m and $m_{PQ} \approx \pm$ 99.0 m. It is much better than TYCHO's result, in spite of the large m_c introduced in the measurement of the base AB. The choice of his base, even between two inaccessible points (spires), and his base extension can therefore not bear the touch of criticism. In this respect the great astronomer TYCHO BRAHE shows himself far inferior to SNELLIUS.

22 Influence of the eccentricity of the observations

As the angular points of TYCHO's network are spires all angles had to be measured outside the centre. As these angles were not corrected for reduction to centre he made an error δ which can be derived from fig. 21. It is zero when the angle is measured e.g. in O, a point of the circumscribed circle of the triangle PQS, the angles of which must be determined. The greatest error occurs when the line which connects centre and observation station goes through the centre of the circle. It is then

$$\delta' = \varrho e\left(\frac{\sin p_1}{l_1} + \frac{\sin p_2}{l_2}\right) =$$

$$= \varrho e \frac{l_2 \sin p_1 + l_1 \sin p_2}{l_1 l_2}$$

with $l_1 \approx 2r\cos p_1$, $l_2 \approx 2r\cos p_2$ and $\varrho = 3438$. δ is therefore

$$\delta' = 3438e \cdot 2r \frac{\sin p_1 \cos p_2 + \cos p_1 \sin p_2}{l_1 l_2}$$

$$= 3438e \cdot 2r \frac{\sin(p_1 + p_2)}{l_1 l_2}$$

As

$$2r = \frac{l_3}{\sin\alpha} \quad \text{and} \quad \alpha \approx p_1 + p_2$$

$$\delta' = 3438e \frac{l_3}{l_1 l_2}$$

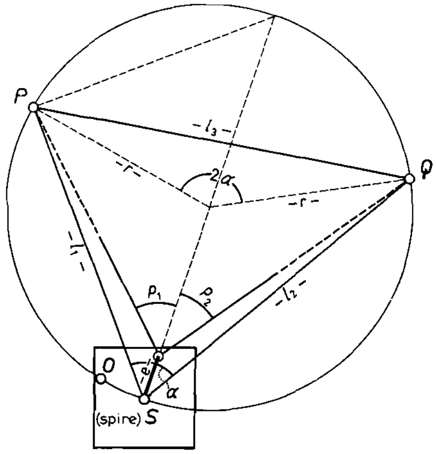


Fig. 21

For a rather large eccentricity $e = 3$ m and for $l_1 = 15$ km, $l_2 = 20$ km and $\alpha = 90°$ ($l_3 = 25$ km) δ is 0.9′. For $l_1 = l_2 = l_3 = 20$ km and $e = 2$ m, $\delta = 0.3'$. These errors are much smaller than the accuracy of the observations so that they can be neglected. Larger errors δ_{max} occur in e.g. the angles 8 and 46 in fig. 15. For $e = 2.5$ m they are about 1.7′ and 1.5′ respectively. They are also negligible.

It is obvious that by the inaccuracy of the observations the spherical shape of the earth can also be neglected. For the spherical excess of one of the greatest spherical triangles of the net, the triangle Landskrona – Malmø – Copenhagen, is only about two seconds of arc. I therefore assumed that the angular points lie in a flat plane. In order to compare the results of the adjustment with the data of the Geodetic Institute this plane is assumed to coincide with the plane of projection in which the co-ordinates of these points have been computed.

23 Condition equations

In the unsurveyable triangulation network of fig. 15 with 36 sides ($L = 36$), 55 angles have been measured for the determination of the mutual position of 11 points ($P = 11$).

CONDITION

	STATION EQUATIONS											ANGLE EQUATIONS								
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*	*r*	*s*	*t*
1																		+1.000		
2																		+1.000		
3																		+1.000		
4																	+1.000		+1.000	
5																	+1.000			
6																	+1.000			
7																+1.000	+1.000			+1.000
8																+1.000				
9																				
10											+1.000									
11											+1.000									
12																+1.000				
13																				
14											-1.000					+1.000				
15									+1.000					+1.000						
16							+1.000	+1.000					+1.000	+1.000						
17							+1.000						+1.000	+1.000						
18										+1.000					+1.000					+1.000
19															+1.000	+1.000				+1.000
20															+1.000					
21																				
22					+1.000	+1.000									+1.000		+1.000			+1.000
23						+1.000											+1.000			+1.000
24						-1.000													+1.000	
25						+1.000														
26																	+1.000	-1.000	+1.000	
27				+1.000	+1.000									+1.000				+1.000		
28					-1.000															
29				+1.000								+1.000								
30			+1.000	-1.000									+1.000							
31			-1.000																	
32			+1.000																	
33																				
34		-1.000																		
35		+1.000																		
36		+1.000										+1.000	+1.000							
37												+1.000								
38																				
39	-1.000											+1.000								
40	+1.000																			
41	+1.000																			
42														+1.000				+1.000		
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52										-1.000										
53									-1.000											
54								-1.000												
55							-1.000													
W	-15.00	0.00	+14.00	+8.00	0.00	-1.00	+5.20	+11.20	+2.50	+1.80	+2.00	+29.00	+13.00	+24.50	-7.00	-3.50	+18.00	+13.00	+11.00	-21.00
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Table 5

EQUATIONS

SIDE EQUATIONS

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
u	v	w	x	y	z	α	β	γ	δ	ε	ζ	η	θ	λ	μ	ν
		+1.238		-0.250												
		-1.259		+2.912												
		-1.259		-6.200												
												+0.971			+0.193	
												+0.971				
												+3.162				
												+0.302	+4.044			
													-0.525			
						-1.924				+1.048			-3.154	-3.094		-4.338
																+15.291
													-0.242			
						+2.061										
						+2.061				-4.448			-0.242			
+0.728	+0.997	-1.114									+0.129					
+0.728	-0.178	+1.259			+2.827	-1.808	-1.808				+0.129					
+0.728	-0.178	+1.259			-6.581	-1.808	-1.808				+0.129					
						-1.662		+1.260		+3.173			+0.580	+0.671		-1.240
								+1.260					+1.169	+1.442		
								+1.260						+1.442		
+0.906		+0.061									+0.129				+0.212	+0.205
											-0.036	+0.953				
								+0.582				+1.184				
							+1.667	+0.594	+3.175						+1.438	
								+0.582								
+7.289		+1.259		+4.474							+1.191	+1.034			+0.193	
-0.215	+0.263								+2.768	-2.771					-0.456	-0.449
	-0.808		+0.049													
						-1.085	-1.085									
				-4.474	-4.950											
			-2.106	-2.155	-2.155											
						+2.158	+2.158									
			+0.747		-0.291											
			+0.529	+0.279	+0.529											
	+0.941			-0.250		+4.258	+4.258									
	-0.386		+0.747	-0.250		-3.108	-3.108									
						+3.108	+3.108									
		+0.060		-0.250												
			+3.346													
			-1.106													
+0.236		+0.060		-0.250					+3.633	-3.633					+3.253	+3.260
-3.210		+0.057		-0.250											-26.451	-26.451
											-0.328	-2.234				
								+4.176			+0.102	+0.677		+4.358		
														-0.315		
							-0.256	-1.986	+2.121						-2.903	
															-15.367	
								+0.323								
							+2.053	+0.323	+4.430						+15.367	
							+2.053									
+27.65	-3.08	+2.14	-27.00	+29.91	+7.67	-22.52	-27.13	+1.52	+54.71	-52.30	+2.62	+33.28	-69.59	-40.97	+45.86	+43.72
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37

Table 5

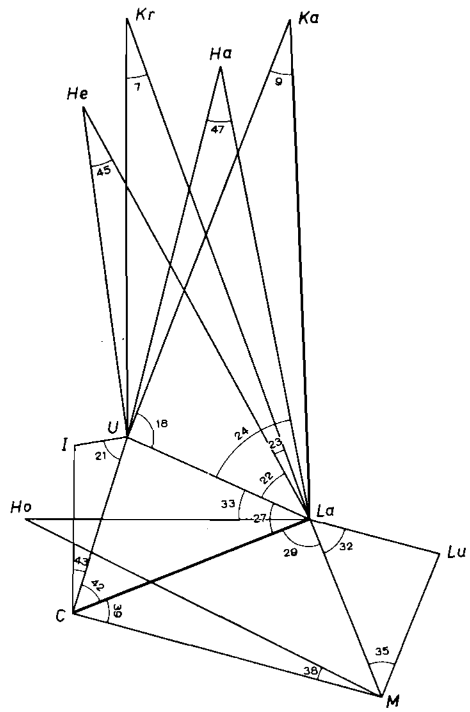


Fig. 22

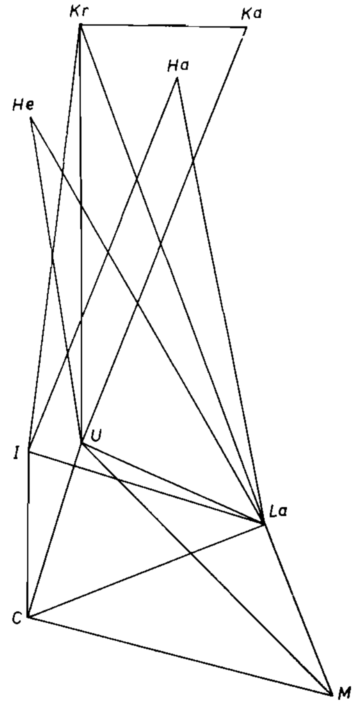


Fig. 23

In fig. 22 I mentioned the 18 angles (19 sides) necessary for the construction of the net. For this construction I started from the arbitrary side *CLa*. As there are $55-18 = 37$ redundant angles there are 37 conditions. $36-19 = 17\,(L-2P+3 = 17)$ of these conditions are side equations [80].

Fig. 23 gives the 17 sides ($l = 17$) of a number of triangles of which all angles were measured. This part of the network has 9 angular points ($p = 9$). The number of angle equations is therefore $l-p+1 = 9$.

The remaining $37-(17+9) = 11$ conditions are therefore station equations. They are indicated as 1, ..., 11 (a, ..., k) in the columns of table 5. An example of such a condition equation (No. 1) relates to the angles 39, 40 and 41, measured in Copenhagen. If the corrections to these angles are p_{39}, p_{40} and p_{41} respectively (p in minutes of arc) then $69°45'+p_{39} = 20°41'+p_{40}+48°49'+p_{41}$ or $-p_{39}+p_{40}+p_{41}-15.00 = 0$.

The angle equations are mentioned in the columns 12, ..., 20 (l, ..., t). The first of these equations runs $p_{29}+p_{36}+p_{37}+p_{39}+29.00 = 0$. It relates to the corrections to the angles of the triangle *LaMC*. The others can easily be found with fig. 23. The closing errors in the triangles – 6 positive and 3 negative – agree rather well with the assumed systematic error of 80′ per 100° in the measured angles (see § 18).

In the side equations 21, ..., 37 (u, ..., v) the connection with fig. 15 is not immediately clear. I therefore give them first in another form

21 (u) $$\frac{LaC \cdot LaI \cdot LaU}{LaI \cdot LaU \cdot LaC} = 1$$

22 (v) $$\frac{CU \cdot CLa \cdot CM}{CLa \cdot CM \cdot CU} = 1$$

23 (w) $$\frac{MC \cdot MI \cdot MU}{MI \cdot MU \cdot MC} = 1$$

24 (x) $$\frac{LuM \cdot LuC \cdot LuLa}{LuC \cdot LuLa \cdot LuM} = 1$$

25 (y) $$\frac{LuM \cdot LuI \cdot LuLa}{LuI \cdot LuLa \cdot LuM} = 1$$

26 (z) $$\frac{LuM \cdot LuU \cdot LuLa}{LuU \cdot LuLa \cdot LuM} = 1$$

27 (α) $$\frac{LaM \cdot LaHo \cdot LaKa \cdot LaU}{LaHo \cdot LaKa \cdot LaU \cdot LaM} = 1$$

28 (β) $$\frac{LaM \cdot LaHo \cdot LaHa \cdot LaU}{LaHo \cdot LaHa \cdot LaU \cdot LaM} = 1$$

29 (γ) $$\frac{LaU \cdot LaHe \cdot LaHa}{LaHe \cdot LaHa \cdot LaU} = 1$$

30 (δ) $$\frac{LaC \cdot LaU \cdot LaHa}{LaU \cdot LaHa \cdot LaC} = 1$$

31 (ε) $$\frac{LaC \cdot LaU \cdot LaKa}{LaU \cdot LaKa \cdot LaC} = 1$$

32 (ζ) $$\frac{HeLa \cdot HeU \cdot HeI}{HeU \cdot HeI \cdot HeLa} = 1$$

33 (η) $$\frac{LaHe \cdot LaU \cdot LaKr \cdot LaI}{LaU \cdot LaKr \cdot LaI \cdot LaHe} = 1$$

34 (θ) $$\frac{LaU \cdot LaKr \cdot LaKa}{LaKr \cdot LaKa \cdot LaU} = 1$$

35 (λ) $$\frac{LaU \cdot LaHe \cdot LaKa}{LaHe \cdot LaKa \cdot LaU} = 1$$

36 (μ) $$\frac{UC \cdot UI \cdot UHa \cdot ULa}{UI \cdot UHa \cdot ULa \cdot UC} = 1$$

37 (ν) $$\frac{UC \cdot UI \cdot UKa \cdot ULa}{UI \cdot UKa \cdot ULa \cdot UC} = 1$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	[a	[b	[c	[d	[e	[f	[g	[h	[i	[j	[k	[l	[m	[n	[o	[p	[q	[r
a]	+3000	–	–	–	–	–	–	–	–	–	–	-1 000	–	–	–	–	–	–
b]	–	+3000	–	–	–	–	–	–	–	–	–	+1 000	+1 000	–	–	–	–	–
c]	–	–	+3000	-1 000	–	–	–	–	–	–	–	–	+1 000	–	–	–	–	–
d]	–	–	-1 000	+3 000	+1 000	–	–	–	–	–	–	+1 000	-1 000	+1 000	–	–	–	+1 000
e]	–	–	–	+1 000	+3 000	+1 000	–	–	–	–	–	–	–	+1 000	+1 000	–	+1 000	+1 000
f]	–	–	–	–	+1 000	+4 000	–	–	–	–	–	–	–	–	+1 000	–	+2 000	–
g]	–	–	–	–	–	–	+3 000	+1 000	–	–	–	–	+2 000	+2 000	–	–	–	–
h]	–	–	–	–	–	–	+1 000	+2 000	–	–	–	–	+1 000	+1 000	–	–	–	–
i]	–	–	–	–	–	–	–	–	+2 000	–	–	–	–	+1 000	–	–	–	–
j]	–	–	–	–	–	–	–	–	–	+2 000	–	–	–	–	+1 000	–	–	–
k]	–	–	–	–	–	–	–	–	–	–	+3 000	–	–	–	–	-1 000	–	–
l]	-1 000	+1 000	–	+1 000	–	–	–	–	–	–	–	+4 000	+1 000	–	–	–	–	–
m]	–	+1 000	+1 000	-1 000	–	–	+2 000	+1 000	–	–	–	+1 000	+4 000	+2 000	–	–	–	–
n]	–	–	–	+1 000	+1 000	–	+2 000	+1 000	+1 000	–	–	–	+2 000	+5 000	–	–	–	+2 000
o]	–	–	–	–	+1 000	+1 000	–	–	–	+1 000	–	–	–	–	+5 000	+1 000	+1 000	–
p]	–	–	–	–	–	–	–	–	–	–	-1 000	–	–	–	+1 000	+5 000	+1 000	–
q]	–	–	–	–	+1 000	+2 000	–	–	–	–	–	–	–	–	+1 000	+1 000	+7 000	-1 000
r]	–	–	–	+1 000	+1 000	–	–	–	–	–	–	–	–	+2 000	–	–	-1 000	+7000
s]	–	–	–	–	–	-1 000	–	–	–	–	–	–	–	–	–	–	+2 000	-1 000
t]	–	–	–	–	+1 000	+2 000	–	–	–	+1 000	–	–	–	–	+3 000	+2 000	+3 000	–
u]	–	–	–	- 215	- 215	–	+1 456	+ 728	+ 728	–	–	–	+1 456	+2 205	–	–	+7 289	-10 478
v]	–	+ 941	–	- 545	+ 263	–	- 356	- 178	+ 997	–	–	- 253	+ 585	+ 904	–	–	–	+ 263
w]	- 60	–	–	–	–	–	+2 518	+1 259	-1 114	–	–	+ 60	+2 518	+1 464	–	–	+1 259	-2 422
x]	+2 240	- 218	-2 106	+ 49	–	–	–	–	–	–	–	+ 796	–	–	–	–	–	–
y]	+ 250	+ 29	+2 319	–	–	–	–	–	–	–	–	- 750	- 250	- 250	–	–	+4 474	-8 512
z]	–	+ 820	+2795	–	–	–	-3754	+2 827	–	–	–	–	-3754	-3754	–	–	–	–
α]	–	+4 258	-1 085	+1 085	–	–	-3 616	-1 808	–	-1 662	-2 061	+1150	- 443	-3 616	-1 662	+2061	–	–
β]	–	+4 258	-1 085	+1 085	–	-1 667	-3 616	-1 808	–	–	–	+1150	- 443	-3 616	–	–	–	–
γ]	–	–	–	–	–	+ 570	–	–	–	+1 260	–	–	–	–	+7956	+1 260	+ 582	–
δ]	–	–	–	+2 768	+2 768	-3 175	–	–	–	–	–	–	–	+6 401	–	–	–	+6 401
ε]	–	–	–	-2 771	-2771	–	–	–	–	+3173	+4448	–	–	-6 404	+3173	-4448	–	-6 404
ζ]	–	–	–	–	- 36	- 36	+ 258	+ 129	+ 129	–	–	–	+ 258	+ 387	+ 66	–	+1155	-1 191
η]	–	–	–	–	+ 953	+2137	–	–	–	–	–	–	–	–	+1630	+ 302	+8 577	-1 034
θ]	–	–	–	–	–	–	–	–	–	+ 580	+ 242	–	–	–	+1 749	+4 204	+4044	–
λ]	–	–	–	–	–	–	–	–	–	+ 671	–	–	–	–	+7913	+1 442	–	–
μ]	–	–	–	- 456	- 456	-1 438	–	–	–	–	–	–	–	+2 797	–	–	+ 386	-23847
ν]	–	–	–	- 449	- 449	–	–	–	–	-1 240	+15 291	–	–	+2 811	-1 240	–	–	-23 640
W	-15000	0	+14000	+8000	0	-1000	+5 200	+11200	+2 500	+1 800	+2000	+29000	+13000	+24500	-7000	-3500	+18000	+13000
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Table 6

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
[s	[t	[u	[v	[w	[x	[y	[z	[α	[β	[γ	[δ	[ε	[ζ	[η	[θ	[λ	[μ	[ν
—	—	—	—	- 60	+2240	+ 250	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	+ 941	—	- 218	+ 29	+ 820	+4258	+4258	—	—	—	—	—	—	—	—	—
—	—	—	—	—	-2106	+2319	+2795	-1085	-1085	—	—	—	—	—	—	—	—	—
—	—	- 215	- 545	—	+ 49	—	—	+1085	+1085	—	+2768	-2771	—	—	—	—	- 456	- 449
—	+1000	- 215	+ 263	—	—	—	—	—	—	—	+2768	-2771	- 36	+ 953	—	—	- 456	- 449
-1000	+2000	—	—	—	—	—	—	—	-1667	+ 570	-3175	—	- 36	+2137	—	—	-1438	—
—	—	+1456	- 356	+2518	—	—	-3754	-3616	-3616	—	—	—	+ 258	—	—	—	—	—
—	—	+ 728	- 178	+1259	—	—	+2827	-1808	-1808	—	—	—	+ 129	—	—	—	—	—
—	—	+ 728	+ 997	-1114	—	—	—	—	—	—	—	—	+ 129	—	—	—	—	—
—	+1000	—	—	—	—	—	—	-1662	—	+1260	—	+3173	—	—	+ 580	+ 671	—	-1240
—	—	—	—	—	—	—	—	-2061	—	—	—	+4448	—	—	+ 242	—	—	+15291
—	—	—	- 253	+ 60	+ 796	- 750	—	+1150	+1150	—	—	—	—	—	—	—	—	—
—	—	+1456	+ 585	+2518	—	- 250	-3754	- 443	- 443	—	—	—	+ 258	—	—	—	—	—
—	—	+2205	+ 904	+1464	—	- 250	-3754	-3616	-3616	—	+6401	-6404	+ 387	—	—	—	+2797	+2811
—	+3000	—	—	—	—	—	—	-1662	—	+7956	—	+3173	+ 66	+1630	+1749	+7913	—	-1240
—	+2000	—	—	—	—	—	—	+2061	—	+1260	—	-4448	—	+ 302	+4204	+1442	—	—
+2000	+3000	+7289	—	+1259	—	+4474	—	—	—	+ 582	—	—	+1155	+8577	+4044	—	+ 386	—
-1000	—	-10478	+ 263	-2422	—	-8512	—	—	—	—	+6401	-6404	-1191	-1034	—	—	-23847	-23640
+6000	—	+7289	—	+1259	—	+4474	—	—	+3464	-1069	+9726	—	+1191	+2005	—	—	+29655	—
—	+5000	—	—	—	—	—	—	-1662	—	+3102	—	+3173	- 36	+2439	+5793	+2113	—	-1240
+7289	—	+65946	+ 410	+10085	—	+33354	-2733	-2632	-2632	—	+ 262	- 262	+9080	+7537	—	—	+87372	+85959
—	—	+ 410	+2814	-1559	- 328	- 139	+ 668	+5850	+5850	—	+ 728	- 729	+ 83	—	—	—	- 120	- 118
+1259	—	+10085	-1559	+10713	—	+9419	-4726	-4553	-4553	—	+ 218	- 218	+1688	+1302	—	—	-1057	-1300
—	—	—	- 328	—	+18252	+4499	+4601	-2322	-2322	—	—	—	—	—	—	—	—	—
+4474	—	+33354	- 139	+9419	+4499	+92050	+26938	- 288	- 288	—	- 908	+ 908	+5329	+4626	—	—	+6663	+5798
—	—	-2733	+ 668	-4726	+4601	+26938	+80813	+6787	+6787	—	—	—	- 484	—	—	—	—	—
—	-1662	-2632	+5850	-4553	-2322	- 288	+6787	+64781	+49822	-2094	—	-16457	- 466	—	+4606	+4838	—	+10407
+3464	—	-2632	+5850	-4553	-2322	- 288	+6787	+49822	+61096	+2162	+13845	—	- 466	—	—	—	+34689	—
-1069	+3102	—	—	—	—	—	—	-2094	+2162	+27385	- 895	+3998	+ 426	+3516	+2204	+22678	+11583	-1562
+9726	—	+ 262	+ 728	+ 218	—	- 908	—	—	+13845	- 895	+55065	-20869	—	—	—	—	+77040	+10601
—	+3173	- 262	- 729	- 218	—	+ 908	—	-16457	—	+3998	-20869	+51828	—	—	- 389	-1113	-10555	-19080
+1191	- 36	+9080	+ 83	+1688	—	+5329	- 484	- 466	- 466	+ 426	—	—	+1604	+1999	—	+ 445	+ 257	+ 26
+2005	+2439	+7537	—	+1302	—	+4626	—	—	—	+3516	—	—	+1999	+20803	+1221	+2950	+ 387	—
—	+5793	—	—	—	—	—	—	+4606	—	+2204	—	- 389	—	+1221	+28397	+11833	—	+12963
—	+2113	—	—	—	—	—	—	+4838	—	+22678	—	-1113	+ 445	+2950	+11833	+33273	—	+12590
+29655	—	+87372	- 120	-1057	—	+6663	—	—	+34689	+11583	+77040	-10555	+ 257	+ 387	—	—	+1193349	+710508
—	-1240	+85959	- 118	-1300	—	+5798	—	+10407	—	-1562	+10601	-19080	+ 26	—	+12963	+12590	+710508	+964697
+11000	-21000	+27650	-3080	+2140	-27000	+29910	+7670	-22520	-27130	+1520	+54710	-52300	+2620	+33280	-69590	-40970	+45860	+43720
19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37

Table 6

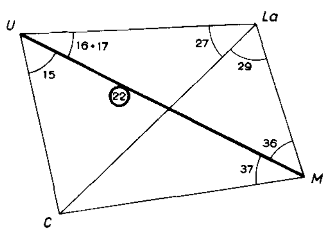


Fig. 24

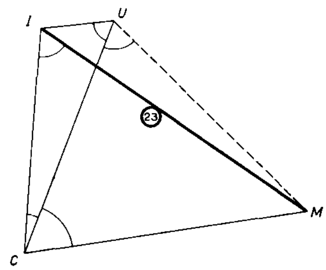


Fig. 25

The second of these equations (22, v) is illustrated in fig. 24. In this figure I marked with thin lines five sides of the triangulation net which were necessary for its construction. The thick line MU which does not come up in fig. 22 but which does in fig. 15 is one of the 17 redundant sides which determine the 17 side equations mentioned above.

If in

$$\frac{CU}{CLa}\cdot\frac{CLa}{CM}\cdot\frac{CM}{CU} = 1$$

the proportion between the three pairs of sides is replaced by the proportion of the sines of the opposite angles, one obtains in a logarithmic form

$$\log\sin(78°16'+p_{27}) + \log\sin\{53°20'+(p_{36}+p_{37})\} + \log\sin(47°03'30''+p_{15}) =$$
$$\log\sin\{82°00'30''+(p_{15}+p_{16}+p_{17})\} + \log\sin(57°24'+p_{29}) + \log\sin(43°36'+p_{37})$$

with e.g.

$$\log\sin(78°16'+p_{27}) = 9.990829+0.0000263p_{27}.$$

Worked out and multiplied by 10000 this equation is mentioned in table 5. Its number 22 is marked upon the side MU in fig. 15. The other side equations are formed in an analogous way. Fig. 25 e.g. refers to equation 23. It is true that this figure has only four lines in common with fig. 22 but as the dashed line MU determined already condition 22, 23 refers to the redundant line MI.

In the sketch of a complicated triangulation net the reference of a redundant side to the number of the relative side equation seems recommendable. It prevents the introduction of dependent equations and it indicates immediately where a missing equation must be found.

24 Normal equations, solution of these equations, corrections to the observations, standard deviations, strength of the triangulation

From the 37 condition equations follow the 37 normal equations. Their general form is $[\phi a]K_a+[\phi b]K_b+\ldots+[\phi\mu]K_\mu+[\phi_v]K_v+W_\phi = 0$ ($\phi = a, \ldots, v$).

The first equation therefore runs:

$$[aa]K_a+[ab]K_b+\ldots+[a\mu]K_\mu+[av]K_v+W_a = 0$$

the 37th:

$$[va]K_a+[vb]K_b+\ldots+[v\mu]K_\mu+[vv]K_v+W_v = 0.$$

The coefficients $[aa]$, $[ab]$, etc. are mentioned in the matrix of table 6.

In order to avoid decimal signs they are multiplied by 1000. At the foot of the columns $a, \ldots, \nu$ the amounts $W_a, \ldots, W_\nu$ are copied from table 5, also multiplied by 1000. The computation of the coefficients $[aa], \ldots, [\nu\nu]$ and the solution of the normal equations from the 37 columns in table 6 was done with the TR4-computer of the Delft Technological University. The result, rounded-off at the third figure behind the decimal point, is mentioned in table 7.

Correlates K_φ ($\varphi = a \longrightarrow \nu$)											
a	+0.855	h	-6.353	o	-7.289	v	+3.052	γ	-5.134	μ	+0.358
b	+1.489	i	+1.801	p	-3.869	w	+1.453	δ	-1.673	ν	-4.333
c	-3.728	j	-4.989	q	-4.542	x	+1.425	ϵ	-1.411		
d	+3.891	k	-1.696	r	-1.084	y	-0.647	ζ	+25.198		
e	+1.944	l	-9.491	s	-3.089	z	+0.394	η	-1.811		
f	-3.877	m	+4.004	t	+17.601	α	-3.102	θ	-0.959		
g	+1.846	n	-8.507	u	+3.117	β	+2.668	λ	+6.200		

Table 7

The corrections $p_i (i = 1, \ldots, 55)$ can then be computed from the equations

$$p_i = a_i K_a + b_i K_b + \ldots + \mu_i K_\mu + \nu_i K_\nu$$

From these amounts follows

$$[pp] = -[KW]$$

which gives an insight into the internal accuracy of the triangulation. These computations were also carried out by the computer.

The result is given in table 8. The amounts p_i (column 3) are rounded-off to a hundredth of a minute. The adjusted angles $\alpha_i + p_i$ are also inserted (column 4).

As

$$[pp] = -[KW] = 1308, \quad m_\alpha^2 = \frac{1308}{37} = 35.4,$$

the internal accuracy of TYCHO BRAHE's trigonometrical network can be characterized by the standard error $m_\alpha = 5.9'$ in the measured angle.

Column 5 gives the values of the angles according to the data of the Geodetic Institute and column 6 the differences v between these angles and the adjusted angles in column 4. As one sees the angles 21, 34, 35 and 49 have very great v's. As the amounts p_i for these angles are small, these v's have contributed to a great extent to the large value $M_\alpha = 17'$ which NØRLUND computed from the v's (see § 18).

As before the adjustment of the triangulation a possible mistake of 1° in the observation of angle 52 ecc. (108° 51' in fig. 16) was already corrected, p_{52} and v_{52} are now reduced to reasonable amounts. As NØRLUND did not do so this "mistake" affected also strongly his computation of M_α.

If one excludes errors in the reconstruction of U or (and) I, the large $v_{21} = -29'$ might, at least partly, be caused by an error of eccentricity, as one of the legs of this angle has a length of only 1287.9 m.

i 1	α_i 2	p_i 3	$\alpha_i + p_i$ 4	Geod. Inst. 5	$\overset{v_i}{5-4}$ 6	i 1	α_i 2	p_i 3	$\alpha_i + p_i$ 4	Geod. Inst. 5	$\overset{v_i}{5-4}$ 6
1	45°34′	+0.′88	45°34.88′	45°39.4′	+4.5′	29	57°24′	−8.′00	57°16.00′	57°16.8′	+0.8′
2	23 27	−4.80	23 22.20	23 29.4	+7.2	30	135 32	−3.14	135 28.86	135 17.1	−11.8
3	11 31	+1.10	11 32.10	11 34.7	+2.6	31	165 41	+4.67	165 45.67	165 45.2	−0.5
4	109 13	−9.32	109 03.68	108 59.6	−4.1	32	30 23	−6.18	30 16.82	30 28.1	+11.3
5	18 20	−6.30	18 13.70	17 56.2	−17.5	33	36 51	−0.94	36 50.06	—	—
6	4 26	−10.27	4 15.73	4 19.8	+4.1	34	77 01	−0.54	77 00.46	77 29.5	+29.0
7	17 21	+4.77	17 25.77	17 32.7	+6.9	35	67 17	+2.27	67 19.27	67 45.2	+25.9
8	67 27	−3.37	67 23.63	67 40.3	+16.7	36	9 44	−2.81	9 41.19	9 44.3	+3.1
9	23 27	−11.48	23 15.52	23 19.8	+4.3	37	43 36	−8.10	43 27.90	43 33.1	+5.2
10	4 40	−2.36	4 37.64	4 40.7	+3.1	38	21 21	−1.35	21 19.65	—	—
11	6 03	−1.70	6 01.30	5 58.8	−2.5	39	69 45	−10.10	69 34.90	69 25.8	−9.1
12	66 41	−3.64	66 37.36	66 15.8	−21.6	40	20 41	+5.62	20 46.62	20 40.9	−5.7
13	17 50	−6.39	17 43.61	—	—	41	48 49	−0.72	48 48.28	48 44.9	−3.4
14	10 41	−2.06	10 38.94	10 39.5	+0.6	42	20 08	−10.01	19 57.99	19 57.3	−0.7
15	47 03.5	−4.30	46 59.20	47 03.8	+4.6	43	2 44	+0.85	2 44.85	2 46.3	+1.4
16	24 05	−4.85	24 00.15	24 08.0	+7.8	44	4 18	−4.22	4 13.78	4 17.1	+3.3
17	10 52	−2.20	10 49.80	10 50.6	+0.8	45	16 50	−0.37	16 49.63	16 52.7	+3.1
18	115 00.5	+3.19	115 03.69	114 51.3	−12.4	46	68 31	−1.95	68 29.05	68 20.0	−9.0
19	17 46.5	+7.79	17 54.29	17 51.7	−2.6	47	23 31	+1.84	23 32.84	23 28.1	−4.7
20	2 08	−4.82	2 03.18	2 09.6	−6.4	48	6 08	−2.41	6 05.59	6 11.3	+5.7
21	86 25	+0.58	86 25.58	85 56.6	−29.0	49	67 14	−1.66	67 12.34	66 21.5	−50.8
22	28 08	+1.20	28 09.20	28 14.7	+5.5	50	10 50	−1.18	10 48.82	10 56.0	+7.2
23	1 23	+4.05	1 27.05	1 29.6	+2.6	51	17 51	+5.48	17 56.48	—	—
24	41 18	−2.61	41 15.39	41 30.3	+14.9	52	114 58.7	+4.99	115 03.69	114 51.3	−12.4
25	11 46	−6.87	11 39.13	11 36.0	−3.1	53	47 01.0	−1.80	46 59.20	47 03.8	+4.6
26	1 27	−2.13	1 24.87	1 27.4	+2.5	54	23 53.8	+6.35	24 00.15	24 08.0	+7.8
27	78 16	−3.15	78 12.85	78 00.3	−12.5	55	34 51.8	−1.85	34 49.95	34 58.6	+8.6
28	106 24	−1.94	106 22.06	106 15.0	−7.1	$[vv] = 7897 \quad M_\alpha^2 \approx 158 \quad M_\alpha \approx 13'$					

Table 8

The large amounts $v_{34} = +29'$ and $v_{35} = +26'$ relate both to angles measured in Malmø with Lund as one of the sighting points. From the similarity of the signs of both v's one might conclude that in Malmø was pointed at the northern tower of Lund's cathedral instead of at the southern one which is used for the computation of the v's. The rather small distance between the two towers, however, does not justify this supposition.

The adjusted angle 49 is almost 51′ larger than the amount which follows from the data of the Geodetic Institute. This very large amount contributes for almost one third to $[vv] = 7897$ ($M_\alpha \approx 13'$) which can be computed from table 8. In my opinion this inadmissable deviation, which finds no expression in the amount $p_{49} = -1.66'$ of the adjusted net, must be attributed to the local bad construction of the net. This construction would have been much better if also the angle *HaHeLa* in fig. 15 – let us call it No. 56 – had been measured. It would have given an extra angle equation:

$$47+50+49+23+25+56+(p_{47}+p_{50}+p_{49}+p_{23}+p_{25}+p_{56}) = 180^\circ$$

and the very complicated second term $\dfrac{LaHe}{LaHa}$ of the side equation 29(γ) in which p_{49} can be found:

$$\frac{LaHe}{LaHa} = \frac{\sin\{(47+49+50)+(p_{47}+p_{49}+p_{50})\}}{\sin[\{180^\circ-(23+25+47+49+50)\}-(p_{23}+p_{25}+p_{47}+p_{49}+p_{50})]}$$

would have been much more simple and much better then, viz.

$$\frac{LaHe}{LaHa} = \frac{\sin\{(47+49+50)+(p_{47}+p_{49}+p_{50})\}}{\sin(56+p_{56})}$$

25 Transformation of the adjusted network to the identical points of the Geodetic Institute

From the length of the base UI and with the adjusted angles α_i+p_i from table 8 one can compute now the lengths of all the sides of the triangulation network. In order to avoid the small top angles mentioned before in § 21 (fig. 18) I used for this computation a method which was not accessible to TYCHO BRAHE. I started from the co-ordinates of the points Copenhagen and Landskrona in table 4.

From these co-ordinates and with the adjusted angles one finds by intersection the co-ordinates of the other angular points of the triangulation network. Rounded-off at dm they are mentioned as X_iY_i in the columns 4 and 5 of table 9.

Points	System $X'Y'$ (Geodetic Institute)		System XY (Tycho Brahe)		System XY brought into sympathy with system $X'Y'$		Differences	
							v_i	w_i
i	X_i'	Y_i'	X_i	Y_i	X_i'	Y_i'	(6-2)	(7-3)
1	2	3	4	5	6	7	8	9
Copenhagen, Frue Kirke	+138130.75	+ 77816.56	+138130.75	+ 77816.56	+138176.1	+ 77807.9	+ 45.3	− 8.7
Landskrona, Skt. Joh. Bapt.	+153638.—	+ 99858.—	+153638.—	+ 99858.—	+153636.3	+ 99853.8	− 1.7	− 4.2
Uraniborg, Obs. Centre	+145097.71	+103506.47	+145101.7	+103541.8	+145101.8	+103521.4	+ 4.1	+ 14.9
Hven, Skt. Ibs (Gamle Kirke)	+143833.97	+103754.77	+143850.0	+103798.4	+143850.8	+103775.9	+ 16.8	+ 21.1
Malmø, Skt. Petri Kirke	+165505.25	+ 70707.39	+165532.0	+ 70622.8	+165564.5	+ 70662.2	+ 59.3	− 45.2
Lund, Domkirke, southern tower	+177026.60	+ 81988.90	+176921.2	+ 81939.6	+176926.6	+ 81986.5	−100.0	− 2.4
Helsingør, Skt. Olai Kirke	+139490.25	+117567.07	+139504.4	+117623.8	+139487.9	+117582.7	− 2.3	+ 15.6
Kronborg, s.e. tower	+139975.60	+117904.20	+139952.0	+117984.9	+139934.6	+117944.1	− 41.0	+ 39.9
Hälsingborg, Maria Kirke	+144543.87	+118899.34	+144605.3	+118880.0	+144582.5	+118845.6	+ 38.6	− 53.7
Hälsingborg, Kärnan	+144628.03	+119133.43	+144632.2	+119190.7	+144608.9	+119156.0	− 19.1	+ 22.6
Højbjerg	——	——	+132364.3	+ 94753.8	+132388.9	+ 94721.7	——	——

Table 9

The columns 2 and 3 give the corresponding co-ordinates $X_i'Y_i'$ of the Geodetic Institute. From the co-ordinates X_iY_i one finds the lengths of all 36 sides of the triangulation network. Rounded-off at dm they are shown as l' in column 3 of table 10.

The numbers 1, ..., 19 in column 1 refer to the sides which were nesessary for the computation of the triangulation (see fig. 22), the numbers 21, ..., 37 to the redundant sides in fig. 15 (the number of the side equations in § 23). Number 20 had therefore to be left out.

For UI one finds 1277.7 m. As the length of the base UI = 1287.90 m one must multiply all distances of column 3 by 1287.90 : 1277.7 = 1.007983 in order to find the lengths of the sides which match this base length (column 4). Column 5 gives the corresponding lengths computed from the co-ordinates of the Geodetic Institute.

From column 6 = 4 − 5 one sees that all amounts l are larger than the corresponding amounts of the Geodetic Institute. According to column 7 these differences fluctuate round about 0.86 m per 100 m. It would be tempting to state that this amount is due to an error of about 11 metres in the length of UI, that is to say to an error in the reconstruction of a (the) base point(s) Skt. Ibs or (and) Uraniborg. One has then not taken into account, however, the standard deviation $m_\alpha = \pm 5.9'$ in all the angles of the network which cause a standard deviation in UI.

N°	Side	Lengths l of the sides in m			Diff. 4–5	Diff. per 100 m
		Tycho Brahe		Geodetic Institute		
		l'	$l = 1.007983\,l'$			
1	2	3	4	5	6	7
1	*C La*	26949.95	27165.1	26949.95	+ 215.1	+ 0.79
2	*C U*	26653.0	26865.8	26617.85	+ 248.0	+ 0.92
3	*La U*	9297.2	9371.5	9286.97	+ 84.5	+ 0.90
4	*U I*	1277.7	1287.9	1287.90	——	——
5	*C I*	26603.9	26816.3	26557.81	+ 258.5	+ 0.96
6	*La Lu*	29379.8	29614.4	29433.51	+ 180.9	+ 0.61
7	*M Lu*	16055.6	16183.8	16124.95	+ 58.8	+ 0.36
8	*La Ho*	21877.4	22052.1	——	——	——
9	*M Ho*	41017.1	41344.5	——	——	——
10	*C M*	28329.8	28556.0	28282.57	+ 273.4	+ 0.96
11	*La M*	31562.0	31814.0	31473.63	+ 340.4	+ 1.08
12	*U He*	15153.6	15274.6	15137.51	+ 137.1	+ 0.90
13	*U Kr*	15333.7	15456.1	15281.71	+ 174.4	+ 1.13
14	*U Ha*	15346.2	15468.7	15402.83	+ 65.9	+ 0.43
15	*U Ka*	15655.9	15780.9	15634.02	+ 146.9	+ 0.93
16	*La He*	22702.1	22883.3	22666.49	+ 216.8	+ 0.95
17	*La Kr*	22713.2	22894.6	22634.63	+ 260.0	+ 1.14
18	*La Ha*	21057.7	21225.8	21101.56	+ 124.2	+ 0.59
19	*La Ka*	21327.4	21497.7	21277.26	+ 220.4	+ 1.03
20						
21	*I La*	10551.4	10635.6	10550.06	+ 85.5	+ 0.80
22	*U M*	38743.5	39052.8	38629.62	+ 423.2	+ 1.08
23	*I M*	39632.4	39948.8	39519.28	+ 429.5	+ 1.08
24	*C Lu*	39008.9	39320.4	39118.99	+ 201.4	+ 0.51
25	*I Lu*	39642.2	39958.7	39692.62	+ 266.1	+ 0.67
26	*U Lu*	38459.5	38766.6	38502.73	+ 263.9	+ 0.68
27	*Ho Ka*	27343.4	27561.7	——	——	——
28	*Ho Ha*	27053.9	27269.9	——	——	——
29	*He Ha*	5253.3	5295.2	5226.28	+ 68.9	+ 1.30
30	*C Ha*	41570.8	41902.6	41580.32	+ 322.3	+ 0.77
31	*C Ka*	41881.9	42216.2	41824.61	+ 391.6	+ 0.93
32	*I He*	14492.3	14608.0	14479.21	+ 128.8	+ 0.88
33	*I Kr*	14712.3	14829.7	14666.06	+ 163.6	+ 1.10
34	*Kr Ka*	4833.1	4871.6	4812.08	+ 59.5	+ 1.22
35	*He Ka*	5361.9	5404.7	5371.24	+ 33.5	+ 0.62
36	*I Ha*	15100.5	15221.1	15161.20	+ 59.9	+ 0.39
37	*I Ka*	15412.2	15535.2	15399.15	+ 136.0	+ 0.88

Table 10

As the very extensive computation of the standard ellipses of U and I seems exaggerated it is still necessary to have an impression whether the deviation of about 11 m is significant. I therefore made some estimations of the standard error in UI in consequence of the standard error $m_\alpha = \pm 5.9'$ in the angles.

I computed this standard error in the triangles IUC, $IUKr$ and $IULa$ in which the side IU is found.

As (see fig. 26):

$$IU = c = \frac{u \sin C}{\sin U},$$

$$\frac{\partial c}{\partial C} = \frac{u \cos C}{\sin U} \quad \text{and}$$

$$\frac{\partial c}{\partial U} = \frac{-u \sin C \cos U}{\sin^2 U} = -c \cot U$$

one has:

$$m_{UI}^{\ 2} = \left(\frac{u \cos C}{\sin U}\right)^2 m_C^{\ 2} + (c \cot U)^2 m_U^{\ 2}$$

with:

$$m_C = m_U = m_\alpha = 5.9' = \frac{5.9}{3437.7}\text{rad.}, \qquad c \approx 1288 \text{ m},$$

$$u \approx 26558 \text{ m}, \quad C = 43 = 2°44' \quad \text{and} \quad U = 21 = 86°25'.$$

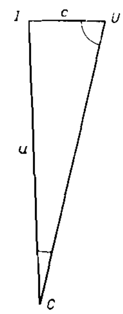


Fig. 26

One finds $m_{UI} \approx 46$ m.

In the triangle *IUKr* (see fig. 27) we find in an analogous way

$$m_{UI}^{\ 2} = \left(\frac{i \cos Kr}{\sin I}\right)^2 m_{Kr}^{\ 2} + (kr \cot I)^2 m_I^{\ 2}$$

with $Kr = 6 = 4°26'$, $I = (5+4+26+55+53+21)-180° \approx 117°18'$, $m_{Kr}^{\ 2} = m_\alpha^{\ 2}$, $m_I^{\ 2} = 6m_\alpha^{\ 2}$, $i \approx 15282$ m and $kr \approx 1288$ m.

The result is now $m_{UI} \approx 30$ m.

In the triangle *IULa* finally we have (fig. 28)

$$m_{UI}^{\ 2} = \left(\frac{u \cos La}{\sin U}\right)^2 m_{La}^{\ 2} + (la \cot U)^2 m_U^{\ 2}$$

with $La = 26 = 1°27'$, $U = 55+53+21 \approx 168°18'$, $m_{La}^{\ 2} = m_\alpha^{\ 2}$, $m_U^{\ 2} = 3m_\alpha^{\ 2}$, $u \approx 10550$ m and $la \approx 1288$ m. This gives $m_{UI} \approx 91$ m.

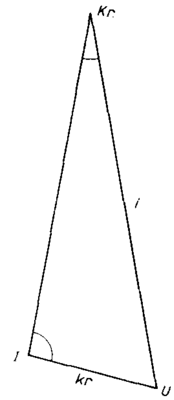


Fig. 27

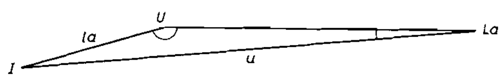


Fig. 28

As one sees even the smallest of these amounts m_{UI} is much larger than the amount of 11 m that was found as an eventual scale error. The eventual possibility of a reconstruction error in *I* or (and) *U* can therefore not be found with the available observations.

The mean difference of about 0.86 m per 100 m in column 7 of table 10 finds of course expression in table 9 where the co-ordinates *XY* in TYCHO BRAHE's system (columns 4 and 5) have been transformed by a similarity transformation to the ten indentical points of the Geodetic Institute (columns 6 and 7). From these columns one computes $UI = 1276.6$ m. The difference with the base length 1287.9 m is now 11.3 m. The remaining errors v_i and w_i in the columns 8 and 9 give an impression of the accuracy of the triangulation network. They are plotted as vectors in fig. 29.

The largest vector – 100 metres – is in Lund, not so large, however, if one takes into consideration that the point is determined by intersection from the 5 stations Landskrona (29.4 km), Malmø (16.1 km), Copenhagen (39.1 km), Skt. Ibs (39.7 km) and Uraniborg (38.5 km). The very large distances, almost as far as 40 km [81] over which had to be pointed

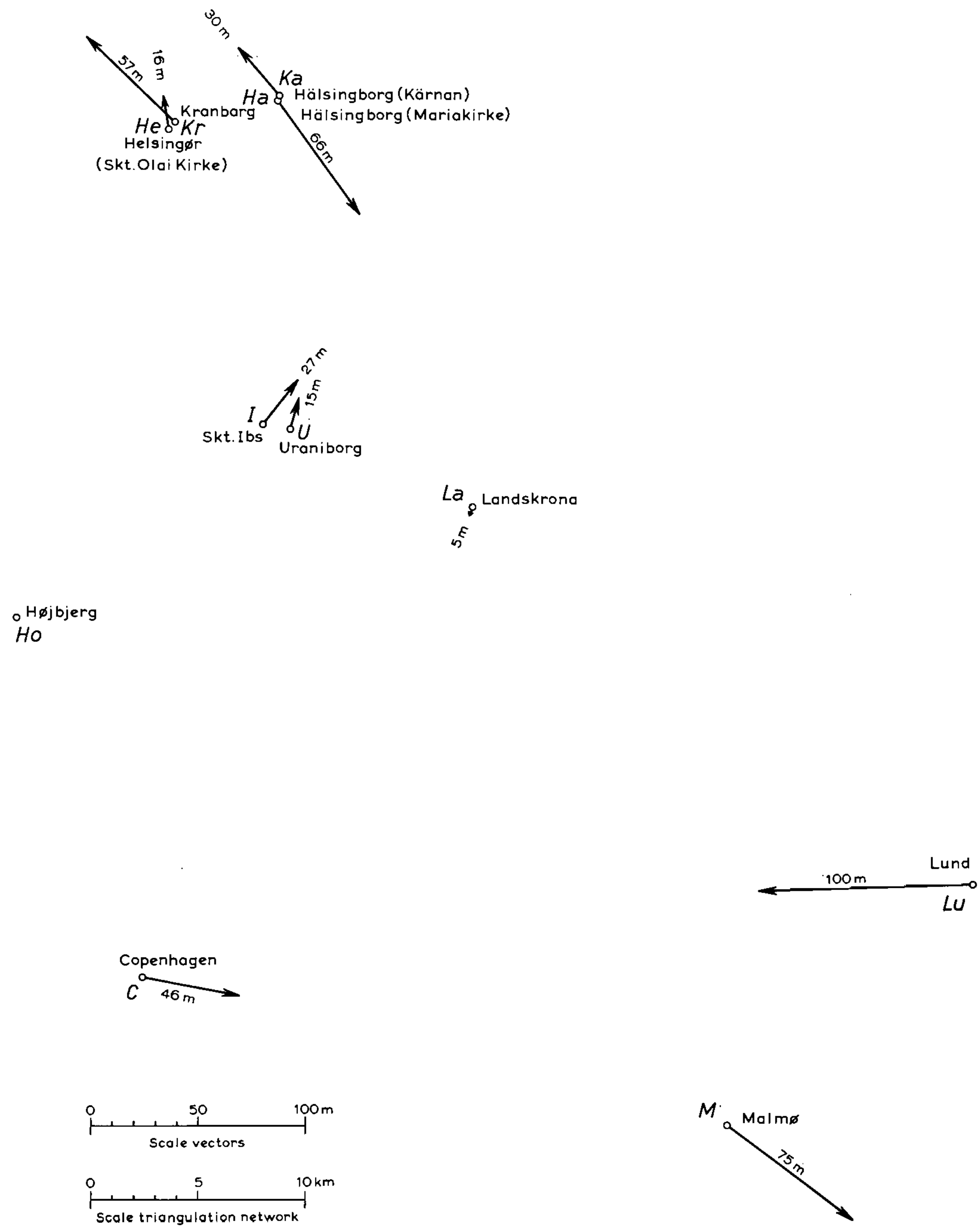

30 m
57 m
16 m
Ka
Ha
Hälsingborg (Kärnan)
Hälsingborg (Mariakirke)
Kronborg
He
Kr
Helsingør
(Skt. Olai Kirke)
66 m
27 m
15 m
I
Skt. Ibs
U
Uraniborg
La
Landskrona
5 m
Højbjerg
Ho
Lund
100 m
Lu
Copenhagen
C
46 m
0 50 100 m
Scale vectors
0 5 10 km
Scale triangulation network
M
Malmø
75 m

Fig. 29

with the primitive instruments make a better result impossible. All other vectors are much smaller, in my opinion very small indeed.

In § 21 I remarked already that because of the very bad construction of the northern part of the triangulation network the mutual position of the towers *He* and *Kr* and *Ha* and *Ka* respectively would be determined very badly. This prediction – not so difficult indeed – can now be affirmed by figures.

For the distance 590.95 m between Helsingør, Skt. Olai kirke (*He*) and Kronborg one finds from the co-ordinates in table 9 (columns 6 and 7))574.6 m, an error of −2.8 percent. For the distance between the two towers Kärnan and Maria kirke in Hälsingborg (248.76 m) one finds 311.5 m. The error is here even +25 percent.

Højbjerg, unknown in the co-ordinate system of the Geodetic Institute, is determined by intersection from the four points Landskrona (*La*), Malmø (*M*), Kärnan (*Ka*) and Hälsingborg Maria kirke (*Ha*). The rays from the latter points, at a distance of about 27 km from *Ho*, almost coincide and the distance Malmø–Højbjerg is about 41 km, one of the largest in the triangulation network. It is therefore no wonder that the co-ordinates $X' = +132388.9$, $Y' = +94721.7$ from table 9, plotted on the topographical map in fig. 17, don't quite agree with the place of the presumable sighting point, the top of the hill, but with a point, the small circle, at about 200 metres south-east of this top.

According to § 21 the standard deviation in the side b of the triangulation network (see fig. 30) can be computed from the standard error in a and the standard errors in the angles A and B with formula (4):

$$m_b^2 = \frac{1}{\sin^2 A}\{\sin^2 B m_a^2 + (b\cos A)^2 m_A^2 + + (a\cos B)^2 m_B^2\}$$

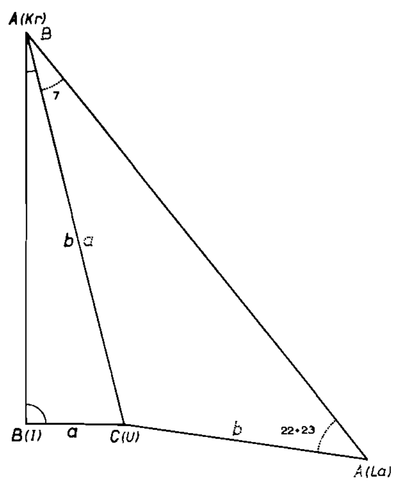


Fig. 30

For $b = UKr$ and $a = UI$, the base of the triangulation, m_b was 332 m. With the adjusted angles in table 8 and the adjusted sides in table 10 (column 4) one finds 361 m.

In order to obtain an insight into the standard deviations of the other sides meeting in Uraniborg – just an insight, because, as I remarked already, I did not compute the standard ellipses of the angular points of the network – I used the same formula. For $a = UKr = 15456$ m, $b = ULa = 9372$ m, $A = 22+23 = 29°36'15''$, $B = 7 = 17°25'46''$, $m_a = 361$ m, $m_A^2 = 2m_\alpha^2 = 0.000006$ and $m_B^2 = m_\alpha^2 = 0.000003$ one finds

$$m_b^2 = \frac{1}{(0.49400)^2}(11693+652+399) = 52218$$

so that $m_b = m_{ULa} = 229$ m.

As in the term in brackets 652+399 is small with respect to 11693 one can say

$$m_b^2 \approx \frac{\sin^2 B}{\sin^2 A} m_a^2 \quad \text{or} \quad m_b \approx \frac{b}{a} m_a \approx 219 \text{ m}.$$

Sides i	Stand. dev. m_i (meters)	Sides i (meters)	$m_i : i$
U Kr	361 (356)	15456	0.0234
U La	229 (219)	9372	0.0244
U He	387 (373)	15275	0.0253
U Ka	393 (386)	15781	0.0249
U Ha	384 (378)	15469	0.0248
U C	669 (656)	26866	0.0249
U M	975 (972)	39053	0.0250
U Lu	979 (965)	38767	0.0253
			0.025

Table 11

These standard deviations, the latter in brackets, are mentioned in table 11. Because of the large amount m_a, caused by the bad base extension, the proportion between m_a and m_b is about the same as the proportion between a and b. This phenomenon repeats itself in the other standard errors in the table. From the mean of the amounts in the last column one sees that $m_i/i \approx 0.025$.

26 Determination of azimuths and systematic errors in these azimuths; determination of latitudes

It is obvious that the very able astronomer Tycho Brahe paid very much attention to the astronomical part of his triangulation, the orientation of his network and the determination of the geographical co-ordinates of one of its angular points.

In table 3 I mentioned already some azimuths measured in Uraniborg. From the differences of these azimuths I computed there the angles 15 up to and including 21. They are used as measured angles in the adjustment of the network.

In table 13 column 5 I mention again the azimuths of series b augmented with the azimuths in the same series to three towers in Copenhagen, which do not belong to the network, and to the Kullen light house (distance about 46.4 km) at the westside of Skälder Bay

Points	Co-ordinates X'	Co-ordinates Y'
Kullen, Lighthouse	+128461.81	+146774.37
Copenhagen, Helligaandskirken	+138431.64	+ 77806.77
Copenhagen, Skt. Petri Kirke	+138023.70	+ 77885.55
Copenhagen, Christiansborg	+138654.95	+ 77488.99

Table 12

Points P	Geodetic Institute: Grid bearings UP	Geodetic Institute: Conv. mer.	Geodetic Institute: Astr. az. UP	Tycho Br.: Astr. az. UP	4–5	ψ_{UP} 5–13'	v 4–7
1	2	3	4	5	6	7	8
Copenhagen, Frue Kirke	195°10.4'	+1°55.3'	197°05.7'	197°18.5'	-12.8'	197°05.5'	+0.2'
Malmø, Skt. Petri Kirke	148 06.6	+1 55.3	150 01.9	150 15	-13.1	150 02	-0.1
Lund, Domkirke	123 58.6	+1 55.3	125 53.9	126 10	-16.1	125 57	- 3.1
Landskrona, Skt. Joh. Bapt.	113 08.0	+1 55.3	115 03.3	115 18	-14.7	115 05	- 1.7
Hälsingborg, Kärnan	358 16.7	+1 55.3	0 12.0	0 17.5	- 5.5	0 04.5	+ 7.5
Kronborg, s.e. tower	340 25.0	+1 55.3	342 20.3	342 31	-10.7	342 18	+ 2.3
Helsingør, Skt. Olai Kirke	338 15.4	+ 1 55.3	340 10.7	340 23	-12.3	340 10	+0.7
Kullen, Lighthouse	338 58.1	+ 1 55.3	340 53.4	341 13	-19.6	341 00	-6.6
Copenhagen, Helligaandskirken	194 32.5	+1 55.3	196 27.8	196 45	-17.2	196 32	- 4.2
Copenhagen, Skt. Petri Kirke	195 26.1	+1 55.3	197 21.4	197 30	- 8.6	197 17	+ 4.4
Copenhagen, Christiansborg	193 54.5	+1 55.3	195 49.8	196 02	-12.2	195 49	+0.8
	$[vv] = 156 \quad m^2 = 15.6 \quad m \approx 4'$				-13.0		+0.2

Table 13

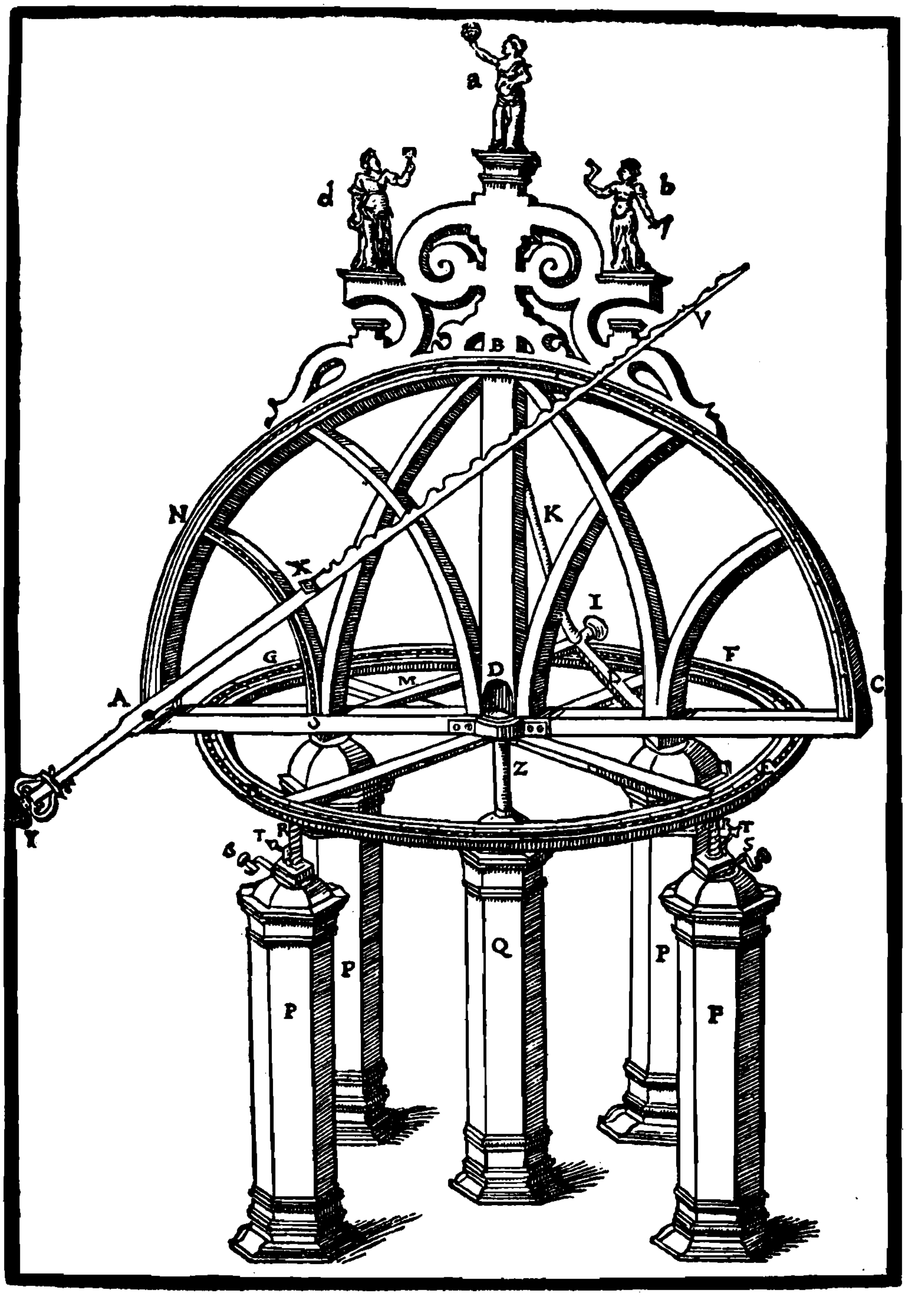


Fig. 31

(Sweden). As the co-ordinates of these towers are known in the system of the Geodetic Institute (see table 12), they give, with the other ones, an impression of the accuracy with which TYCHO BRAHE carried out these measurements.

Column 2 of table 13 gives the grid bearings, computed from the co-ordinates. The convergence of meridians in Uraniborg being $+1°55.3'$ (column 3) – I borrowed this value from NØRLUND – the astronomical azimuths are the amounts in column 4. They all are smaller than TYCHO's azimuths. Apparently there is a systematic error of about $-13'$. In order to give a possible explanation of this systematic error I refer to fig. 31, a representation of one of the instruments with fixed setting-up which were used in Uraniborg for the determination of azimuths [82]. Its horizontal circle (centre D) had a diameter of 4 cubits (1.55 m); its zero point was in the meridian of D. Perpendicular to this azimuth circle was a vertical circle ABC with the same centre D and a diameter of 5 cubits (2.33 m). It could rotate round D. YAV (turning point in A) is the alhidade. The vertical limb was provided with a calibration between 0° (in C) and about 65° (top B: 45°). On the top of the instrument are three figures "artfully carved out of strong wood. Their purpose is not only for ornament but they should represent a symbolic meaning. The figure that is placed highest (a) is Urania, representing Astronomy herself. She is a beautiful shapely virgin, turning her face towards the sky and contemplating the stars" [83]. The other two women represent Geometry (b) and Arithmetic (d).

There could be measured heights and azimuths with this instrument. The zero point of the horizontal circle was brought into the meridian by taking the mean of an observation of the pole star in its eastern and western elongation. The advantage of this method is that during these elongations the star's movement in azimuth is zero so that a time observation was not necessary. As a matter of fact time observations in those days were very difficult on account of the lacking of reliable time keepers [84]. A disadvantage, however, is shown in fig. 32 (see also fig. 31).

In this figure CAX represents the horizontal plane of the azimuth circle, ABC the plane of the "vertical" circle with the alhidade YAV. It is pointed at Polaris in eastern or western elongation. The height h of the star is then about 56° [84]. I assumed that the requirement of adjusting the instrument is not quite satisfied and that the angle between both planes is $90°-\beta$ instead of 90°. If B' is an arbitrary point of YAV, W' the foot of the perpendicular from B' on CAX and $B'W'D'$ perpendicular to AC then:

$B'D' = AD' \tan h$, $D'W' = B'D' \sin\beta = AD' \tan h \sin\beta$ and:

$\tan x = D'W'/AD' = \tan h \sin\beta$ or, as x and β are very small:

$$x = \beta \tan h \approx 1.5\beta$$

By the great elevation of the pole star an error β in the non-perpendicular position of the two planes passes therefore enlarged into the azimuth. For $\beta = 0.002$ rad, that is to say if in fig. 32 $DW = 2.3$ mm – a small amount for a primitive instrument – $x = 0.003$ rad $\approx 10'$, which means that the systematic error of 13′ has already been made clear for the greater part. The zero point of the azimuth circle will point west of the north; all azimuths are too large. The instrumental error does not influence the observations to the towers in table 13 as all of them lie about in the horizontal plane ($h \approx 0$).

The standard deviation $m = \pm 4'$ in an azimuth which can be computed from column 8 of table 13 is very good. Still better is the determination of several latitudes in a number of towns in Denmark and Norway, carried out by TYCHO BRAHE and his assistants ELIAS OLSEN MORSING and PEDER JAKOBSEN FLEMLØS. On page 31 of his publication NØRLUND compares 23 of TYCHO's results with the present values and derives from these amounts a standard deviation $m = \pm 2'$. The latitudes of Uraniborg, Helsingør, Landskrona and Hälsingborg, all points of the triangulation network, agree even exactly with the present values when rounding them off at minutes. The latitudes of Copenhagen, Malmø and Lund differ 2′, 2′ and 3′ respectively.

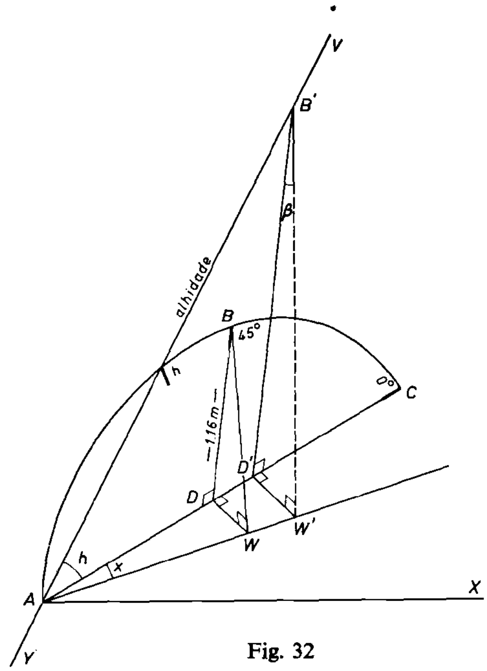


Fig. 32

Apart from the systematic error in the azimuths the determination of TYCHO BRAHE's azimuths is much better than the result obtained by SNELLIUS, which will be discussed in § 46. SNELLIUS made even an inexplicable error of more than two degrees in the determination of his azimuth. The fact that TYCHO BRAHE had an observatory at his disposal and an instrument with a fixed setting-up whilst SNELLIUS had to do his observations on the roof of his house, is no excuse for SNELLIUS' failing in this respect.

The errors which TYCHO BRAHE made in the determination of his latitudes (the largest, in Aarhus, is 6′) don 't differ much from those measured by SNELLIUS (in Alkmaar $2\frac{1}{2}'$, in Leiden 1′ and in Bergen op Zoom 1′; see § 46) .The accuracy of SNELLIUS' determination of latitudes, however, was of paramount influence on the accuracy of his final result, the determination of the earth's circumference. We will see later (in § 46) that these errors resulted in an error of about five percent. Apparently he can not be blamed for that; even the great astronomer TYCHO BRAHE was no more successful in this respect.

27 Speculation on the triangulation

It is very difficult to give a final judgement on the value of TYCHO BRAHE's triangulation. It was not difficult to prove that the base measurement on Hven was absolutely insufficient to derive from it the lengths of the real sides of the triangulation network with a reasonable degree of accuracy. The estimated standard deviations of almost 1 km, which I derived in § 21 for the sides Uraniborg – Malmø and Uraniborg – Lund (see table 11), have proved it. Nor was it difficult to find that the astronomical part of the triangulation, the determination of azimuths and latitudes, is very good.

To give a judgement on the whole triangulation, however, is not so easy as TYCHO BRAHE, apart from some incidental computations, left us but observations and we don't know at all what he might have done with these really plentiful observations which make an impression of being gathered without thinking seriously of how to work them out. The triangulation was never computed completely, which might be motivated by the fact that a map came never about. Another presupposition, however, is that TYCHO BRAHE saw no opportunity to fix with these observations the mutual position of the 11 points of the network, especially the four northern points, with an accuracy which could also satisfy him, the able, accurate and faithful astronomer.

As far as I can see there are no obvious reasons for observations in Hven South (see § 20 fig. 16). I could use these observations for the computation of the triangulation network because I reduced them to the centre Uraniborg. This method, however, was unknown in those days and if so TYCHO could not have used it because he did not know the distances from Uraniborg to Kärnan, Landskrona, Lund, Malmø and Copenhagen, as these distances just follow from the computation of the triangulation.

It is senseless to speculate how TYCHO BRAHE might have computed the network and how he might have used the 37 redundant angles in order to check his measurements and his computations. It must be said, however, that the adjustment of the measurements has given very good results. A standard deviation $m_\alpha = \pm 5.9'$ with a primitive instrument as the cross-staff could hardly be expected.

It is not quite clear whether the determination of the great number of latitudes relates to the measurement of the triangulation network. NØRLUND says "that for a part of the trigonometrical stations TYCHO BRAHE has derived the longitudes from the observations mentioned above. He started therefore from the longitude 36° 45′ fixed for Uraniborg [85]. But this longitudes are not very accurate and there is no reason to go into their way of derivation" [86].

The method for the computation of differences in longitude was already given by GEMMA FRISIUS in his *Libellus* (see §§ 5–7). For from the astronomical azimuth ψ_{UP} (U = Uraniborg, P = Copenhagen, Malmø, etc., see § 26 table 13 column 7), the distance UP (see § 25 table 10, column 4) the latitude ϕ_P of P and the earth's radius, the sought difference in longitude can approximately be found by plane geometry:

$$\lambda_P - \lambda_U = \Delta\lambda_P \approx \frac{UP \sin \psi_{UP}}{r \cos \phi_P} \varrho \quad \text{(8)}$$

In the same way the difference in latitude:

$$\phi_P - \phi_U = \Delta\phi_P \approx \frac{UP \cos \psi_{UP}}{r} \varrho \quad \text{(9)}$$

If $\Delta\lambda_P$ and $\Delta\phi_P$ are expressed in minutes of arc then $\varrho = 3438$ [87]. From the measured latitudes ϕ_P and the amounts $\Delta\phi_P$ computed with (9) one can determine ϕ_U with $\phi_U = \phi_P - \Delta\phi_P$.

As the latitude of Uraniborg is also measured one can fix then a "mean" value for ϕ_U. It is dependent on the weights which must be given to the several ϕ_U's. From this amount follow the latitudes $\phi_P = \phi_U$ (mean) $+\Delta\phi_P$, whereafter follows the computation of $\Delta\lambda_P$ with (8). It is obvious that the amounts $\Delta\phi$ and $\Delta\lambda$ for Højbjerg must be determined in a devious way. The accuracy with which $\Delta\phi$ and $\Delta\lambda$ can be computed depends on the accuracy with which UP, ψ_{UP}, ϕ_P and r are known in (9) and (8). Application of the law of propagation of errors gives:

$$m_{\Delta\phi_P}{}^2 = (\Delta\phi_P)^2 \left\{ \left(\frac{m_{UP}}{UP} \right)^2 + (\tan\psi_{UP} m_{\psi_{UP}})^2 \right\} \quad \ldots\ldots\ldots\ldots \quad (10)$$

and

$$m_{\Delta\lambda_P}{}^2 = (\Delta\lambda_P)^2 \left\{ \left(\frac{m_{UP}}{UP} \right)^2 + (\cot\psi_{UP} m_{\psi_{UP}})^2 + (\tan\phi_P m_{\phi_P})^2 \right\} \quad \ldots\ldots \quad (11)$$

with $m_{UP}/UP \approx 0.025$ (see § 25 table 11), $m_{\psi_{UP}} = 4'$ (§ 26, table 13 column 8) $= 0.0012$ rad and $m_{\phi_P} = 2' = 0.0006$ rad. The uncertainty in the earth's radius is not included in these formulae.

In his *Opera Omnia* II [88] TYCHO BRAHE assumes for the earth's circumference 5400 german miles and for its radius 900 german miles, which stands for $\pi = 3$. All these rounded-off values prove that they are estimated. According to GRAY [89] $r = 860$ german miles, a value, which, for $\pi = 3.14159$ corresponds approximately with the circumference 5400 german miles mentioned before. As 1 german mile = 4.611 english miles and 1 english mile = 1609.341 m one finds for $r = 6382$ km, an amount almost equal to the value $r = 6366$ km which we know now. According to NØRLUND, however, TYCHO BRAHE's standard measure, the *passus geometricus*, is a natural measure for r. As 1 *passus* = 1.552 m (see § 20) r is about 4.11 million *passus* and this is not a natural measure. If 4 million *passus* should be meant $r \approx 6200$ km, about 2.5 percent too small.

Points P	Tycho Brahe φ_P	UP (km)	ψ_{UP}	$\Delta\varphi_P$	$m_{\Delta\varphi_P}$	$\varphi_U = \varphi_P - \Delta\varphi_P$ 2−5	$\varphi_P = \varphi_U + \Delta\varphi_P$ 7+6	Geod. Inst. φ_P	v 9−8	$\Delta\lambda_P$	$m_{\Delta\lambda_P}$
1	2	3	4	5	6	7	8	9	10	11	12
U	55°54.5'	——	——	——	——	55°54.5'	55°55.5'	55°54.5'	-1'	——	——
C	55 43	26.866	197°05.5'	-13.8'	0.3'	55 56.8	55 41.7	55 41	-1	- 7.5'	0.2'
M	55 38.5	39.053	150 02	-18.2	0.5	55 56.7	55 37.3	55 36	-1	+18.6	0.5
Lu	55 45	38.767	125 57	-12.3	0.3	55 57.3	55 43.2	55 42	-1	+30.0	0.8
La	55 52.5	9.372	115 05	- 2.1	0.1	55 54.6	55 53.4	55 53	0	+ 8.2	0.2
Ka	56 03	15.781	0 04.5	+ 8.5	0.2	55 54.5	56 04.0	56 03	-1	+ 0.0	0.0
Kr	——	15.456	342 18	+ 7.9	0.2	——	56 03.4	56 02	-1	- 4.5	0.1
He	56 02	15.275	340 10	+ 7.7	0.2	55 54.3	56 03.2	56 02	-1	- 5.0	0.1
						55 55.5					

Table 14

Table 14 gives the results of the computation of the amounts $\Delta\phi_P$ (column 5) and $\Delta\lambda_P$ (column 11). For these computations I used $r = 6382$ km (860 german miles). For $r \approx 6200$ km (4 million *passus*) these amounts must be multiplied by about $(1 + {}^{180}/_{6382}) = 1.028$ and for $r = 6679$ km (900 german miles) by $(1 - {}^{297}/_{6382}) = 0.954$. In column 2 are the latitudes ϕ_P, determined by TYCHO BRAHE. Columns 3 and 4 give the lengths UP (in km; see § 25 table 10 column 4) and the astronomical azimuths ψ_{UP} (see § 26 table 13 column 7) which were necessary for the computation of $\Delta\phi_P$ and $\Delta\lambda_P$. In columns 6 and 12 are the amounts $m_{\Delta\phi_P}$ and $m_{\Delta\lambda_P}$. As $m_{\Delta\phi_P}{}^2$ is very small with respect to the square of the standard error with which the latitudes ϕ_P are determined, the weights with which ϕ_U mean = = 55° 55.5′ is computed in column 7 are taken alike. The latitudes $\phi_P = \phi_U$ mean $+ \Delta\phi_P$ in column 8 deviate all about $-1'$ (column 10) from the geographical latitudes in column 9.

From the exposition I gave in this paragraph about TYCHO's triangulation the reader will have an impression rather of what could have be done with his observations than of the results which he attained himself. As his work was unfinished these speculations were, unfortunately, necessary.

In the discussion of SNELLIUS' work which follows now all disputable points disappear. It is built up completely logical and it was a justifiable introduction to all triangulations after him.

WILLEBRORD SNEL VAN ROYEN (SNELLIUS) (1580–1626)

Fig. 33

28 His parents and the place of his parental home – 29 His youth, marriage and appointment to professor in Leiden – 30 His meridian chain, described in his "Eratosthenes Batavus" – 31 Unit of length – 32 Expatiations in "Eratosthenes Batavus" – 33 The Brussels' copy of "Eratosthenes Batavus" and the revision of Snellius' work in Van Musschenbroek's "De Magnitudine Terrae" – 34 Snellius' base lines ae and ig and the computation of the side Leiden–The Hague of his network – 35 Speculations on the base lines ae and ig – 36 Van Musschenbroek's computation of the side Leiden–The Hague – 37 Speculations on Snellius' base line km in Van Musschenbroek's "De Magnitudine Terrae" – 38 Speculations on Snellius' base line bd in "De Magnitudine Terrae" – 39 Speculations on Snellius' base line of in "De Magnitudine Terrae" – 40 Speculations on the base lines bd and of for b = f – 41 The triangulation network and its computation by Snellius – 42 The adjustment of the triangulation – 43 Speculations on the strength of the triangulation and on Van Musschenbroek's bad revision – 44 Computation of the *lengths of the sides in Snellius' adjusted triangulation, the lengths of the sides in the R.D- co-ordinate system and the transformation of Snellius' network to the identical points of the R.D. – 45 Computation of the length Alkmaar–Bergen op Zoom – 46 Determination of latitudes and determination of the azimuth Leiden–The Hague – 47 Computation of the azimuth Alkmaar–Bergen op Zoom and of the length of one degree upon the meridian of Alkmaar – 48 Comparison between Snellius' results in § 47 and the R.D.-data – 49 Snellius' solution of the resection problem – 50 Final speculations; Snellius' death.*

28 His parents and the place of his parental home

SNELLIUS' family comes from Oudewater, a small town about 11 km east of Gouda at the boundary of the two Dutch provinces Zuid Holland and Utrecht. His father, RUDOLPH SNEL VAN ROYEN, was born there on October 8th, 1546. Already in 1561 – only 15 years old then – he studied Hebrew and mathematics. Later on he worked in Marburg where he taught Greek, Latin and Hebrew. In 1575 he came back to Oudewater where he married MACHTELD CORNELISDOCHTER. On August 2, 1581 he was appointed extraordinary professor in mathematics in the young Leiden university (founded 1575) at an annual salary of two hundred guilders (twenty pounds). He lived with his wife, his son WILLEBRORD (born autumn 1580) and 22 boarders (students) in a house at Pieterskerkhof in Leiden. Besides WILLEBRORD two other sons, JACOB and HENDRIK, were born of the marriage but both of them died at an early age.

On July 21th, 1601 RUDOLPH SNEL VAN ROYEN buys from BARBARA CORNELISDOCHTER, widow of MICHIEL GERRITSZOON, a house and a garden, 2 roods and 9 inches broad (7.77 m) on the east side of the Koepoortsgracht, the present Doezastraat in Leiden [90]. It was the fifth house north of the corner of Lange Raamsteeg (steeg = alley) with an exit to that alley. The lot is represented on the map in fig. 34 [91].

As the house had five fire places [92] it was a substantial building, which can be proved by its price, three thousand guilders (300 pounds), a large amount for those days.

Presumably for lack of money he paid the purchase-money in terms, the first term, 600 guilders, on November 1st, 1601, the rest in eight annual terms of 300 guilders on the first of November of the years 1602–1609.

On April 13th, 1612, about a year before his death on March 1st, 1613, he buys from PIETER KLAASZOON GRAEFF for 600 guilders an adjacent open yard of 2 × 5 roods (7.5 × 18.8 metres) of the adjacent premises, the fourth house north of Lange Raamsteeg [93].

I gave such a detailed description of these purchases by SNELLIUS' father because, after his death, the property of the house passed to his son WILLEBRORD who determined for the first time in history of geodesy (in 1615) a point by resection on its roof (see § 49).

I must disappoint the interested reader who would like to visit the building. It does not exist any longer since it was destructed together with a great many other houses in the neighbourhood on Januay 12th, 1807 by the explosion of a gunpowder ship. As the cadastral maps of Leiden were not yet made at that moment there is no map available with which the place of the side-walls of the house could be determined. Sketch fig. 34 gives also no help. Though the present corner *A* of Lange Raamsteeg on this sketch is about the same as in SNELLIUS' days, lack of the width of the first house north of this alley prevents the recon-

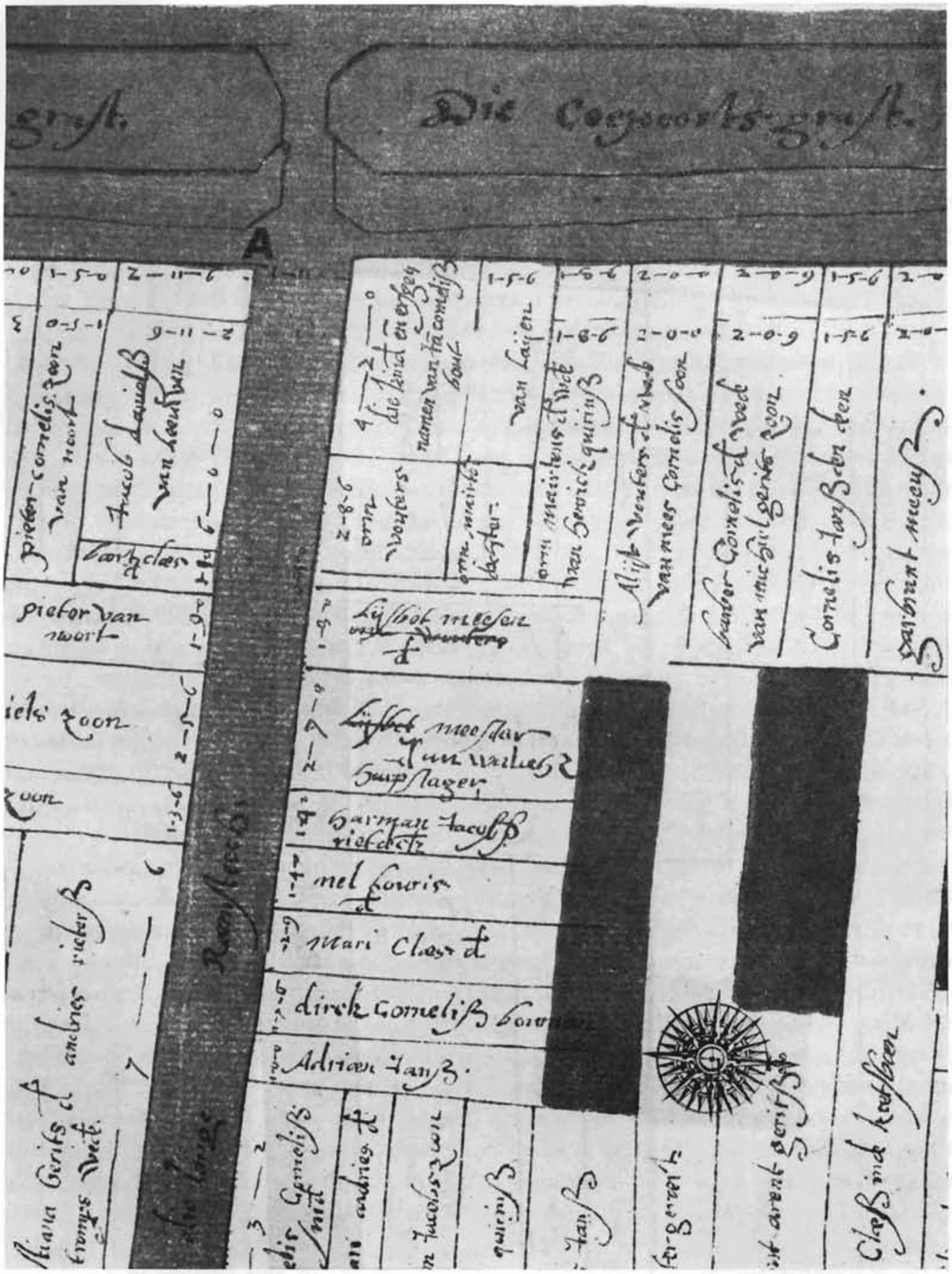


Fig. 34

struction of the limits of SNELLIUS' property. Fortunately, as we shall see, reconstruction of the point where the resection took place is possible.

29 His youth, marriage and appointment to professor in Leiden

At the early age of ten – on the first of September 1590 – WILLEBRORD was matriculated as a student into Leiden University, at first to go in for law. Very soon this study was changed into that for mathematics. LUDOLPH VAN CEULEN (1540–1610) was his teacher. We know of VAN CEULEN that he computed the number π faultless in 35 figures behind the decimal point. They were all mentioned on his sepulchral monument which was formerly in the Pieterskerk (kerk = church) in Leiden [94].

Already in his youth SNELLIUS was an able mathematician for already on May 7th, 1600 – only 19 years old then – he got permission to give lectures on mathematics and astronomy in Leiden university. Where astronomy is concerned these lectures will have been confined to the discussion of the famous *Almagest* by PTOLEMY (second century A.D.) in which the earth is considered the centre of the universe. In our opinion it may be queer that during his life SNELLIUS remained a supporter of this geocentric structure of the universe. Apparently COPERNICUS' theories could not convince him. It must be said, however, that COPERNICUS (1473–1543) convinced but a few radical thinkers of the correctness of his heliocentric system and TYCHO BRAHE did not belong to them either. His sensational book *De revolutionibus orbium coelestium* (On the revolution of the heavenly bodies) was even placed on the index in 1616, a year before the publication of SNELLIUS' *Eratosthenes Batavus.*

SNELLIUS' lectures in Leiden did not last very long as he leaves soon for abroad where he meets KEPLER and TYCHO BRAHE (see § 13). It seems, however, that he is back in Leiden in 1604 where he translates SIMON STEVIN's *Wisconstighe gedachtenissen* (Mathematical thoughts) in Latin. It was published in 1608 under the title *Hypomnemata mathematica.* With its appearance STEVIN's work became accessible for scientists from abroad.

Between 1615 and 1619 SNELLIUS made also a Latin translation of VAN CEULEN's publications. Moreover he gave an important improvement to VAN CEULEN's computation of π. In those days he had already a great scientific fame: KEPLER calls him even *geometrarum nostri seculi decus* (an ornament of the geometricians of our century).

On July 12th, 1608 SNELLIUS was promoted in Leiden *magister artium*, a degree corresponding with the present doctors degree and on the first of August of that year he married MARIA DE LANGE, daughter of LAURENS ADRAENSE DE LANGE, burgomaster of Schoonhoven, and JANNEKE SYMONS. From this marriage 18 children were born. Only three of them, two sons and a daughter, survived their parents. The sons died unmarried; the daughter, JOHANNA, became the second wife of ADRIAEN ADRIAENSE VROESEN, a member of a Rotterdam regents-family and a few times burgomaster of Rotterdam. They had four children who all married and got children. In JOHANNA and her descendants the SNEL VAN ROYEN family lived on.

On November 5th, 1609 SNELLIUS gets again permission to give lectures in Leiden on mathematics and astronomy and on February 9th, 1613, some weeks before his father's death, he is appointed his successor at an annual salary of 300 guilders (30 pounds), already raised to 400 guilders in 1614. On February 8th, 1615 he is appointed ordinary professor. In February 1616 his salary was once raised to 500 guilders and in May 1618 to 600 guilders, a considerable amount in those days.

30 His meridian chain, described in his "Eratosthenes Batavus"

SNELLIUS appears to have paid little attention to his Latin translations. They are badly edited and badly printed. They give the impression that this was due to his being engaged with many other things. A great deal of his available time will have been taken up by the measurement of his meridian chain between Alkmaar and Bergen op Zoom (distance about 130 km) which served the determination of the earth's circumference. Of course SNELLIUS knew the determinations of the earth's size which were done before him; in the first place the attempt by ERATOSTHENES of Cyrene (276–194 before Christ), director of the library of Alexandria. From the difference in latitude between Syene, the present Assuan, and Alexandria (7°12′ = 0.02 of the circle's circumference), both situated approximately on the same meridian, and the linear distance between these two places (5000 stadia) he computed the earth's circumference (250,000 stadia) [95]. SNELLIUS knew also the attempt by JEAN FERNEL who, in 1525, determined the difference in latitude between Paris and Amiens, also situated approximately on the same meridian. The linear distance between the two places was found by counting the number of revolutions of the wheel of the carriage in which the distance was covered.

Quite rightly SNELLIUS considers the attempst of his predecessors as little reliable as he says in the first part of his book which appeared in 1617 under the title *Eratosthenes Batavus, de terrae ambitus vera quantitate.* The title page of the book is reproduced as fig. 35.

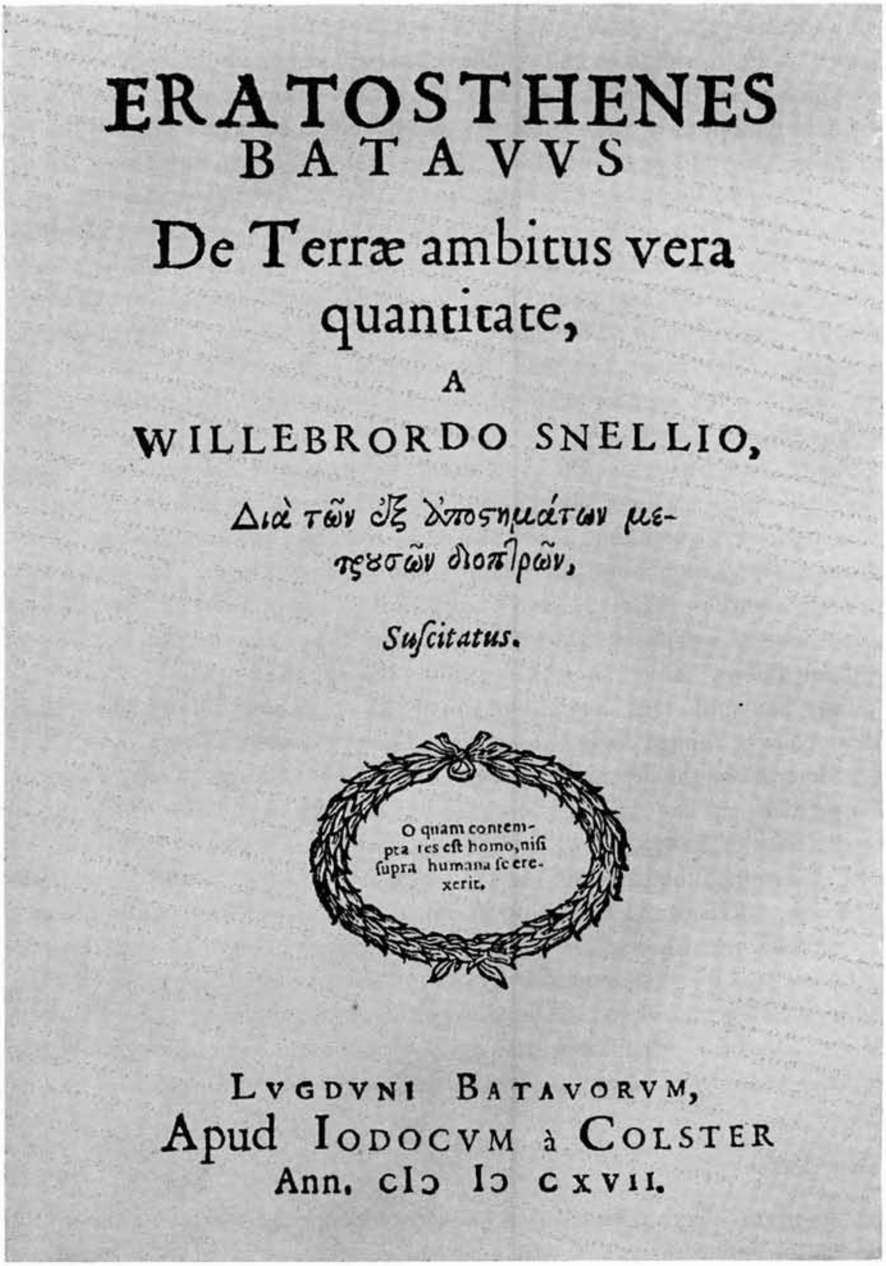
The English translation runs: "The Dutch Eratosthenes. On the real dimensions of the earth's circumference by WILLEBRORD SNELLIUS with the aid of distances which are borrowed from measurements with instruments. O, what a worthless thing is man, if he has not risen above human things. Leiden, at JODOCUS VAN COLSTER, 1617" [96].

Essentially SNELLIUS' measurement of a part of a meridian differs but little from the method which was applied by his great predecessor in ancient Greece. Just like ERATOSTHENES he determined the difference in latitude between two places which lie approximately on the same meridian and the linear distance between the two places. The great difference with the methods of ERATOSTHENES and FERNEL, however, is that SNELLIUS, as the first in the world, determined the length of the arc of the meridian by a triangulation, a net of triangles between Alkmaar and Bergen op Zoom. From the measured angles of the triangles and the length of one side one can compute the distance between Alkmaar and Bergen op Zoom. He was also the first geodesist who determined the length of a side of the network in a manner which is considered the only correct one even in our days. For by the measurement of a rather short, well-chosen base and by the measurement of angles he transferred the measured length by computation to a side of the triangulation network.

31 Unit of length

SNELLIUS' great merit is that he saw that his unit of length the Rijnlandse roede (Rhineland rood) had to be defined accurately. He gave special attention to this important part of his work, much more than anyone before him.

Nevertheless its length is not precisely known. The official length is fixed at 3.76736 metres by Royal Degree of February 8th, 1808. VAN DER PLAATS [97], however, thinks that it was somewhat smaller in SNELLIUS' time; he calculates it at about 3.7635 m. JORDAN [98] mentions an amount of 3.7662420 m. Fortunately the differences lie between narrow limits; at any rate they are not of paramount influence on the determination of the accuracy


ERATOSTHENES
BATAVVS
De Terræ ambitus vera
quantitate,
A
WILLEBRORDO SNELLIO,
Διὰ τῶν ἐξ ἀποστημάτων με-
τρουσῶν διοπτρῶν,
Suſcitatus.

O quam contem-
pta res eſt homo,niſi
ſupra humana ſe ere-
xerit.

LVGDVNI BATAVORVM,
Apud IODOCVM à COLSTER
Ann. cIↃ IↃ CXVII.

Fig. 35

of SNELLIUS' triangulation [99]. In order to compare the distances in roods found by SNELLIUS with the present measures in metres I used 1 rood = 3.766 m.

Though the rood was divided into 12 feet and a foot into 12 inches SNELLIUS works with tenth and hundredth parts of roods. Like SIMON STEVIN he shows himself here a follower of a decimal system which facilitated much his computations. This "modern" method contrasts strongly with his now out of date world view, which also finds expression on page 11 of his *Eratosthenes Batavus*: the earth is a sphere and is the centre of the universe.

32 Expatiations in "Eratosthenes Batavus"

The first part of his book (pages 1–116) is historical for the greater part. In the second part SNELLIUS describes the measurement of his meridian chain. He does not do that in a way to which we are accustomed in scientific works nowadays. He falls very often into speculations and expatiations which we, men of the 20th century, think little concise. In SNELLIUS' days, however, these expatiations were very common.

As an illustration I give a free translation of some pages [100]. They relate to the arrival of the barons STERRENBERG to whom the second part of the book is dedicated. It seems that they helped SNELLIUS with several things but it is not sure that they also took observations:

> "As the noble Austrian barons, the brothers Erasmus and Caspar Sterrenberg were already penetrated very far into the knowledge of arithmetic and geometry and were skilled in the laws of the tangents of the circle (we call this usually the theory of trigonometry) they should like to prove, as Alexander the Great, their vigour and ingenuity, not in the soft sand [101] but in a worthier material which would extent its use and advantage to many people.
> Then, tired by a protracted and diligent exertion, they decided to relax their minds during the summer holidays and to free them from their hard studies. The very learned man Joannes Philemon, in those days their teacher, great in ingenuity and science, said that he contrived a journey to the neighbouring provinces in order that these holidays should be free from care but should not flow away entirely into idleness. In this manner they would learn to know the adjacent districts and be able to pass their judgement on them. I praise this greatly in him but I appreciate also very highly their decision. And they asked me unanimously to accompany them on this trip and they would not accept a refusal; yes, even against my will they led me away from my house and my family. Especially because in former days I had mentioned geodesy and had said how thankful future generations would be if one has determined in this way the exact circumference of the earth. They had tasted already a great deal of spherical trigonometry. I should perform now what I mentioned once incidentally but what they took in earnest. We prepared ourselves therefore very carefully for the journey and took out the instruments for such a great undertaking. A semi-circle with a diameter of $3\frac{1}{2}$ Rhineland-feet (about 1.10 m) for the measurement of angles on towers from which distances can be determined in a geodetic manner. A very large iron quadrant, mounted with bronze, greater than $5\frac{1}{2}$ feet (about 1.75 m), for the measurement of the elevation of the pole.

> Thus we travelled to Oudewater in order to take there a few days rest and in order to choose an observation-station in that isolation. I had also the intention to visit there the graves of my father and my grandfather who, buried there, await the day of resurrection and especially to pay my respects to my old mother who is a widow. In the preceding year, after the death of my dear father, the noble Rudolph Snellius, she had consented to finish there the last years of her life near the graves of her beloved deceased".

Then SNELLIUS writes about Oudewater, "very famous for the plainness and the zest for work of its inhabitants", he tells of the siege by the Spaniards, of a handfull soldiers who, helped by women and "martial girls", harassed the ennemy with boiling tar, of the capture of the town, the atrocities commited and the destruction [102]. "In such a clear light this town got up in flames that it announced, as from a pharos, the arrival of the most cruel ennemy to all the people of Holland, so that it is a fact that it could be seen not only in Amsterdam but even in Noord Holland, Hoorn and Enkhuizen which is a very great distance" [103].

After having told that he wandered from his subject by his patriotism and the memory of his family he does not resume his subject until page 179.

33 The Brussels' copy of "Eratosthenes Batavus" and the revision of Snellius' work in Van Musschenbroek's "De Magnitudine Terrae"

SNELLIUS' triangulation network is represented in fig. 36. A map of Holland in that time serves as base map.

The work carried out can be divided into the following parts:

a The measurement of the base and its extension to the side Leiden – The Hague (*LHg*) of the triangulation network,

b The measurement of the triangulation network,

c The computation of the triangulation network and the computation of the side Alkmaar–Bergen op Zoom (*AlBz*),

d The astronomical measurements, that is to say the determination of the latitudes of *Al* and *Bz* and his house in Leiden and the determination of the astronomical azimuth from his house to the Jacobstoren (toren = tower) in The Hague,

e The transfer of this azimuth to the side Leiden Stadhuis (stadhuis = townhall) – The Hague Jacobstoren (*LHg*) of the triangulation network,

f The computation of the length along the meridian of Alkmaar from Alkmaar to the intersection point with the parallel circle of *Bz*.

From the results in *d* and *f* follows the circumference of the earth.

In the following paragraphs I shall treat these operations in detail and also analyse SNELLIUS' observations after the publication of his *E(ratosthenes) B(atavus)* for he had detected errors in his original observations when carrying out measurements with his students in the surroundings of Leiden.

The changes he made – sometimes improvements, sometimes deteriorations – were very extensive. He even extended his triangulation network with some triangles in the south as far as Malines (Mechelen) in Belgium.

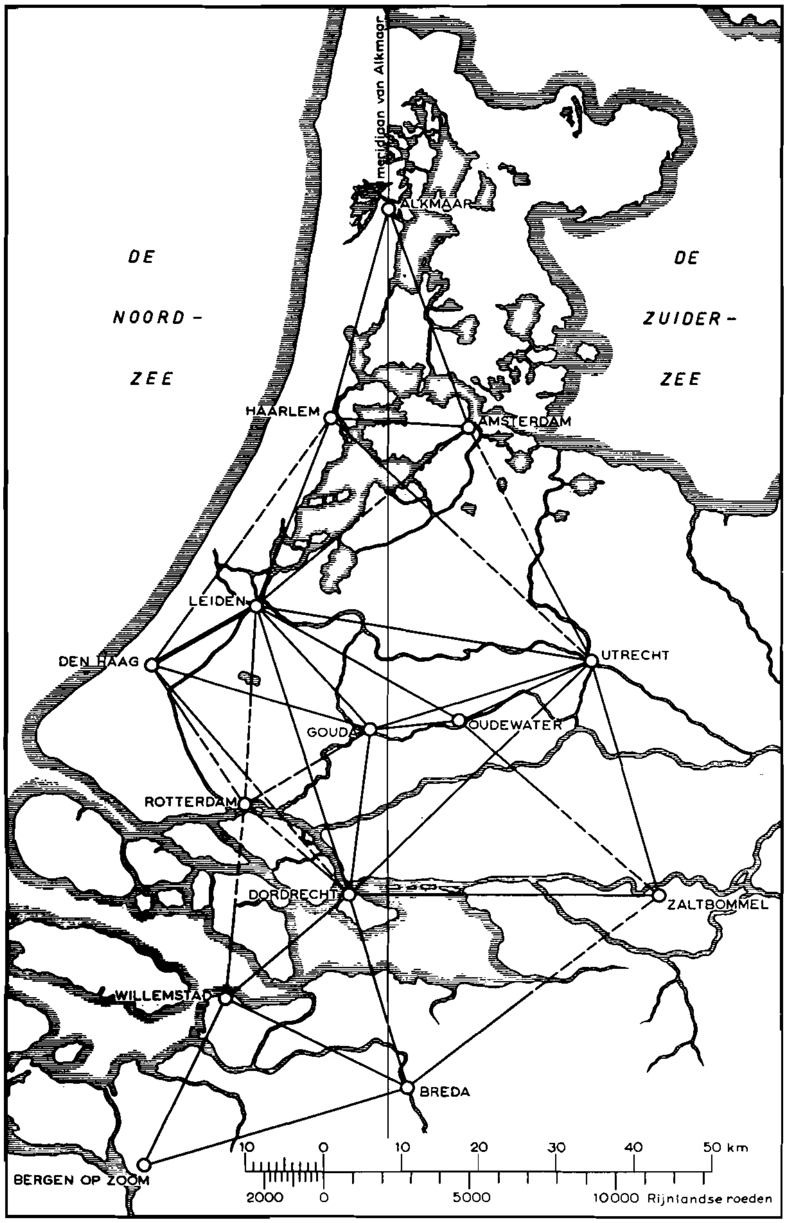
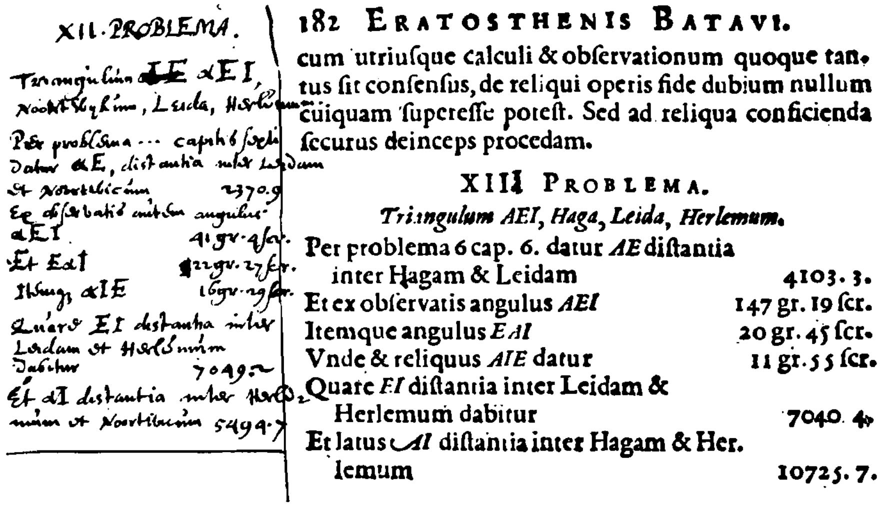


Fig. 36

These changes in his work were never published by SNELLIUS and his death on October 30th, 1626 made an end of his intentions in this respect. In his own copy of *E.B.*, however, he made changes on several places and on 24 additional pages he informs us of the extension which he gave to his triangulation.

After SNELLIUS' death this book was sold. In 1717 it was in the monastery of the Franciscans in Louvain. Later on it came into the possession of C. VAN BAVIÈRE in Brussels. In 1817


182 ERATOSTHENIS BATAVI.

cum utriusque calculi & observationum quoque tantus sit consensus, de reliqui operis fide dubium nullum cuiquam supeesse potest. Sed ad reliqua conficienda securus deinceps procedam.

XIII PROBLEMA.

Triangulum AEI, Haga, Leida, Herlemum.

Per problema 6 cap. 6. datur *AE* distantia inter Hagam & Leidam	4103. 3.
Et ex observatis angulus *AEI*	147 gr. 19 scr.
Itemque angulus *EAI*	20 gr. 45 scr.
Vnde & reliquus *AIE* datur	11 gr. 55 scr.
Quare *EI* distantia inter Leidam & Herlemum dabitur	7040. 4.
Et latus *AI* distantia inter Hagam & Herlemum	10725. 7.

Fig. 37

it was bought by the well-known bibliophile C. VAN HULTHEM whose library was bought in 1837 by the Belgium Government. Since then it is in the Royal Library in Brussels.

It is peculiar that with all these annotations and additions no observations can be found relating to a base extension net measured in 1622. Only from a note on page 182 of the "Brussels' copy" of the book one can see that these observations must have been carried out, as SNELLIUS mentions there that the distance Leiden–Noordwijk (see fig. 38) = 2370.9 roods. He borrows this distance from chapter VI of his book but neither there nor elsewhere can be found how it was computed. As a curiosity and because this distance is so important in the following considerations, I reproduce the upper part of this page as fig. 37. The note in question can be found in the margin at the left.

About 100 years after SNELLIUS' death his notes came into the hands of PETRUS VAN MUSSCHENBROEK (1692–1761), then a professor in Utrecht but since 1740 professor of mathematics and philosophy in the Leiden university. He checked SNELLIUS' notes and made a great number of changes in them, as is said on the ground of own observations. On these corrections he based new computations which were published in his *De magnitudine terrae*, forming part of *Physicae experimentalis et geometricae de magnete* (Leiden 1729).

In the first part of this book VAN MUSSCHENBROEK copies the *E.B.* of 1617, to which are added the changes made by SNELLIUS in the "Brussels' copy". In the second part (pages 398–420) he recomputes the triangulation net between Alkmaar and Bergen op Zoom, using now the base extension net measured by SNELLIUS on the ice in January and February 1622. The observations which should be made then come therefore only to light with VAN MUSSCHENBROEK. As the reader will see I doubt in many cases the authenticity of these observations or to say it less euphemistically, I shall be able to prove that they are often falsified in a serious manner in order to get a closing computation.

34 Snellius' base lines *ae* and *ig* and the computation of the side Leiden - The Hague of his network

There are five base lines, all situated in the surroundings of SNELLIUS' dwelling place Leiden.

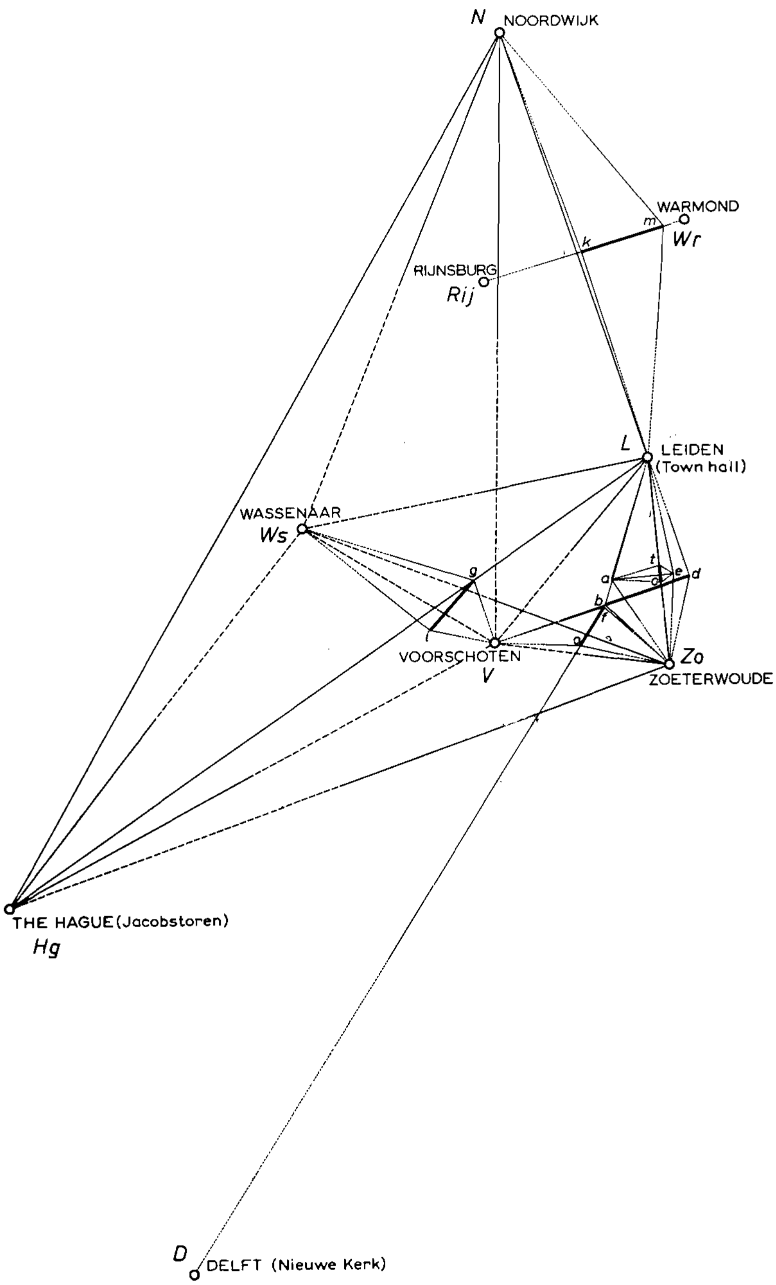


Fig. 38

They are indicated in fig. 38. Leiden (stadhuis) – The Hague (Jacobstoren) is the side of the triangulation network which must be computed from the measured lengths and angles.

A base line in the surroundings of Oudewater is left out of consideration. It is described on the pages 179 and 180 of his *Eratosthenes Batavus* and it was meant for a check on the side Gouda–Oudewater of his network. By the poor construction of this base extension net the check closes very badly. Quite rightly SNELLIUS has rejected its results.

The base lines *tc* (*ae*) and *ig* have been measured in 1615 and 1616 with a surveyor's chain, the angles of the base extension nets with a quadrant (radius 2¼ feet = 0.69 m) or a semi-circle (radius 1¾ feet = 0.55 m). With these instruments, which were not yet fitted with optics, readings in minutes were possible. They were made by the famous instrument-maker WILLEM JANSZOON BLAEU whom I mentioned already in § 11 as TYCHO BRAHE's companion during the years 1594–1596.

The bases *tc* (*ae*) and *ig* are used for the computations in SNELLIUS' *E.B.*; *bd*, *of* and *km* date from January and February 1622. They were measured on the ice and form the basis of the computations in VAN MUSSCHENBROEK's *M.T.* The place-names in figure 38 are spires of church towers in those places. Provisional computations and a local investigation have shown that these towers still exist nowadays with the exception of the Leiden townhall, which was destructed by fire on February 12th, 1929. The co-ordinates $X'Y'$ of the points in the system of the Netherlands' *R(ijks) D(riehoeksmeting)* – those of Leiden before the fire – are mentioned in table 15.

Points	X'	Y'
Leiden	- 61342.49	+ 725.08
Zoeterwoude	- 60964.00	- 3396.20
The Hague	- 74077.22	- 8106.07
Wassenaar	- 68218.96	- 630.28
Voorschoten	- 64425.14	- 2934.98
Delft	- 70504.42	- 15420.22
Noordwijk	- 64185.08	+ 9189.33
Warmond	- 60568.00	+ 5438.88
Rijnsburg	- 64552.42	+ 4238.61

Table 15

The base *tc* and its extension to the side *LHg* of the network in fig. 38 is once more given at a larger scale in fig. 39. From *tc* = 87.05 roods (327.8 m) the extended base *ae* = 326.45 roods (1229 m) can be computed, from *ae* the side *LZo* = 1093.55 roods (4118 m) and from *LZo* and the angles *L* and *Zo* of the triangle *LZoHg* the distance *LHg* = 4107.87 roods (15470 m). Though the base *tc* is very short, it is, with a very good feeling for practice, excellently chosen between *L* and *Zo*. I discussed this manner of base extension already in § 21 (see fig. 19 and 20). SNELLIUS measured also *ae* with a surveyor's chain. He preferred, however, the results of the computation from *tc* on account of a check which will be discussed presently. The results of the computation are given in the numbers 1–8 and 16 of table 16. Columns 2 and 3 refer to the places where the several problems can be found in *E.B.*, columns 4, 5 and 6 to the various triangles and the angles of these triangles according to

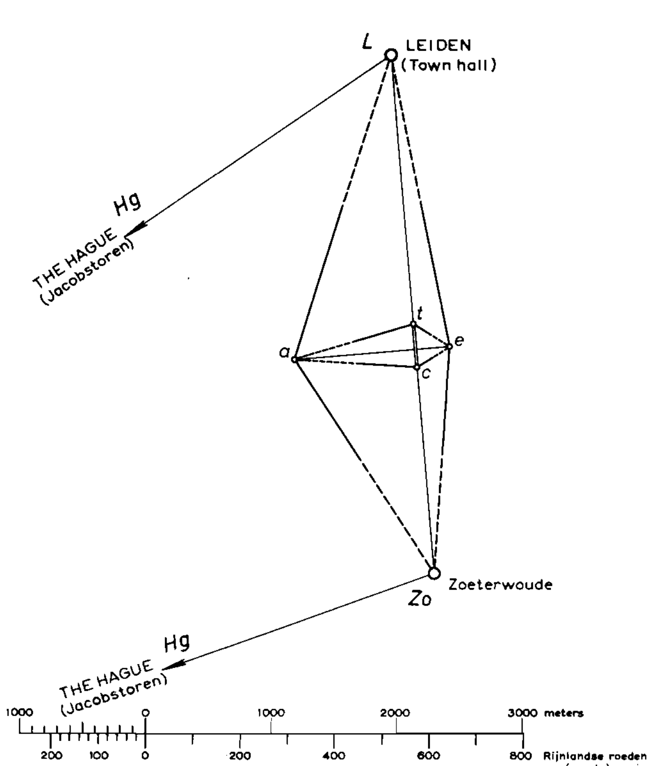


Fig. 39

nº	E.B.		Tri-angle	Angles		v	from	Opposite sides (roods)		
	prob.	page		E.B.	R.D.	6−5		E.B.	checked	R.D.
1	2	3	4	5	6	7	8	9	10	11
1	—	159	*t*	54°00'	——	——		*79.66*	*79.66*	——
			c	63 52	——	——		*88.40*	*88.40*	——
			e				base	87.05	87.05	——
2	—	159	*t*	78 30	——	——		*256.30*	*257.34*	——
			c	82 08½	——	——		*260.15*	*260.15*	——
			a				base	87.05	87.05	——
3	—	159	*a*				1	88.40	88.40	——
			t	132 30	——	——		*326.43*	*326.45*	——
			e				2	260.15	260.15	——
4	—	——	*a*				1	79.66	79.66	——
			c	146 00½	——	——		——	*326.45*	——
			e				2	256.30	257.34	——
5	—	159/160	*a*	67 44	——	——		*624.30*	*624.45*	——
			e	83 20	——	——		*670.19*	*670.20*	——
			L				3	326.43	326.45	——
6	—	160	*a*	61 38	——	——		*478.58*	*478.60*	——
			e	81 29	——	——		*537.90*	*537.91*	——
			Zo				3	326.43	326.45	——
7	—	160	*L*				6	537.90	537.91	——
			a	128 52	——	——		*1092.33*	*1093.55*	1098.94
			Zo				5	670.19	670.20	——
8	—	——	*L*				6	478.58	478.60	——
			e	164 49	——	——		——	*1093.55*	1098.94
			Zo				5	624.30	624.45	——
9	I	161	*Zo*	63 57	63°53.0'	−4.0		*1853.63*	*1855.69*	1861.06
		162	*L*	84 05	84 05.8	+0.8		*2052.12*	*2054.51*	2061.69
			Ws				7	1092.33	1093.55	1098.94
10	II	162	*Zo*	77 12	77 09.8	−2.2		*1263.68*	*1265.10*	1270.65
			L	45 21	45 21.2	+0.2		*921.91*	*922.94*	927.17
			V				7	1092.33	1093.55	1098.94
11	III	162	*L*	38 45	38 44.7	−0.3		*1174.41*	*1175.72*	1178.70
		163	*Ws*				10	1263.68	1265.10	1270.65
			V				9	1853.63	1855.69	1861.06
12	IV	163	*i*	92 10	——	——		*938.72*	*938.71*	——
			g	66 05	——	——		*858.73*	*874.65*	——
			Ws				base	348.10	348.10	——
13	IV	164	*i*	60 11	——	——		*347.06*	*347.06*	——
			g	59 20	——	——		——	——	——
			V				base	348.10	348.10	——
14	IV	164	*Ws*				13	347.06	347.06	——
			g	125 25	——	——		*1174.42*	*1174.40*	1178.70
			V				12	938.72	938.71	——
15	V	166	*L*	23 36	23 35.4	−0.6		——	——	——
			Hg	17 09	17 10.6	+1.6	9	1853.63	1855.69	1861.06
			Ws					*4103.36*	*4107.98*	4115.02
16	VI	166	*L*	60 32	60 30.4	−1.6		——	——	——
		167	*Zo*	104 32	104 30.6	−1.4		*4103.21*	*4107.87*	4115.02
			Hg				7	1092.33	1093.55	1098.94
17	—	165	*L*	6 12	15 09.8	——		——	——	——
			Hg	15 10	6 33.6	——	10	1263.68	1265.10	1270.65
			V					——	——	1098.94

Table 16

E.B. and *R.D.* Column 7 gives the differences *v* between these angles in minutes and columns 9 and 11 the opposite sides in roods according to SNELLIUS and *R.D.* The upright numbers are given values, the italic numbers computed values. The lengths in column 11 refer to distances in the plane of (the stereographic) projection. In this area they are about 6.5 mm per 100 m or 0.065 roods per 1000 roods smaller than the distances on earth.

As SNELLIUS was a shoddy calculator I verified his computations in column 10. In SNELLIUS' days such computations must have been a tremendous work as they had to be carried out without computers and calculating machines, even without logarithms, because the "invention" of logarithms by JOHN NAPIER in 1614 did not lead immediately to a common use of them.

The table will be clear. Already in No. 2 one sees that SNELLIUS made a mistake in the computation of *ac*. Fortunately his computation of *ae* in No. 3 is good, though he has not used the check which I give in No. 4.

No. 7 shows again SNELLIUS' shoddy way of computing: $LZo = 1092.33$ roods – according to page 160 "accurate and absolutely correct" – must be $LZo = 1093.55$ roods. He could have found this mistake if he had checked this distance as I did in No. 8.

In No. 16 one finds the computation of *LHg* in triangle *LZoHg*. SNELLIUS finds $LHg = 4103.21$ roods. Had he not made a mistake in the computation of *LZo* he would have found 4107.87 roods. The length of *LHg* is checked in the numbers 9–15 of table 16. In Nos. 9 and 10 SNELLIUS computes the lengths *LWs* and *LV* respectively and from these amounts and the angle *L* in No. 11 the distance $VWs = 1174.41$ roods. This distance is checked by the measurement of a second base *ig* between *V* and *Ws* and the extension of this base to the side *VWs* (see the numbers 12, 13 and 14 of the table). To SNELLIUS' joy it proved that the result $VWs = 1174.42$ roods (in 14) differed but 0.01 rood from 11. The difference in column 10 is somewhat bigger: 1.32 roods $\approx$ 5 metres. Finally *LHg* is computed in 15 from *LWs* and the angles *L* and *Hg* of triangle *LHgWs*. The result is almost the same as that in No. 16.

In a second check of *LHg* (No. 17 of the table) SNELLIUS failed. It was his intention to compute this distance from the length $LV = 1263.68$ roods and the angles $L = 6°12'$ and $Hg = 15°10'$ of triangle *LHgV*. $Hg = 15°10'$, however, must have been measured in *L*, and $L = 6°12'$ in *Hg*. Moreover the latter value is wrong; according to the data of *R.D.* it must be $6°33.6'$.

Apart from this mistake it must be said that the results 4103.36 and 4103.21 roods (must be 4107.98 and 4107.87 respectively) are excellent. The mean amount of 4107.92 roods differs but 7.10 roods ($\approx$ 27 metres) or 0.2 percent from the correct distance 4115.02 roods. It is in my opinion the best result that could be obtained in those days and it contrasts favourably, also by its excellent checks, with TYCHO BRAHE's primitive base extension in § 21.

35 Speculations on the base lines *ae* and *ig*

In the base points *a* and *e* in fig. 39 SNELLIUS measured not only the angles mentioned in Nos. 5 and 6 of table 16 (column 5) but also the base angles of five other triangles. The tops of these triangles were towers in Leiden. In the problems II, III, IV, V and VI on the pages 201 and 202 of his book he calls them *o*, *u*, *y*, *s* and *r* respectively. *o*, *u* and *y* still exist nowadays. They are identical with the spires of Lodewijkskerk (*Lo*), Hooglandse kerk (*Ho*) and Pieterskerk (*P*) respectively. They are marked on the map of a part of the present Leiden in fig. 40, together with the base points *c*, *t*, *a* and *e*.

The observations relating to these towers are mentioned in the numbers 5, 6 and 7 of table 17 which is arranged in the same way as table 16. Those relating to *s* and *r* are left out.

Table 17

nº	E.B.		tri-angle	Angles		v	from	Opposite sides (roods)		
	prob	page		E.B.	R.D.	6−5		E.B.	checked	R.D.
1	2	3	4	5	6	7	8	9	10	11
1	—	159	*t*	54°00'				*79.66*	*79.66*	
			c	63 52				*88.40*	*88.40*	
			e				base	87.05	87.05	87.5
2	—	159	*t*	78 30				*256.30*	*257.34*	
			c	82 08½				*260.15*	*260.15*	
			a				base	87.05	87.05	87.5
3	—	159/160	*a*	67 44	67°44'	0		*624.30*	*624.45*	627.7
			e	83 20	83 21	+1'		*670.19*	*670.20*	673.7
			L				tab.16/3	326.43	326.45	328.0
4	—	160	*a*	61 38	61 38	0		*478.58*	*478.60*	480.9
			e	81 29	81 28	−1		*537.90*	*537.91*	540.4
			Zo				tab.16/3	326.43	326.45	328.0
5	II	201	*a*	64 34	64 34	0		——	*557.63*	559.3
			e	83 31	83 27	−4		——	*613.53*	615.3
			Lo				tab.16/3	326.43	326.45	328.0
6	III	201	*a*	62 29	62 29	0		*603.1*	*603.20*	608.0
			e	88 50	88 56	+6		*679.9*	*680.00*	685.4
			Ho				tab.16/3	326.43	326.45	328.0
7	IV	201	*a*	71 30	71 30	0		*620.0*	*620.10*	622.3
			e	78 33	78 30	−3		*640.7*	*640.87*	643.0
			P				tab.16/3	326.43	326.45	328.0
8	VIII	202	*a*	5 15				*62.6*	*62.61*	63.27
			L				6	679.9	680.00	685.4
			Ho				3	670.19	670.20	673.7
9	IX	203	*a*	3 46				*52.0*	*52.12*	53.06
			L				7	640.7	640.87	643.0
			P				3	670.19	670.20	673.7
10	X	203	*a*	9 01				*110.9*	*110.91*	112.63
			Ho				7	640.7	640.87	643.0
			P				6	679.9	680.00	685.4

Points	Co-ordinates (roods)	
	X	*Y*
c	0.00	0.00
t	+ 87.05	0.00
a	+ 35.18	+254.93
e	+ 35.09	− 71.52
L	+655.34	+ 0.80
Zo	−438.21	− 0.50
Lo	+589.17	− 8.72
Ho	+638.17	− 59.41
P	+642.88	+ 51.40

Table 18

The angles in *t* and *c* with tops in *e* and *a* and those in *a* and *e* with tops in *L* and *Zo* from table 16 are once more given in the numbers 1–4.

For reasons which will be discussed in § 49 SNELLLIUS computed also the lengths of the sides of triangle *PHoL*. I give his computation in the numbers 8, 9 and 10 of table 17.

Fortunately he made no mistakes in these computations though they were not checked. He could have checked them by computing the same distances in the triangles *eLHo, eSP* and *eHoP*.

Points	System Snellius		System R.D.		System *XY* brought into sympathy with system *X'Y'*		Differences	
							v_i	w_i
i	X_i	Y_i	X_i'	Y_i'	X_i''	Y_i''	(5−3)	(6−4)
	1	2	3	4	5	6	7	8
c	0.0	0.0			−61115.1	− 1745.1		
t	+ 327.8	0.0			−61145.3	− 1417.0		
a	+ 132.5	+ 960.1			−62088.3	− 1699.8		
e	+ 132.2	− 269.3			−60857.9	− 1588.4		
L	+ 2468.0	+ 3.0	−61342.49	+ 725.08	−61342.8	+ 724.6	−0.4	−0.6
Zo	− 1650.3	− 1.9	−60964.00	−3396.20	−60963.6	− 3396.6	+0.4	−0.4
Lo	+ 2218.8	− 32.8	−61286.17	+ 473.98	−61284.3	+ 478.4	+1.9	+4.4
Ho	+ 2403.4	÷ 223.8	−61107.18	+ 687.57	−61110.0	+ 680.5	−2.8	−7.1
P	+ 2421.1	+ 193.6	−61530.23	+ 656.64	−61529.3	+ 660.3	+0.9	+3.7
							0.0	0.0

Table 19

It will be clear that, instead of SNELLIUS' way of computing, it is much easier to determine the checked lengths in column 10 by computing on a calculating machine the co-ordinates of the nine points *c*, *t*, *a*, *e*, *L*, *Zo*, *Lo*, *Ho* and *P* in a local co-ordinate system. The distances required follow then from the co-ordinates. As the origin of this co-ordinate system I used *c*. *ct* was the positive *X*-axis. The co-ordinates (in roods) are given in table 18 and once more (in metres) in table 19 (columns 1 and 2).

As the *Y*-co-ordinates of *L* and *Zo* are very small the base *tc* must have been set out between the towers of Leiden and Zoeterwoude. The small difference between the *X*-co-ordinates of *a* and *e* proves that *ae* and *tc* are perpendicular. The sketch on page 157 of SNELLIUS' *E.B.* agrees with these conclusions. The Latin text l.c, however, is not quite clear.

By a similarity transformation the co-ordinates in columns 1 and 2 are brought into sympathy with those of *R.D.* in columns 3 and 4 (co-ordinates $X'Y'$). The result of the computation ($X''Y''$) is mentioned in columns 5 and 6. The remaining differences v and w in columns 7 and 8 give an impression of the accuracy of SNELLIUS' measurements.

If one assumes that the co-ordinates of *a* and *e* in columns 5 and 6 are reasonably reliable, one can compute from these co-ordinates and the *R.D.*-co-ordinates of the five towers in columns 3 and 4, the angles which SNELLIUS should have found in *a* and *e*. They are, rounded-off in minutes, mentioned in column 6 of table 17. The differences v between these angles and those measured by SNELLIUS, are remarkable. In the western base point *a* all v's are zero; the differences in columns 7 and 8 of table 19 must therefore only be imputed to the inaccurate observations in *e*. The amount $+6'$ in that point is a striking example.

In my opinion the explanation of this phenomenon must be found in the influence of the prevailing western wind on the observations. In *a* this influence on the large instrument could be screened off; in *e* this is not possible.

In my opinion it is also striking that the base angles in *e* with tops in *L* and *Zo* are much better than the three other ones. I think, as they had to be used for SNELLIUS' "first order triangulation", that they were measured with a "first order" instrument and on an earlier date. At any rate they are mentioned on quite different places in his *E.B.*

Thanks to the kind co-operation of the surveying department of the municipality of Leiden I could plot the base points *c*, *t*, *a* and *e* on a photogrammetric map at a scale of 1 to 1000 in the co-ordinate system of *R.D.* A reduced reproduction of this map, to which is added a southern part of the present town, is reproduced as fig. 40.

All points lie in a rural area of the cadastral municipality of Zoeterwoude and in the plots indicated on the map. The terrain has changed very little during the last few centuries.

The choice of *t* and *c* is very plausible; *t* on the south side, *c* on the north side of the same crooked watercourse, *t* in the grass land, close by the water side, *c* at the foot of a talus which runs from the higher situated water-course to the lower grass land. *tc* lies therefore in flat terrain so that it could easily be measured. *a* and *e* lie far from ditches. The terrain, until now intact, will in future change completely as it lies in the town development plan of Leiden.

From the co-ordinates of *a* and *e* in columns 5 and 6 of table 19 and those of the five towers in columns 3 and 4 follow the distances in the *R.D.*-system which I mentioned in column 11 of table 17 (Nos. 3–7). They are all larger than the corresponding amounts in column 10. As the base length *tc*, which can be computed from the co-ordinates $X''Y''$ of the terminal points in table 19, is 87.5 roods instead of the measured length 87.05 roods,

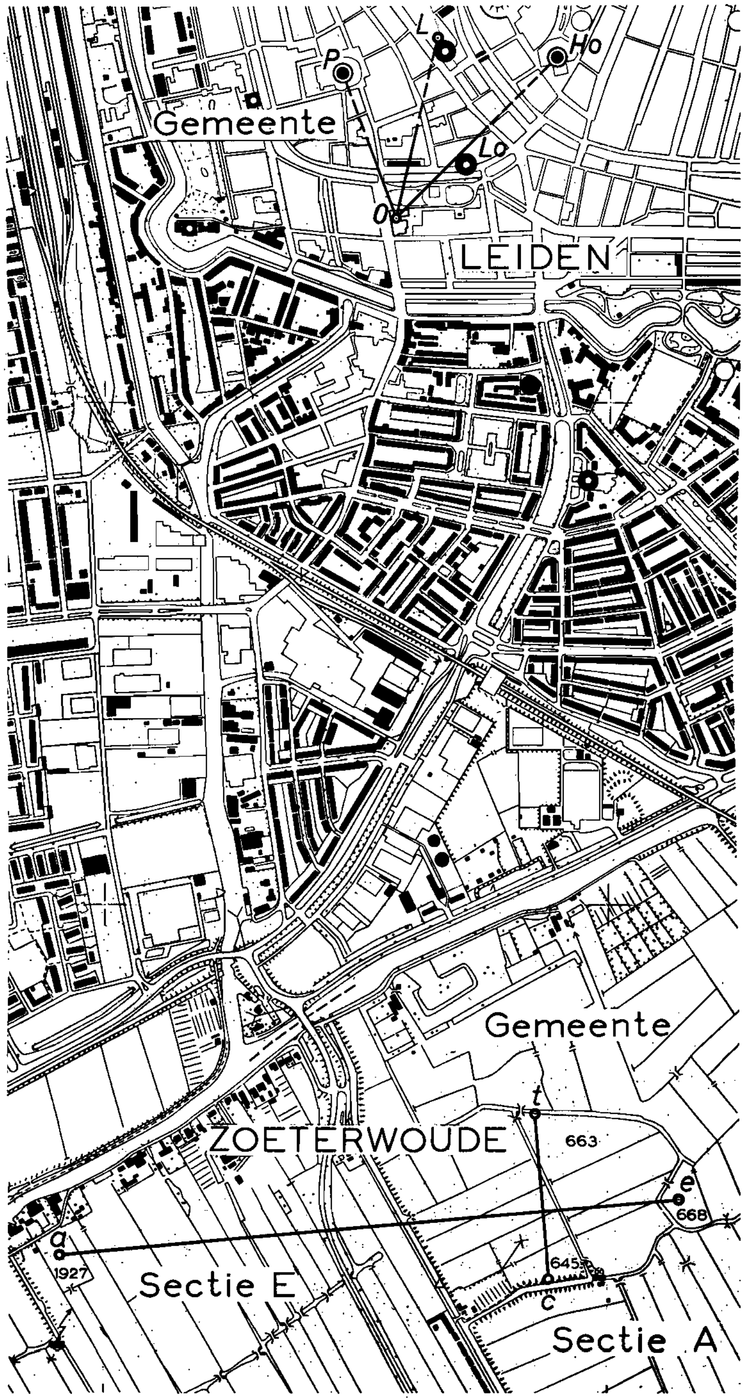

Gemeente
LEIDEN
Gemeente
ZOETERWOUDE
Sectie E
Sectie A
P
L
Ho
Lo
O
t
663
e
668
a
1927
645
c

Fig. 40

Fig. 41

it might be possible that the difference of 0.45 roods (about 0.5 percent) is caused by the use of an old surveyor's chain. Because of the large number of worn-out points in such a chain it becomes too long; the measured distance is therefore too short.

One must take into account, however, that each of the distances has a standard error caused by the standard errors in the measured base tc and the angles α.

According to formula (6) in § 21 this standard error for ae is

$$m_{ae}^2 = \left(\frac{ae}{tc}\right)^2 m_{tc}^2 + \left(\frac{tc^2}{2ae}\right)^2 m_\alpha^2 [A_1^2 + B_1^2 + C_1^2 + D_1^2] : \varrho^2$$

$$= 14.1 m_{tc}^2 + 0.101 m_\alpha^2$$

(m_α in minutes for $\varrho = 3437.7468$) and for LZo

$$m_{LZo}^2 = \left(\frac{LZo}{ae}\right)^2 m_{ae}^2 + \left(\frac{ae^2}{2LZo}\right)^2 m_\alpha^2 [A_2^2 + B_2^2 + C_2^2 + D_2^2] : \varrho^2$$

$$= 11.2 m_{ae}^2 + 0.389 m_\alpha^2$$

If one takes $m_{tc} = 0$ and $m_\alpha \approx 2.5'$, an amount which can be derived from the v's in table 17, $m_{ae} \approx 0.80$ roods and $m_{LZo} \approx 3.1$ roods. The difference 5.39 roods between *R.D.* and SNELLIUS' computation of the side Leiden (townhall) – Zoeterwoude (see table 16 No. 7) can therefore be explained by the standard error in the measurement of the angles α of his base extension net. As all the distances, however, are larger, also the distances *LHo*, *LP*, *HoP* (see table 17 Nos. 8–10) and *VWs*, the latter derived from the base *ig* (see table 16 No. 14), a systematic error in the surveyor's chain seems likely.

As, according to (6) $m_{VWs} = 2.5$ roods the difference $1178.70 - 1174.40 = 4.30$ roods in table 16 can also be imputed to m_α.

The *R.D.*-co-ordinates of *i* and *g* can be found by a similarity transformation on the points *V* and *Ws*. The result is $X_i'' = -65699.8$ $Y_i'' = -2677.0$, $X_g'' = -64832.1$ $Y_g'' = -1687.9$. They are plotted on a topographical map 1 to 10,000. A reproduction of this map at a reduced scale is given in fig. 41. Both points lie in Voorschoten. As one can see, the choice of the base is very well adapted to the shape of the plots.

36 Van Musschenbroek's computation of the side Leiden - The Hague

Van Musschenbroek's computation of the side *LHg* can be found in table 20; it is similar to table 16 in which Snellius' computation is given. Columns 2 and 3 refer now to the problems and pages of *M(agnitudine) T(errae)*. Van Musschenbroek computes first the distance *LZo* (No. 1–3), as Snellius does in his *E.B.* The base $tc = 87.05$ roods of the year 1615 is now replaced by the much longer and therefore much better base *bd*. Its length is 475.00 roods. It is pointed at Voorschoten and, as already said, measured on the ice in January 1622. It is represented once more in fig. 42 with the points *L*, *Zo* and *V*. The computation $LZo = 1097.10$ roods gives an excellent result.

LV is computed twice. The first computation is mentioned in No. 4 of the table. The angle *L* in that computation is $VLZo\text{-}bLZo = 45°21' - 22°07'40'' = 23°13.3'$. *VLZo* is borrowed from Snellius (table 16 No. 10) and *bLZo* is computed from the data in quandrangle *bLdZo* of fig. 42. Step 5 in table 20 is the same as step 10 in table 16. Probably in order to get a closure Snellius' $Z = 77°12'$ is changed into 77°10.5′ which is indeed somewhat better. $LV = 1268.63$ roods differs but very little from the amount of 1270.65 roods given by the *R.D.*

In No. 5 Van Musschenbroek computes also the side $ZoV = 925.60$ roods. This distance is checked by measuring the base $of = 250.00$ roods and the base angles of the triangles *ofZo* and *ofV* (see fig. 38 and table 20 Nos. 6–8). Wrongly he says that this check should close ($ZoV = 925.60$) but, as it will be shown in § 39, the closure is made in a rather primitive manner. In step 9, which is similar to step 9 in table 16, *LWs* and *ZoWs* are computed.

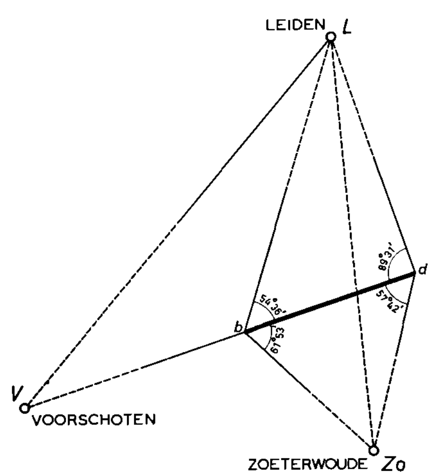


Fig. 42

Phase 10 is the same as phase 11 of Snellius.

In the phases 11–16 of his computation Van Musschenbroek uses the tower of Noordwijk (*N*), incorporated in Snellius' base extension net by the measurement of the base *km* (see fig. 38).

In No. 11 he computes $NL = 2338.22$ roods from *VL* and the angles of triangle *NVL* and in Nos.

nº	M.T.		Tri-angle	Angles		v	from	Opposite sides (roods)		
	prob.	page		M.T.	R.D.	6 – 5		M.T.	checked	R.D.
1	2	3	4	5	6	7	8	9	10	11
1	I	398	b	54°36'	54°36.5'	+0.5'		*660.57*	*660.54*	661.75
	see also table 21		d	89 31	89 30.4	-0.6		*810.36*	*810.36*	811.73
			L				base	475.00	475.00	475.82
2	II	398	b	61 53	61 52.6	-0.4		*481.74*	*481.75*	482.55
		399	d	57 42	57 42.3	+0.3		———	———	462.51
	see also table 21		Zo				base	475.00	475.00	475.82
3	III	399	L				2	481.74	481.75	482.55
	see also table 21		d	147 13	147 12.7	-0.3		*1097.10*	*1097.05*	1098.94
			Zo				1	660.57	660.54	661.75
4	IV	400	V				1	810.36	810.36	811.73
			L	23 13.3	23 13.4	+0.1		———	———	———
			b	125 24	125 23.5	-0.5		*1268.63*	*1268.63*	1270.65
5	V	400	Zo	77 10.5	77 09.7	-0.8		*1268.63*	*1268.65*	1270.65
	VI	401	L	45 21	45 21.1	+0.1		*925.60*	*925.63*	927.17
			V				3	1097.10	1097.05	1098.94
6	VII	401	o	70 34	70 31.4	-2.6		———	———	———
	see also table 21		f	78 38	78 40.6	+2.6		*478.66*	*478.67*	479.42
			Zo				base	250.00	250.00	250.36
7	VII	401	o	119 44	119 46.6	+2.6		———	———	———
	see also table 21		f	40 02.6	39 32.9	-29.7		*461.66*	*465.29*	451.51
			V				base	250.00	250.00	250.36
8	VII	401	V				6	478.66	478.67	479.42
	see also table 21		o	169 42 *	169 42.0	0.0		*925.60*	*940.15*	927.17
			Zo				7	461.66	465.29	451.51
9	VIII	402	Zo	63 57	63 53.0	-4.0		*1861.58*	*1861.63*	1861.06
	IX	402	L	84 05	84 05.8	+0.8		*2061.04*	*2061.09*	2061.69
			Ws				3	1097.10	1097.05	1098.94
10	X	402	L	38 45	38 44.7	-0.3		*1179.49*	*1179.56*	1178.70
			Ws				5	1268.63	1268.65	1270.65
			V				9	1861.58	1861.63	1861.06
11	XI	403	N	19 57	19 41.9	-15.1	5	1268.63	1268.65	1270.65
			V	38 58	38 58.3	+0.3		*2338.22*	*2338.26*	2370.90
			L	121 05	121 19.8	+14.8		———	———	———
12	XII	403	k	88 26	88 23.4	- 2.6		*1219.64*	*1219.66*	1222.45
			m	68 50	68 43.7	- 6.3		*1137.71*	*1137.79*	1139.61
			L				base	471.50	471.50	475.53
13	XII	403	k	91 56	91 58.6	+ 2.6		———	*1314.93*	1335.15
			m	67 04	67 10.3	+ 6.3		*1210.50*	*1211.69*	1231.31
			N				base	471.50	471.50	475.53
14	XII	403	N				12	1137.71	1137.79	1139.61
			k	179 38	179 38.0	0.0		*2338.00*	*2349.48*	2370.90
			L				13	1210.50	1211.69	1231.31
15	XII	403	N				12	1219.64	1219.66	1222.45
			m	135 54	135 54.0	0.0		———	*2349.48*	2370.90
			L				13	———	1314.93	1335.15
16	XIII	403	L	106 03	106 10.6	+ 7.6		———	———	———
			N	48 42	48 19.9	-22.1		*4118.07*	*4118.06*	4115.02
			Hg	25 15	25 29.5	+14.5	11	2338.22	2338.26	2370.90
17	XIV	404	L	60 32	60 30.4	- 1.6		———	———	———
			Zo	104 32	104 30.6	- 1.4		*4120.81*	*4121.02*	4115.02
			Hg				3	1097.10	1097.05	1098.94

* Van Musschenbroek mentions 159°42'; this is not a printer's error but a mistake in the calculation.

Table 20

12–14 he "checks" this distance with the base *km* and the base angles in the triangles *kmL* and *kmN* (*NL* = 2338.00 roods).

In No. 16 *LHg* is found from *LN* and the angles of triangle *LNHg*. Finally the result *LHg* = 4118.07 roods is checked in No. 17 and well in the same triangle and with the same angles SNELLIUS used in No. 16 of table 16. The result, 4120.81 roods, differs but 2.74 roods or about 10.3 metres from the first computation.

37 Speculations on Snellius' base line *km* in Van Musschenbroek's "De Magnitudine Terrae"

The reader will have noticed that in the numbers 11–16 of table 20, that is to say in those triangles in which *N* is used, the results of the computation deviate very much from the data of *R.D.* In No. 14 *NL* is wrongly computed. VAN MUSSCHENBROEK could have checked this distance as indicated in No. 15. Nevertheless his wrong result is almost exactly the same as the amount found in No. 11. It is clear that this is a question of falsification of the observations. Since in No. 16 the falsified *NL* = 2338.22 roods is also used, the angles of triangle *LNHg* had also to be falsified in order to obtain a length *LHg* which deviates but little from the result in No. 17.

The falsification is to such a great extent at variance with our present scientific views that it is hardly credible. It presumes a fictitious tower of Noordwijk on a spot about 130 metres south-south-east of the place where it stood already in the 13th century and where it is still present nowadays.

Apart from VAN MUSSCHENBROEK's cheating in the numbers 11 and 16 of the table it is interesting that SNELLIUS found *LN* = 2370.9 roods (see the reproduction of the Brussels' copy of *E.B.* in fig. 37). This amount happens to agree exactly with the *R.D.*-distance in column 11 of table 20. From *km* = 471.50 roods and the base angles of the triangles *kmL* and *kmN* this value for the distance *LN* cannot be found unless SNELLIUS would have made a large error in his calculation; for according to the numbers 14 and 15 of the table it is 2349.48 roods.

The correctness of the angles in question and the base length *km* = 471.50 roods imply an enlargement λ_{km} = 2370.90 : 2349.48 = 1.0091 of the base net between Leiden and Noordwijk. Apart from the correctness of the angles this factor means an unlikely large systematic error in the measurement of *km* of more than 0.9 percent. If this systematic error would have been zero, *km* = 475.8 roods. I am of the opinion that SNELLIUS measured *km* = 475.00 roods and that VAN MUSSCHENBROEK altered this length into 471.50 roods in order "to get a better result". If this conclusion would be correct SNELLIUS should have found *LN* = 2366.92 roods provided VAN MUSSCHENBROEK did not alter the base angles as well. In that case the difference of 4.0 roods might be imputed to SNELLIUS' shoddy way of calculating. The amount would be in harmony with the round values 475.00 roods and 250.00 roods respectively of the base lines *bd* and *of*, also measured on the ice in January–February 1622.

My supposition is confirmed by the following consideration.

It is clear that by the similarity transformation with which the quadrangle *LkNm* (see fig. 43) can be transferred to the points *L* and *N* of the *R.D.*-triangulation network, the place of *k* and *m* is exclusively determined by the four angles in these points. Only in case

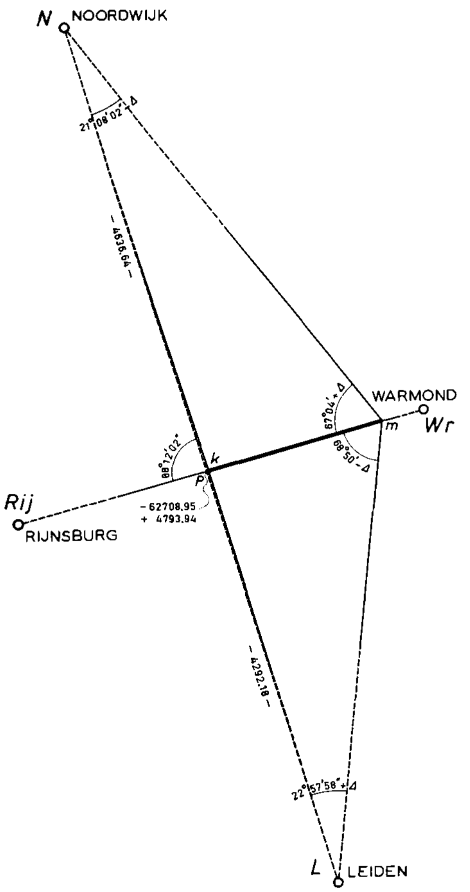


Fig. 43

of an "ideal" measurement k and m would fall on the connecting line of the towers Rijnsburg and Warmond between which they are assumedly set out with a sufficient accuracy.

Provisional computations have shown that both points lie north of this connecting line. In order to bring them on *RyWr* one must alter the angles k and m in the triangles *kmN* and *kmL*. I did that in such a manner that, both in k and in m, the positive correction Δ in *kmN* is equal to the negative correction in *kmL*. The sum of the angles in k (180° 22') and m (135° 54') remains therefore unaltered. The computation of Δ_m is elucidated in fig. 43. In that figure P is the intersection point of *NL* and *RyWr*. Its co-ordinates according to the data of *R.D.* are

$$X'_P = -62708.95 \qquad Y'_P = +4793.94$$

The distances *NP* and *PL* are 4636.64 and 4292.18 metres respectively. Angle *RyPN* is 88°12'02".

From triangle *PmN* follows:

$$Pm = \frac{4636.64 \sin(21°08'02'' - \Delta)}{\sin(67°04' + \Delta)}$$

and from triangle *PmL*:

$$Pm = \frac{4292.18 \sin(22°57'58'' + \Delta)}{\sin(68°50' - \Delta)}$$

From the equality of the first terms of these equations follows the equality of the second terms from which Δ can be resolved. From the result $\Delta = +6'20'' \approx +6.3'$ follows (with a check) $Pm \approx 1805.10$ metres and for the co-ordinates of m

$$X'_m = -60980.6 \qquad Y'_m = +5314.6$$

In a similar way in triangle *PkN*:

$$k = 88°0.4' - 2'36'' \approx 88°01.4'$$

and in triangle *PkL*:

$$k = 91°34' + 2'36'' \approx 91°36.6'$$

so that $Pk \approx 14.27$ metres and

$$X'_k = -62695.3 \qquad Y'_k = +4798.1$$

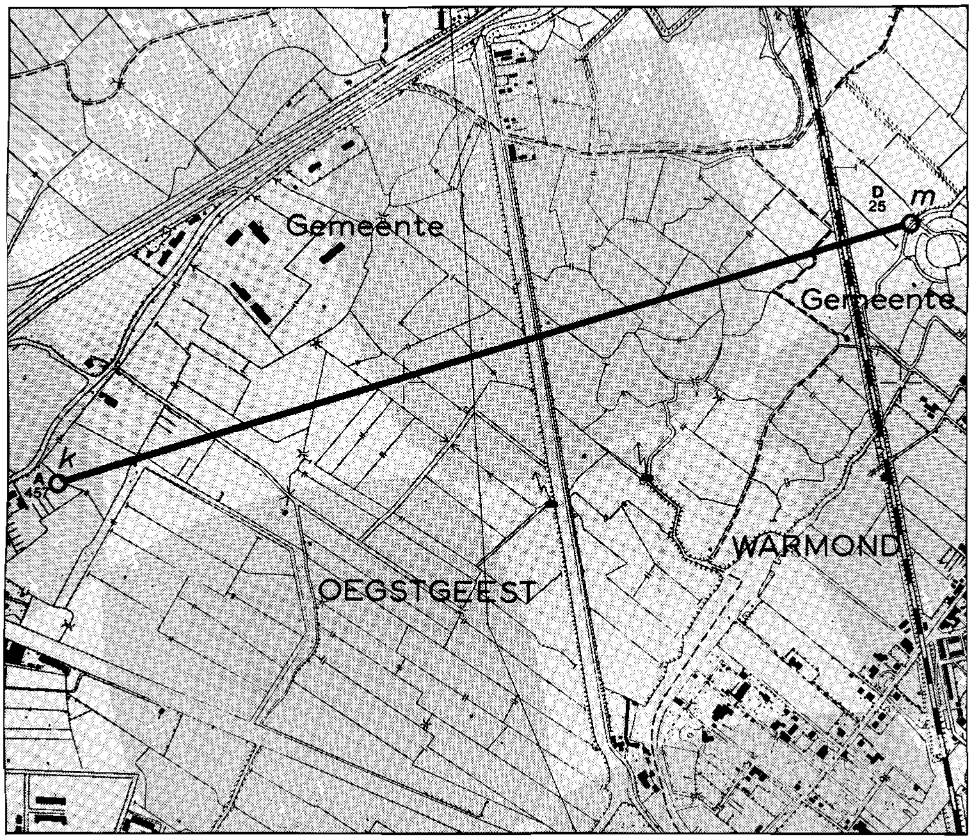


Fig. 44

From $km = Pm - Pk \approx 1790.8$ metres = 475.53 roods one sees that *km* must have been longer than the amount of 471.50 roods mentioned by VAN MUSSCHENBROEK. Assuming that $km = 475.00$ roods, the enlargement factor λ_{km} is 475.53 : 475.00 = 1.00112.

In table 20 the columns 6 and 11 are brought into agreement with the results just mentioned. Though the corrections $\Delta_m = -6.3'$ and $+6.3'$ in 12 and 13 are rather big, they can be tolerated in my opinion. I found a similar amount in table 17 (No. 6).

km has been projected on the topographical map 1 to 10,000. A reproduction of this map is given in fig. 44. In contradistinction to fig. 41 in § 35 there is here no adaption to the form of the plots as the ice coat which covered the flooded land had wiped out the boundaries.

38 Speculations on Snellius' base line *bd* in "De Magnitudine Terrae"

The *R.D.*-co-ordinates of the base points *b* and *d* (see fig. 42) can be computed by a similarity transformation of quadrangle *bLdZo* (table 20 Nos. 1–3). The result is

$$X'_b = -62230.21 \qquad Y'_b = -2200.29$$

$$X'_d = -60531.05 \qquad Y'_d = -1631.13$$

Fig. 45

The enlargement factor $\lambda_{bd} = 1098.94 : 1097.05 = 1.00172$ is almost the same as $\lambda_{km} = 1.00112$ found for the base net between Leiden and Noordwijk if $km = 475.00$ roods. Though one cannot attach too great a value to this phenomenon on account of the influence of the standard deviation in the measured angles, it corresponds much better than $\lambda_{km} = 1.0091$ found for $km = 471.50$ roods.

From the co-ordinates of b, d and V follow the distances $Vb = 2314.6$ metres and $Vd = 4106.6$ metres and the gridbearings $\overline{Vb} = 71°29.6'$ and $\overline{Vd} = 71°29.3'$, which differ but 0.3'. Therefore bd satisfies almost the condition that it is pointed at Voorschoten. From the gridbearing $\overline{Vb} = \overline{Vd} = 71°29.4'$ and the distances Vb and Vd follow the co-ordinates

$$X'_b = -62230.3 \qquad Y'_b = -2200.2$$

$$X'_d = -60531.0 \qquad Y'_d = -1631.3$$

They enabled me to plot them on the topographical map 1 to 10,000 (see fig. 45). Here too there is no connection between the topography and the position of the base. North of the base one finds the base points t, c, a and e, already discussed in § 35 (see also fig. 40).

The *R.D.*-results in columns 6 and 11 of the table have been computed by means of the co-ordinates of b and d just mentioned.

39 Speculations on Snellius' base line *of* in "De Magnitudine Terrae"

The last base which must be analysed is *of*. It is the base which served to check the distance $ZoV = 925.60$ roods in No. 5 of table 20. In § 36 we find already that VAN MUSSCHENBROEK made mistakes in the calculation of this check and that he falsified SNELLIUS' observations in order to get a closing result.

The base is indicated in fig. 46. It is measured on the ice on February 3, 1622 and it is pointed at the tower of the Nieuwe Kerk (New church) in Delft. The angles in the drawing are borrowed from the data in the numbers 6 and 7 of table 20.

One sees immediately that the "observation" 40°02′35″ in f (in No. 7 of the table is mentioned 40°02.6′) cannot be made by SNELLIUS. In his *E.B.* he gives but very rarely observations to half a minute of arc (see e.g. table 16 No. 2). An observation to 5″ is impossible. His instruments were not accurate enough for such a reading. The amount must therefore be imputed to VAN MUSSCHENBROEK's phantasy. With this invented observation, the amount $of = 250.00$ roods and the angle $o = 119°44'$ he computes $oV = 461.66$ roods in triangle ofV (No. 7). The result is wrong; it should be 465.29 roods. With this wrong amount, the correct distance oZo from No. 6 and the angle contained he determines the length oZo in No. 8. Again he makes a mistake by taking for the angle $o = 159°42'$ instead of 169°42′. Two of the elements of triangle $VoZo$ are therefore wrong. Nevertheless he finds – and this time he *ciphers* correctly – $VZo = 925.60$ roods, an amount which agrees exactly with the length found in No. 5 of the table. With this falsification VAN MUSSCHENBROEK's work is fully condemned; it is entirely unreliable and it contrasts very badly with the

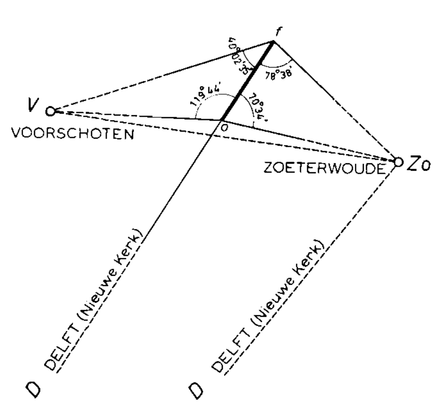


Fig. 46

faithful work carried out by SNELLIUS a century earlier. It is therefore incomprehensible that it was considered as an improvement of SNELLIUS' work for more than two centuries.

It is a precarious venture to examine how the co-ordinates of the base points *o* and *f* might possibly be, the more precarious because, here too, there is no connection between the topography and the position of the base line. I did, however, an attempt, making use of the fact that *fo* is pointed at Delft.

A more accurate determination of the co-ordinates of *o* from the two angles measured in that point is not possible as *o*, *V*, *Zo* and *D* lie almost on the same circle. A small change in the angles causes therefore a great change in the co-ordinates.

In the first phase of my computation I computed an approximate length of the side *oZo* in triangle *ofZo* from the data $o = 70°34'$, $f = 78°38'$ and $of = 250.36$ roods. This base length can be found by multiplying the measured length of 250.00 roods with the mean of the enlargement factors $\lambda_{km} = 1.00112$ and $\lambda_{bd} = 1.00172$ for the base lines *km* and *bd* respectively (see §§ 37 and 38). In triangle *DoZo* one can compute now the angle *D*. By this computation the sides *oZo* and *Dof* can be oriented provisionally in the *R.D.*-co-ordinate system. Now follows the computation of the approximate co-ordinates of *o* and from these and those of *V* the angle $Vof = 119°49'16''$. It agrees very well with the measured amount of 119°44'. In my opinion one can conclude from it that the three angles used until now do agree with the observations.

I divided the difference 5'16'' equally over the two angles *Vof* and *foZo* so that $Vof = 119°46'38''$ and $foZo = 70°31'22''$. If in triangle *ofZo* the angle *Zo* remains unaltered, that is to say if angle *f* is corrected with +2'38'', one can compute another approximate length *oZo* and with this value another amount for angle *D* in triangle *DoZo*. From new approximate co-ordinates of *o* – one sees that the computation is an iteration process – follows a third value for angle *Vof*.

One finds $Vof = 119°46'40''$. As it differs but 2'' from the just mentioned amount of 119°46'38'', the approximate co-ordinates of *o* are now definitive. They are:

$$X'_o = -62726.1 \qquad Y'_o = -3002.8$$

Those of *f* are:

$$X'_f = -62225.6 \qquad Y'_f = -2203.8$$

From these co-ordinates and those of the points *Zo* and *V* one can compute the angles mentioned in column 6 of table 20 (Nos. 6–8). As one sees the falsified angle *f* in No. 7 has an error of about 30'. Unless SNELLIUS' original observations come to light, no one will be able to examine whether the solution given is correct. In my opinion it has the advantage that, with the exception of the falsified angle, the angles agree very well with the *R.D.*-data and that the base length, by about the same enlargement factors, is comparable with the base lines *bd* and *km*.

As these enlargement factors, however, are also influenced by the errors in the measured angles, $\lambda_{of} = 1.00142$ is rather arbitrary. If this factor must be larger, e.g. $\lambda_{of} = 1.005$, the distance *oZo* becomes larger, which means that *o* moves to the west, approximately along he common circle through *V*, *o*, *Zo* and *D* (see fig. 38). This shifting of *o* will be followed by a shifting of f. As of is very small with respect to oD, the place of the "new" f will be about at the intersection point of Zof and the line through the "new" o parallel to of.

The base is indicated in fig. 45.

40 Speculations on the base lines *bd* and *of* for $b = f$

It appears that b and f lie close to each other. Their distance, computed from the co-ordinates, is about 5.9 metres or 1.6 roods. Their mutual position is once more given in fig. 47 at a scale of 1 to 200 with the distances from b to the towers that were used for their determination.

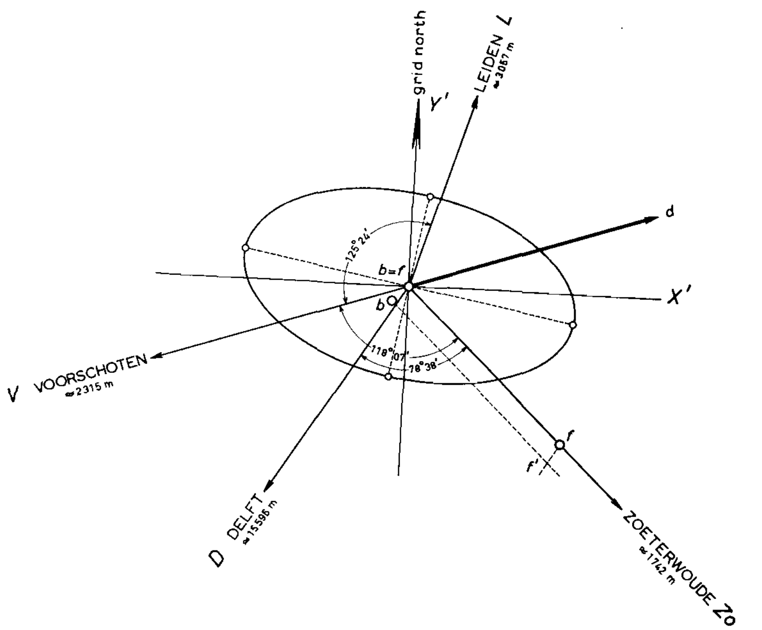


Fig. 47

It is tempting to assume that SNELLIUS chose coinciding points b and f. Though VAN MUSSCHENBROEK says nothing concerning this question it seems improbable, however, that on the vast ice sheet which covered the pastures between Leiden, Zoeterwoude and Voorschoten in January and February 1622, two points should only happen to lie at so small a distance from each other.

In figure 47 f lies approximately on the line bZo. According to the end of § 39 the difference between b and f can be explained by too small a λ_{of}. A better amount can even be computed: as bf' (see fig. 47) $\approx$ 6.0 metres and $bZo \approx 1742$ m, $\lambda_{of} \approx 1.00142\,(1+6.0 : 1736) \approx 1.0049$.

It is plausible to compute the point $b = f$ by resection from the angles measured in b between Voorschoten and Leiden (125° 24'; see table 20 No. 4) and between Voorschoten and Zoeterwoude (180° − 61° 53' = 118° 07'; see fig. 42 and table 20 No. 2) and from the angle measured in f between Zoeterwoude and Delft (78° 38'; see fig. 46 and table 20 No. 6). As the problem is already determined by two angles there is even a redundant observation which gives the possibility for an adjustment. Though the results of such an adjustment are not very reliable, they give, however, some insight. One finds

$$X'_{b=f} = -62229.8 \qquad Y'_{b=f} = -2199.8$$

The corrections to the three angles have the small amounts

$$v_{LV} = -0.5', \qquad v_{ZoV} = -1.1' \text{ and } v_{ZoD} = +1.4'$$

respectively from which one computes a standard deviation $m_\alpha = \pm 1.8'$. The length of the semi long axis of the standard ellipse is 0.9 m, the length of the semi short axis is 0.5 m. The grid bearing of the long axis is 100°13'. It is drawn in fig. 47 at the scale 1 to 40 [104].

One sees that the co-ordinates of the adjusted point deviate but $\Delta X' = +0.5$ m and $\Delta Y' = +0.4$ m from those of b (see § 38).

In the way as described in § 37 (fig. 43) one can compute now a correction Δ to the angles d in each of the triangles bdL and $bdZo$. One finds $-18'' = -0.3'$ and $+0.3'$ respectively. From these amounts follows $bd = fd \approx 1791.68$ m ≈ 475.70 roods and $\lambda_{bd} = \lambda_{fd} = 475.70 : 475.00 = 1.00147$ which agrees still better with $\lambda_{km} = 1.00112$ in § 37.

The co-ordinates

$$X'_d = -60531.1 \qquad Y'_d = -1630.9$$

differ but slightly (0.1 m and 0.4 m respectively) from those in § 38.

The results of this alternative determination of $b = f$ and d are mentioned in table 21 Nos. $1-3$.

nº	M.T.		Tri-angle	Angles		v	from	Opposite sides (roods)		
	prob.	page		M.T.	R.D.	6−5		M.T.	checked	R.D.
1	2	3	4	5	6	7	8	9	10	11
1	I	398	*b=f*	54°36'	54°36.5'	+0.5'		*660.57*	*660.54*	661.65
			d	89 31	89 30.7	−0.3		*810.36*	*810.36*	811.60
			L				base	475.00	475.00	475.70
2	II	398	*b=f*	61 53	61 54.1	+1.1		*481.74*	*481.75*	482.64
		399	*d*	57 42	57 42.3	+0.3		——	——	462.50
			Zo				base	475.00	475.00	475.70
3	III	399	*L*				2	481.74	481.75	482.64
			d	147 13	147 13.0	0.0		*1097.10*	*1097.05*	1098.94
			Zo				1	660.57	660.54	661.65
6	VII	401	*o*	70 34	70 31.5	−2.5		——	——	462.50
			f=b	78 38	78 39.4	+1.4		*478.66*	*478.67*	480.98
			Zo				base	250.00	250.00	251.31
7	VII	401	*o*	119 44	119 46.5	+2.5		——	——	614.75
			f=b	40 02.6	39 26.5	−36.1		*461.66*	*465.29*	449.95
			V				base	250.00	250.00	251.31
8	VII	401	*V*				6	478.66	478.67	480.98
			o	169 42	169 42.0	0.0		*925.60*	*940.15*	927.17
			Zo				7	461.66	465.29	449.95

Table 21

They can be compared with those in table 20 Nos. 1–3. In the numbers 6–8 of table 21 are the results of the alternative determination of o from $f = b$. Here too the angles $foZo$ and foV have equal corrections which differ only in sign.

The corrected angles determine the distance $fo = bo \approx 946.45$ metres ≈ 251.31 roods and the co-ordinates

$$X'_o = -62732.0 \qquad Y'_o = -3002.0$$

The enlargement factor $\lambda_{fo} = 251.31 : 250.00 = 1.0052$ deviates very much from the amounts λ found for the other base lines. It agrees fairly, however, with $\lambda_{of} \approx 1.0049$ in this paragraph.

The amounts v for the angles o (Nos. 6 and 7) are almost equal in both tables. According to my computation the difference v for the falsified angle $ofV = obV$ is now even 36′.

One might compute the co-ordinates of o also from those of $f = b$ and from the length $fo = bo = 250.00\lambda$ with

$$\lambda = (\lambda_{km}+\lambda_{bd}) : 2 = (1.00112+1.00147) : 2$$

In that case one finds:

$$X'_o = -62730.0 \qquad Y'_o = -2998.9$$

The angle o in triangle ofV, however, is then 9.0′ larger than the observation 119°44′. o in triangle $ofZo$ is 4.3′ larger. I have rejected this solution.

Whether one prefers the solution in table 20 or the one in table 21 is a question of taste. My preference, mentioned before already, goes out to that in table 21 though the factor $\lambda_{fo} = 1.0052$ is in my opinion very large.

And so after the analysis of SNELLIUS' base line nets there remains doubt. Not only the doubt which is inevitable on account of the defectiveness of the surveying instruments of 350 years ago, but above all the doubt of the authenticity of the observations mentioned by VAN MUSSCHENBROEK. If one has caught him several times falsifying, there is always a risk of using once more falsifications for observations.

The last word concerning these doubts can only be spoken if SNELLIUS' observations would come to light. It will not have been VAN MUSSCHENBROEK's interest that they were preserved.

It will be clear that in the discussion on SNELLIUS' triangulation network which follows now VAN MUSSCHENBROEK's determination of the length LHg must be left out of consideration.

41 The triangulation network and its computation by Snellius

The triangulation network is represented in fig. 48. The 14 angular points are the spires of the towers mentioned underneath with the epoch in which they were built:

Alkmaar (*Al*); Grote- or St. Laurenskerk (1470–1520),
Haarlem (*Hl*); Grote kerk (St. Bavo) (13th–16th century),
Amsterdam (*Am*); Oude- or St. Nicolaaskerk with a spire dating from 1565,
Leiden (*L*); Townhall with a spire dating from 1599,
Utrecht (*U*); Tower of the cathedral (14th century),
Gouda (*G*); Grote- or St. Janskerk (16th century),

Oudewater (*O*); saddle roof tower (13th century),
Den Haag (The Hague, *Hg*); Grote- or St. Jacobstower (14th century); the present shape dates from after the fire in 1536,
Rotterdam (*R*); Grote- or St. Laurenskerk (15th century),
Zaltbommel (*Z*); St. Martinuskerk (15th century),
Breda (*B*); Grote- or Onze Lieve Vrouwekerk (15th century),
Willemstad (*W*), Hervormde kerk (reformed church), about 1596,
Dordrecht (*D*); Grote kerk with tower from the 14th century,
Bergen op Zoom (*Bz*); Hervormde kerk (15th century).

	° '		° '
1	97 11	28	68 04
2	32 25	29	65 25
3	50 23	30	82 31
4	25 49	31	62 13
5	25 50	32	44 20
6	128 22	33	73 29
7	85 51	34	72 15
8	71 31	35	70 14
9	90 18	36	54 12
10	53 40	37	48 15
11	43 36	38	86 19
12	80 00	39	41 10
13	37 40	40	66 11
14	114 48	41	67 51
15	27 32	42	45 59
16	63 26	43	27 11
17	54 08	44	43 18
18	62 28	45	172 11
19	20 26	46	50 38
20	33 53	47	54 00
21	125 43	48	67 45
22	17 23	49	77 55
23	125 42	50	34 22
24	36 53	51	116 23
25	147 19	52	89 25
26	20 45	53	43 24
27	77 50	54	47 15

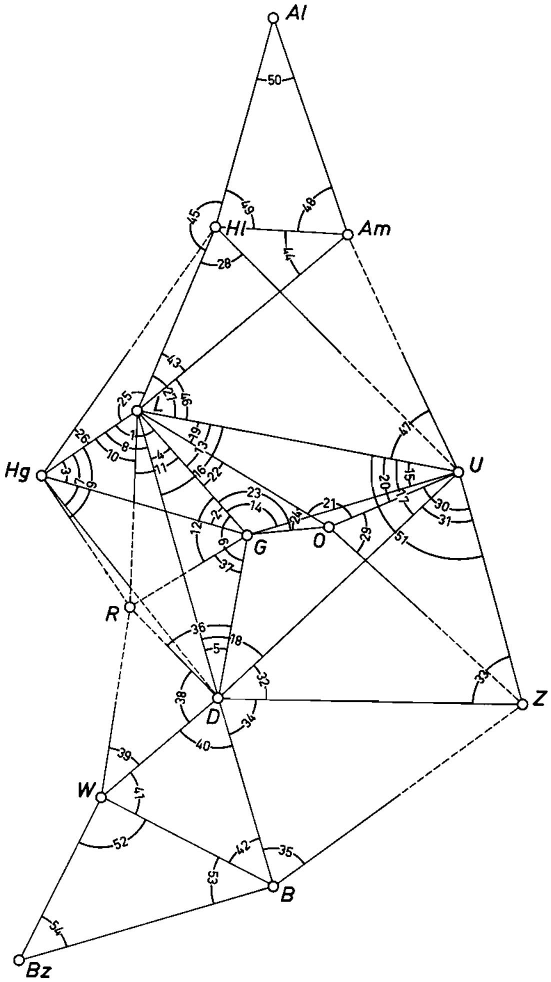


Fig. 48

With the exception of the Leiden townhall (*L*), which was wrecked by fire on February 12th, 1929, all these towers remained unaltered since SNELLIUS' time. The spires are known in the *R.D.*-co-ordinate system. Apart from small changes SNELLIUS must have known them in the state in which they still exist nowadays. Zaltbommel's tower, however, was much higher in the beginning of the 17th century, Breda's spire burned down in 1702 and the tower of Bergen op Zoom was heavily damaged by several sieges, especially the capture of the town by the French armies in 1747. With regard to Oudewater there is some doubt. On the saddle roof of the tower are two spires about five metres apart and it is not known whether SNELLIUS used the northern or the southern one as sighting point or perhaps (and probably) the centre of the tower. Though both spires are now known in the *R.D.*-co-ordinate system, I could only use the southern one in my computations as the co-ordinates of the northern one have only been determined very recently. The doubt is not of any practical importance on account of the inaccuracy of the observations. The unchanged place of the towers in this paper, however, is not open to any doubt. The co-ordinates $X'Y'$ of the spires which will be used in the analysis of the triangulation network are mentioned in table 28 (columns 2 and 3).

The 54 measured angles of the triangulation network are indicated in fig. 48. They are all borrowed from the *Eratosthenes Batavus* with the exception of angle 44 which was measured later; the magnitude of the angle (43°18′) can be found in the "Brussels' copy" of the book. Angle 9 = 39° 53′, a printer's error for 89° 53′ (page 173) is 90°18′ according to the Brussels' copy. From the same manuscript I used the improved angles in triangle *LGU*. The amount 90°12′ for angle $BzBD = 53+42$ had to be rejected.

As all the angular points of the network are spires, the angles had to be measured outside the centre. Just as TYCHO BRAHE, SNELLIUS did not introduce corrections for reduction to centre as they are smaller than the accuracy of his observations (see § 22). For the same reason all his computations were carried out with plane trigonometry. For the spherical excess of the largest triangle of the network (*LDU* with sides of about 44 km) is only 4″.

It is SNELLIUS' great merit that he mentioned faithfully the observed amounts of the measured angles and the corrections to these angles on account of the $54-24 = 30$ redundant data in the network. For in order to determine the mutual position of the 14 angular points 24 independent angles are necessary. SNELLIUS' corrections are limited to the condition that in every triangle the sum of the angles had to be 180°.

After this primitive adjustment and with $LHg = 4103.3$ roods (table 16 Nos. 15 and 16) he could compute now the lengths of a number of sides in different ways. As side equations were unknown in those days the results of these computations could not be alike. In triangle *LHgG* e.g. he finds $LG = 5897.8$ roods and then in triangle *LGD*, $LD = 10633.1$ roods (see table 26 Nos. 1 and 2, column 12). The side *LD*, however, can also be computed from the three data in triangle *LHgD*. The result is now $LD = 10634.7$ roods (table 26 No. 3). In this case he uses the former result for his further computations (see No. 7 of the table). In another case he takes the mean of two computations. In the triangles around Dordrecht e.g. he finds $BD = 7005.7$ roods from *DZ* in triangle *DZB* (see No. 17) but also $BD = 6998.0$ roods from *DW* in triangle *DWB* (see No. 20). The rounded-off mean value $BD = 7000.0$ roods is used for his computations in No. 28 of table 26.

As this primitive adjustment method is out of date nowadays and since I could make use of a computer, I adjusted the triangulation network with its 30 condition equations according to the method of the least squares. Such an adjustment has the advantage that an insight is obtained into the accuracy of the triangulation by the computation of the standard error m_α in the measured angle.

An attempt for such a computation has already been made in the past. JORDAN e.g. mentions [105] that from the station equations in Leiden where 13 angles have been mea-

CONDITION

	STATION EQUATIONS										ANGLE			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*
1				− 1.000	− 1.000			+ 1.000			+ 1.000			
2											+ 1.000			
3											+ 1.000			
4	+ 1.000			+ 1.000								+ 1.000		
5							− 1.000					+ 1.000		
6						− 1.000						+ 1.000		
7														
8				+ 1.000										
9														
10					+ 1.000									
11					+ 1.000									
12						+ 1.000								
13	+ 1.000	− 1.000						+ 1.000					+ 1.000	
14													+ 1.000	
15													+ 1.000	
16	− 1.000													+ 1.000
17			− 1.000						+ 1.000					+ 1.000
18							+ 1.000							+ 1.000
19		+ 1.000												
20			+ 1.000											
21														
22		+ 1.000												
23														
24														
25								+ 1.000						
26														
27								+ 1.000		− 1.000				
28														
29														
30			+ 1.000											
31			− 1.000						+ 1.000					
32							+ 1.000							
33														
34							+ 1.000							
35														
36							+ 1.000							
37						+ 1.000								
38							+ 1.000							
39														
40							+ 1.000							
41														
42														
43										+ 1.000				
44														
45														
46										+ 1.000				
47														
48														
49														
50														
51									− 1.000					
W	+ 3.00	+ 9.00	+ 3.00	+ 9.00	+ 5.00	− 7.00	− 5.00	0.00	− 2.00	− 1.00	− 1.00	+ 1.00	0.00	+ 2.00
	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Table 22

EQUATIONS

EQUATIONS						SIDE EQUATIONS								
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
o	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*	*α*	*β*	*γ*
						+ 1.990				+ 1.154		- 0.131		
						- 1.044				- 1.510		- 1.773		
							- 2.619					- 2.611	- 3.939	
							+ 4.311				- 1.702		+ 2.340	
								+ 0.639			- 0.639	- 0.869		
												+ 1.681		
												+ 0.422		
										+ 1.517				
										+ 0.930				
										- 1.328			+ 2.612	
										- 0.932			+ 0.222	
							+ 1.637							
						+ 0.582		+ 0.639	- 6.550		- 7.189			
						+ 2.418	- 4.943	+ 2.525	+11.352					
							+ 2.525	- 2.525			+ 2.015			
							- 1.702				+ 1.702			
+ 1.000									- 3.394					
+ 1.000									- 9.473		- 2.015			
+ 1.000											+ 4.027			
	+ 1.000								+ 4.027					
	+ 1.000								+ 7.459		+ 6.550			
	+ 1.000										+ 4.027			
						+ 5.986								
						+ 9.321								
						+ 1.867								- 0.273
						+ 2.375								+ 1.920
											+ 2.593			
		+ 1.000						+ 0.665			+ 2.015			
		+ 1.000									- 1.293			
		+ 1.000						- 0.376			+ 0.376			
								- 1.642						
								- 2.095						
								- 0.278					- 1.429	
								- 1.405					- 1.127	
								+ 0.966						
								+ 2.411						
			+ 1.000											
			+ 1.000					- 0.513						
			+ 1.000					+ 1.223						
				+ 1.000										
				+ 1.000										- 0.691
				- 1.000										+ 1.411
														+ 2.057
														+ 1.020
					+ 1.000									
				- 1.000	+ 1.000									+ 1.411
					+ 1.000									
+ 2.00	- 2.00	+ 2.00	+ 1.00	+23.00	+ 2.00	+20.11	-32.47	+13.97	+19.22	- 0.37	-33.81	- 5.50	- 8.77	-11.00
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29

Table 22

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	[a	[b	[c	[d	[e	[f	[g	[h	[i	[j	[k	[l	[m	[n
a]	+ 3000	- 1000	—	+ 1000	—	—	—	+ 1000	—	—	—	+ 1000	+ 1000	- 1000
b]	- 1000	+ 3000	—	—	—	—	—	- 1000	—	—	—	—	- 1000	—
c]	—	—	+ 4000	—	—	—	—	—	- 2000	—	—	—	—	- 1000
d]	+ 1000	—	—	+ 3000	+ 1000	—	—	- 1000	—	—	- 1000	+ 1000	—	—
e]	—	—	—	+ 1000	+ 3000	—	—	- 1000	—	—	- 1000	—	—	—
f]	—	—	—	—	—	+ 3000	—	—	—	—	—	- 1000	—	—
g]	—	—	—	—	—	—	+ 7000	—	—	—	—	- 1000	—	+ 1000
h]	+ 1000	- 1000	—	- 1000	- 1000	—	—	+ 4000	—	- 1000	+ 1000	—	+ 1000	—
i]	—	—	- 2000	—	—	—	—	—	+ 3000	—	—	—	—	+ 1000
j]	—	—	—	—	—	—	—	- 1000	—	+ 3000	—	—	—	—
k]	—	—	—	- 1000	- 1000	—	—	+ 1000	—	—	+ 3000	—	—	—
l]	+ 1000	—	—	+ 1000	—	- 1000	- 1000	—	—	—	—	+ 3000	—	—
m]	+ 1000	- 1000	—	—	—	—	—	+ 1000	—	—	—	—	+ 3000	—
n]	- 1000	—	- 1000	—	—	—	+ 1000	—	+ 1000	—	—	—	—	+ 3000
o]	—	+ 1000	+ 1000	—	—	—	—	—	—	—	—	—	—	—
p]	—	+ 1000	—	—	—	—	—	—	—	—	—	—	—	—
q]	—	—	- 1000	—	—	—	+ 1000	—	+ 1000	—	—	—	—	—
r]	—	—	—	—	—	—	+ 1000	—	—	—	—	—	—	—
s]	—	—	—	—	—	—	—	—	—	+ 1000	—	—	—	—
t]	—	—	—	—	—	—	—	—	—	—	—	—	—	—
u]	—	—	—	—	—	—	—	+ 7853	—	- 1867	+ 946	—	+ 3000	—
v]	- 982	- 1637	- 2525	- 2619	—	—	- 6013	+ 1637	+ 2525	—	—	+ 1692	- 3306	+ 823
w]	—	—	+ 1860	—	—	- 2044	- 954	—	- 1860	—	—	+ 639	+ 3164	- 2525
x]	—	+ 633	- 9473	—	—	—	—	—	—	—	—	—	+ 4802	—
y]	—	—	—	—	- 398	- 932	—	—	—	—	- 356	—	—	—
z]	—	—	- 6045	—	—	+ 639	+ 2111	—	+ 4030	—	—	- 2341	- 7189	+ 3717
α]	- 2611	—	—	- 2189	—	+ 869	—	—	—	—	- 1904	- 3480	—	—
β]	- 3939	—	—	- 3939	+ 2612	- 905	- 3769	—	—	—	—	- 1599	—	—
γ]	—	—	—	—	—	—	—	- 273	—	+ 2330	—	—	—	—
W	+ 3000	+ 9000	+ 3000	+ 9000	+ 5000	- 7000	- 5000	0	- 2000	- 1000	- 1000	+ 1000	0	+ 2000
	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Table 23

15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
[o	[p	[q	[r	[s	[t	[u	[v	[w	[x	[y	[z	[α	[β	[γ
—	—	—	—	—	—	—	- 982	—	—	—	—	- 2611	- 3939	—
+ 1000	+ 1000	—	—	—	—	—	- 1637	—	+ 633	—	—	—	—	—
+ 1000	—	- 1000	—	—	—	—	- 2525	+ 1860	- 9473	—	- 6045	—	—	—
—	—	—	—	—	—	—	- 2619	—	—	—	—	- 2189	- 3939	—
—	—	—	—	—	—	—	—	—	—	- 398	—	—	+ 2612	—
—	—	—	—	—	—	—	—	- 2044	—	- 932	+ 639	+ 869	- 905	—
—	—	+ 1000	+ 1000	—	—	—	- 6013	- 954	—	—	+ 2111	—	- 3769	—
—	—	—	—	—	—	+ 7853	+ 1637	—	—	—	—	—	—	- 273
—	—	+ 1000	—	—	—	—	+ 2525	- 1860	—	—	+ 4030	—	—	—
—	—	—	—	+ 1000	—	- 1867	—	—	—	—	—	—	—	+ 2330
—	—	—	—	—	—	+ 946	—	—	—	- 356	—	- 1904	—	—
—	—	—	—	—	—	—	+ 1692	+ 639	—	—	- 2341	- 3480	- 1599	—
—	—	—	—	—	—	+ 3000	- 3306	+ 3164	+ 4802	—	- 7189	—	—	—
—	—	—	—	—	—	—	+ 823	- 2525	—	—	+ 3717	—	—	—
+ 3000	—	—	—	—	—	—	—	—	-12867	—	+ 2012	—	—	—
—	+ 3000	—	—	—	—	—	—	—	+11486	—	+10577	—	—	—
—	—	+ 3000	—	—	—	—	—	+ 289	—	—	+ 1098	—	—	—
—	—	—	+ 3000	—	—	—	—	+ 710	—	—	—	—	—	—
—	—	—	—	+ 4000	- 1000	—	—	—	—	—	—	—	—	- 3513
—	—	—	—	- 1000	+ 3000	—	—	—	—	—	—	—	—	+ 1411
—	—	—	—	—	—	+143075	-11952	+ 6477	+23637	+ 3873	- 4184	+ 1590	—	+ 4050
—	—	—	—	—	—	-11952	+61829	-18857	-56113	—	- 5146	+ 6838	+20404	—
—	—	+ 289	+ 710	—	—	+ 6477	-18857	+31793	+24478	—	- 8891	- 555	+ 1981	—
-12867	+11486	—	—	—	—	+23637	-56113	+24478	+344881	—	+115032	—	—	—
—	—	—	—	—	—	+ 3873	—	—	—	+ 9410	—	+ 2526	- 3676	—
+ 2012	+10577	+ 1098	—	—	—	- 4184	- 5146	- 8891	+115032	—	+153937	+ 555	- 3983	—
—	—	—	—	—	—	+ 1590	+ 6838	- 555	—	+ 2526	+ 555	+13737	+10285	—
—	—	—	—	—	—	—	+20404	+ 1981	—	- 3676	- 3983	+10285	+31175	—
—	—	—	—	- 3513	+ 1411	+ 4050	—	—	—	—	—	—	—	+13492
+ 2000	- 2000	+ 2000	+ 1000	+23000	+ 2000	+20110	-32470	+13970	+19220	- 370	-33810	- 5500	- 8770	-11000
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29

Table 23

sured between 8 sighting points (see fig. 48) a standard deviation $m_\alpha = 3'58''$ can be derived and that, from the closing errors in 12 triangles and from 9 side equations, the same standard deviation of about 3′ or 4′ can be found. An adjustment of the whole triangulation, however, was never done before.

42 The adjustment of the triangulation

Just like SNELLIUS and for reasons already mentioned, I presupposed that his measurements took place in a flat plane. In order to examine how the triangulation network fits on the 14 angular points of the *R.D.*-co-ordinate system after the adjustment, I started from the fiction that this flat plane coincides with the plane of projection of the *R.D.* The errors made – the reader will see that presently – are insignificant compared with the errors caused by the primitive instruments with which SNELLIUS had to work.

From fig. 48 one sees that in the south the triangle *WBBz* is built upon the side *WB* with no other check than that the sum of the corrected angles 52 up to and including 54 must be 180°.

The condition in this triangle is therefore:

$$(89°25'+p_{52}) + (43°24'+p_{53}) + (47°15'+p_{54}) = 180°$$

or, if the corrections p are expressed in minutes of arc:

$$p_{52}+p_{53}+p_{54}+4.00 = 0$$

$p_{52} = p_{53} = p_{54} = -1.33'$ are now fixed. The other corrections p_i $(i = 1, \ldots, 51)$ are connected in the condition equations 1 up to and including 29 in the part of the network north of the line Willemstad-Breda. They are arranged in table 22 in the same way as those in TYCHO BRAHE's triangulation (see table 5 in § 23).

The 10 station equations are indicated by $a, \ldots, j$, the 10 angle equations by $k, \ldots, t$ and the 9 side equations by $u, \ldots, y$. The latter group has been derived from the relations mentioned underneath:

$$u \quad \frac{LHg \cdot LHl \cdot LU \cdot LG}{LHl \cdot LU \cdot LG \cdot LHg} = 1$$

$$v \quad \frac{GL \cdot GU \cdot GD}{GU \cdot GD \cdot GL} = 1$$

$$w \quad \frac{DW \cdot DR \cdot DG \cdot DU \cdot DZ \cdot DB}{DR \cdot DG \cdot DU \cdot DZ \cdot DB \cdot DW} = 1$$

$$x \quad \frac{OG \cdot OL \cdot OU}{OL \cdot OU \cdot OG} = 1$$

$$y \quad \frac{RHg \cdot RL \cdot RG}{RL \cdot RG \cdot RHg} = 1$$

$$z \quad \frac{UZ \cdot UO \cdot UG \cdot UD}{UO \cdot UG \cdot UD \cdot UZ} = 1$$

$$\alpha \quad \frac{DHg \cdot DL \cdot DG}{DL \cdot DG \cdot DHg} = 1$$

$$\beta \quad \frac{RL \cdot RG \cdot RD}{RG \cdot RD \cdot RL} = 1$$

$$\gamma \quad \frac{UL \cdot UHl \cdot UAm}{UHl \cdot UAm \cdot UL} = 1$$

Correlates K_φ ($\varphi = a \rightarrow \gamma$)									
a	− 6.8623	*g*	+ 2.9782	*m*	+ 4.4161	*s*	−10.1944	*y*	+ 0.9880
b	− 2.9687	*h*	+ 0.6353	*n*	− 6.2693	*t*	− 2.8627	*z*	+ 0.9118
c	− 1.6914	*i*	+ 1.1941	*o*	− 2.0005	*u*	+ 0.0128	*α*	− 1.4287
d	− 2.1226	*j*	+ 5.9362	*p*	+ 0.4732	*v*	+ 0.0253	*β*	+ 0.7392
e	− 1.9347	*k*	− 2.0244	*q*	− 2.9293	*w*	− 0.2662	*γ*	− 2.5558
f	+ 4.4436	*l*	+ 4.6261	*r*	− 1.2631	*x*	− 0.5306		

Table 24

The tables with the matrix of coefficients of the normal equations (table 23) and the correlates K_ϕ ($\phi = a, \ldots, \gamma$) (table 24) are arranged in the same way as the tables 6 and 7 of TYCHO BRAHE's triangulation. The corrections p_i ($i = 1, \ldots, 51$) are gathered in table 25.

Corrections p_i ($i = 1 \rightarrow 51$) in minutes of arc											
1	+ 2.668	10	− 1.016	19	− 3.168	28	− 4.877	37	+ 3.984	46	+ 0.679
2	− 0.671	11	− 1.316	20	− 0.503	29	+ 2.364	38	+ 2.721	47	− 2.607
3	− 0.997	12	+ 3.687	21	+ 1.671	30	− 1.691	39	− 0.642	48	− 2.863
4	− 3.606	13	+ 1.199	22	− 4.632	31	+ 1.616	40	+ 1.715	49	+ 3.725
5	+ 1.935	14	+ 1.174	23	+ 2.487	32	− 1.130	41	− 1.126	50	− 2.863
6	+ 0.671	15	− 2.373	24	+ 4.145	33	− 2.486	42	− 1.589	51	− 1.194
7	− 2.402	16	+ 0.593	25	+ 0.712	34	+ 3.415	43	− 4.258		
8	− 2.726	17	− 0.811	26	+ 0.119	35	+ 0.558	44	− 8.428		
9	+ 1.499	18	− 1.782	27	− 4.579	36	+ 1.996	45	+ 6.588		

Table 25

From $[pp] = -[KW] = 409.8$ follows $m_\alpha^2 = 14.13$ or $m_\alpha = \pm 3.76' = 3'46''$. It agrees excellently with the amount of $m_\alpha = 3'58''$ mentioned by JORDAN and it is better than TYCHO BRAHE's result $m_\alpha = 5.9'$ derived from the p_i's in table 8.

A survey of the measured angles of the several triangles, of the corrections p and the adjusted angles can be found in the columns 6, 7 and 8 of table 26. Column 5 refers to the

Nº	E.B.		Tri-	Angles					v	Opposite sides (roods)			
	prob.	page	angle	nº	E.B.	corr. *p*	adjusted 6 + 7	R.D.	9 − 8 *E*	from	E.B.	adjusted	R.D.
1	2	3	4	5	6	7	8	9	10	11	12	13	14
					° '	'	° '	° '	'				
1	I	169	*L*	1	97 11	+ 2.67	97 13.67	97 11.34	− 2.33		*7594.3*	7604.4	7619.4
			Hg	3	50 23	− 1.00	50 22.00	50 24.52	+ 2.52		*5897.8*	5903.4	5918.1
			G	2	32 25	− 0.67	32 24.33	32 24.16	− 0.17	base	4103.3	4107.92	4115.3
							180 00.00	180 00.02	+ 0.015				
2	II	172	*L*	4	25 49	− 3.61	25 45.39	25 41.03	− 4.36		*5897.8*	5880.3	5879.3
			G	6	128 22	+ 0.67	128 22.67	128 27.04	+ 4.37		*10633.1*	10608.1	10623.7
			D	5	25 50	+ 1.94	25 51.94	25 51.95	+ 0.01	1	5897.8	5903.4	5918.1
							180 00.00	180 00.02	+ 0.017				
3	III	172	*L*	8	71 31	− 2.73	71 28.27	71 30.31	+ 2.04		*10112.7*	10085.1	10102.5
			Hg	7	85 51	− 2.40	85 48.60	85 46.18	− 2.42		*10634.7*	10608.1	10623.7
			D				22 43.13	22 43.53	+ 0.40	base	4103.3	4107.92	4115.3
							180 00.00	180 00.02	+ 0.025				
4	IV	173	*Hg*	9*	90 18	+ 1.50	90 19.50	90 17.49	− 2.01		*6972.3*	6984.5	6997.5
			L	10	53 40	− 1.02	53 38.98	53 41.16	+ 2.18		*5616.8*	5625.4	5638.5
			R				36 01.52	36 01.36	− 0.16	base	4103.3	4107.92	4115.3
	*Bruss.	copy					180 00.00	180 00.01	+ 0.014				
5	V	173	*L*	11	43 36	− 1.32	43 34.68	43 30.17	− 4.51		*4883.1*	4888.0	4890.1
			G	12	80 00	+ 3.69	80 03.69	80 04.74	+ 1.05				
			R				56 21.63	56 25.10	+ 3.47	1	5897.8	5903.4	5918.1
							180 00.00	180 00.01	+ 0.017				
6	VI	173	*L*	13*	37 40	+ 1.20	37 41.20	37 43.60	+ 2.40		*7847.5*	7817.6	7844.9
			G	14*	114 48	+ 1.17	114 49.17	114 46.97	− 2.20		*11628.8*	11606.5	11639.9
			U	15*	27 32	− 2.37	27 29.63	27 29.45	− 0.18	1	5897.8	5903.4	5918.1
	*Bruss.	copy					180 00.00	180 00.02	+ 0.025				
7	VII	174	*L*	16	63 26	+ 0.59	63 26.59	63 24.63	− 1.96		*11732.5*	11711.1	11732.5
			D	18	62 28	− 1.78	62 26.22	62 31.28	+ 5.06		*11631.8*	11606.5	11639.9
			U	17	54 08	− 0.81	54 07.19	54 04.15	− 3.04	2	10633.1	10608.1	10623.7
							180 00.00	180 00.06	+ 0.066				
8	VIII	174	*L*	19	20 26	− 3.17	20 22.83	20 25.24	+ 2.41		*5000.6*	4980.1	5004.6
			O	21	125 43	+ 1.67	125 44.67	125 45.38	+ 0.71	7	11631.8	11606.5	11639.9
			U	20	33 53	− 0.50	33 52.50	33 49.40	− 3.10		*7981.8*	7970.7	7984.0
							180 00.00	180 00.02	+ 0.018				
9	IX	175	*L*	22	17 23	− 4.63	17 18.37	17 18.36	− 0.01		*2934.6*	2921.3	2923.4
			O	24	36 53	+ 4.14	36 57.14	37 01.66	+ 4.52	1	5897.8	5903.4	5918.1
			G	23	125 42	+ 2.49	125 44.49	125 39.99	− 4.50		*7975.1*	7970.7	7984.0
							180 00.00	180 00.01	+ 0.009				
10	XII	182	*Hg*	26	20 45	+ 0.12	20 45.12	20 46.74	+ 1.62		*7040.4*	7047.3	7097.9
			L	25	147 19	+ 0.71	147 19.71	147 21.09	+ 1.38		*10725.7*	10736.8	10793.7
			Hl				11 55.17	11 52.18	− 2.99	base	4103.3	4107.92	4115.3
							180 00.00	180 00.01	+ 0.010				
11	XIII	182	*Am*				75 23.93	75 12.26	−11.67	7	11631.8	11606.5	11639.9
			L	46	50 38	+ 0.68	50 38.68	50 40.82	+ 2.14		*9201.0*	9274.0	9313.7
			U	47	54 00	− 2.61	53 57.39	54 06.97	+ 9.58		*9725.8*	9697.9	9754.1
							180 00.00	180 00.05	+ 0.052				

Table 26

Nº	E.B.		Tri-	Angles					V	Opposite sides (roods)			
	prob.	page	angle	nº	E.B.	corr. *p*	adjusted 6 + 7	R.D.	9 − 8 *E*	from	E.B.	adjusted	R.D.
1	2	3	4	5	6	7	8	9	10	11	12	13	14
12	XIV	182	*L*	27	77 50	− 4.58	77 45.42	77 43.97	− 1.45		*12257.7*	12234.6	12278.4
			Hl	28	68 04	− 4.88	67 59.12	67 52.43	− 6.69	7	11631.8	11606.5	11639.9
			U				34 15.46	34 23.65	+ 8.19		*7030.1*	7047.3	7097.9
							180 00.00	180 00.05	+ 0.049				
13	XV	183	*L*	43	27 11	− 4.26	27 06.74	27 03.15	− 3.59		*4730.0*	4695.3	4712.3
			Hl		*110 03*		109 43.69	109 42.48	− 1.21	11	9725.8	9697.9	9754.1
			Am	44	43 18* *42 46*	− 8.43	43 09.57	43 14.39	+ 4.82	12	7030.1	7047.3	7097.9
	*Bruss. copy						180 00.00	180 00.02	+ 0.018				
14	XVI	184	*Hl*	49	77 55	+ 3.72	77 58.72	77 50.19	− 8.53		*8193.0*	8145.3	8116.7
			Am	48	67 45	− 2.86	67 42.14	67 35.13	− 7.01		*7754.2*	7705.2	7675.8
			Al	50	34 22	− 2.86	34 19.14	34 34.70	+ 15.56	13	4730.0	4695.3	4712.3
							180 00.00	180 00.02	+ 0.021				
15	XVIII	185	*O*	29	65 25	+ 2.36	65 27.36	65 24.10	− 3.26		*8548.4*	8535.3	8571.9
			U	30	82 31	− 1.69	82 29.31	82 32.13	+ 2.82		*9338.8*	9302.6	9347.6
			Z				32 03.33	32 03.79	+ 0.46	8	5000.6	4980.1	5004.6
							180 00.00	180 00.02	+ 0.026				
16	XIX	185	*D*	32	44 20	− 1.13	44 18.87	44 24.67	+ 5.80		*8552.6*	8535.3	8571.9
			U	31	62 13	+ 1.62	62 14.62	62 17.38	+ 2.76		*10826.0*	10811.9	10844.2
			Z	33	73 29	− 2.49	73 26.51	73 18.01	− 8.50	7	11732.5	11711.1	11732.5
							180 00.00	180 00.06	+ 0.053				
17	XXII	186	*D*	34	72 15	+ 3.42	72 18.42	72 15.09	− 3.33		*10956.2*	10944.8	10961.7
			Z				37 27.02	37 19.55	− 7.47		*7005.7*	6985.7	6978.7
			B	35	70 14	+ 0.56	70 14.56	70 25.40	+ 10.84	16	10826.0	10811.9	10844.2
							180 00.00	180 00.04	+ 0.043				
18	XXIII	187	*D*	36	54 12	+ 2.00	54 14.00	54 15.35	+ 1.35		*4888.8*	4888.0	4890.1
			G	37	48 15	+ 3.98	48 18.98	48 22.30	+ 3.32		*4506.1*	4499.0	4503.5
			R				77 27.02	77 22.36	− 4.66	1	5897.8	5903.4	5918.1
							180 00.00	180 00.01	+ 0.013				
19	XXIV	187	*D*	38	86 19	+ 2.72	86 21.72	86 16.69	− 5.03		*6831.2*	6822.5	6834.9
			W	39	41 10	− 0.64	41 09.36	41 06.64	− 2.72	18	4506.1	4499.0	4503.5
			R				52 28.92	52 36.69	+ 7.77		*5432.0*	5422.3	5442.0
							180 00.00	180 00.02	+ 0.017				
20	XXV	188	*D*	40	66 11	+ 1.72	66 12.72	66 08.87	− 3.85		*6912.1*	6902.5	6899.5
			W	41	67 51	− 1.13	67 49.87	67 40.98	− 8.89		*6998.0*	6985.7	6978.7
			B	42	45 59	− 1.59	45 57.41	46 10.17	+ 12.76	19	5432.0	5422.3	5442.0
							180 00.00	180 00.02	+ 0.017				
21	XXVI	188	*W*	52	89 25	− 1.34	89 23.66	89 20.60	− 3.06		*9414.7*	9402.6	9394.1
			B	53	43 24	− 1.33	43 22.67	43 24.02	+ 1.35		*6467.2*	6458.1	6455.0
			Bz	54	47 15	− 1.33	47 13.67	47 15.41	+ 1.74	20	6912.1	6902.5	6899.5
							180 00.00	180 00.03	+ 0.027				

Table 26

Nº	E.B.		Tri-	Angles					V	Opposite sides (roods)			
	prob.	page	angle	nº	E.B.	corr. p	adjusted 6 + 7	R.D.	9 − 8 E	from	E.B.	adjusted	R.D.
1	2	3	4	5	6	7	8	9	10	11	12	13	14
22	XVII	184	Am	44,48	111° 03' 110 31	− 11.29'	110° 51.71'	110° 49.52'	− 2.19'		14750.0	14719.3	14741.7
			L		31 $21\frac{3}{4}$		31 08.27	30 58.36	− 9.91	14	8193.0	8145.3	8116.7
			Al		38 $07\frac{1}{4}$		38 00.02	38 12.17	+ 12.15	11	9725.8	9697.9	9754.1
							180 00.00	180 00.05	+ 0.044				
23	XXX	192	L		4 06		4 01.52	3 55.21	− 6.31	14	7754.2	7705.2	7675.8
			Hl	45	172 11	+ 6.59	172 17.59	172 27.33	+ 9.74		14749.7	14719.3	14741.7
			Al		3 43		3 40.89	3 37.54	− 3.35	12	7030.1	7047.3	7097.9
							180 00.00	180 00.08	+ 0.073				
24	XXXI	193	L	-	81 56		81 46.95	81 39.17	− 7.78		17455.2	17393.4	17406.4
			U		56 47		56 53.04	56 55.43	+ 2.39	22,23	14749.0	14719.3	14741.7
			Al		41 17		41 20.01	41 25.50	+ 5.49	7	11631.8	11606.5	11639.9
							180 00.00	180 00.10	+ 0.103				
25	XX	186	L		26 25		26 24.82	26 26.41	+ 1.59	16	8552.6	8535.3	8571.9
			U		116 23		116 21.80	116 21.53	− 0.27		17250.7	17191.4	17250.1
			Z		37 12		37 13.38	37 12.12	− 1.26	7	11631.8	11606.5	11639.9
							180 00.00	180 00.06	+ 0.053				
26	XXI	186	Al		39 $02\frac{1}{4}$		39 06.71	39 12.57	+ 5.86	25	17250.7	17191.4	17250.1
			L		108 $24\frac{1}{4}$		108 11.78	108 05.58	− 6.20		25996.0	25889.0	25938.8
			Z		32 $33\frac{1}{2}$		32 41.51	32 41.99	+ 0.48	22,23	14750.0	14719.3	14741.7
							180 00.00	180 00.14	+ 0.146				
27	XXXII	193	Al		2 $15\frac{5}{6}$		2 13.31	2 12.92	− 0.39	15,16	8550.0	8535.3	8571.9
			U		173 07		173 14.83	173 16.97	+ 2.14		25963.6	25889.0	25938.8
			Z		4 $37\frac{1}{6}$		4 31.86	4 30.12	− 0.74	24	17455.2	17393.4	17406.4
							180 00.00	180 00.01	+ 0.010				
28	XXVII	189	D		53 15		53 49.12	53 40.18	− 8.94	21	9414.7	9402.6	9394.1
			B	42,53	89 23 90 12	− 2.92	89 20.08	89 34.18	+ 14.10		11751.7	11648.4	11660.5
			Bz		36 33		36 50.80	36 45.68	− 5.12	17,20	7000.0	6985.7	6978.7
							180 00.00	180 00.04	+ 0.039				
29	XXVIII	189	B	53,42, 35	159 37 160 26	− 2.36	159 34.64	159 59.59	+ 24.95		20076.4	20027.0	20048.2
			Z		8 48		9 25.75	9 13.52	− 12.23	21	9414.7	9402.6	9394.1
			Bz		10 46		10 59.61	10 46.91	− 12.70	17	10956.2	10944.8	10961.7
							180 00.00	180 00.02	+ 0.021				
30	XXIX	190	Computation of ZBz from nº 29 in quadrangle $DZBBz$ (see text)								20076.8	20027.0	20048.2
31	XXXIII	194	Bz		47 $56\frac{1}{2}$		47 59.15	48 01.83	+ 2.68	26,27	25966.0	25889.0	25938.8
			Z		97 $01\frac{1}{2}$		96 55.92	96 53.92	− 2.00		34710.6	34590.1	34635.9
			Al		35 02		35 04.93	35 04.56	− 0.37	29,30	20076.8	20027.0	20048.2
							180 00.00	180 00.31	+ 0.309				

Table 26

number used for the angles in fig. 48. Column 9 gives the amounts of the angles computed from the data of *R.D.* They have been corrected for the small angle between arc and chord in the stereographic map projection. In every triangle their sum is therefore larger than 180°. The spherical excess is once more mentioned in column 10. In this column are also the differences v between the *R.D.*-amounts and the adjusted angles in column 8. Though these differences are much smaller than the corresponding amounts in table 8 column 6, they are rather large. Apart from possible identification errors they must be imputed to the rather weak construction of several parts of the triangulation network.

43 Speculations on the strength of the triangulation and on Van Musschenbroek's bad revision

13 out of 54 angles, a disproportional large number of almost a fourth part, have been measured in SNELLIUS' dwelling place Leiden. In Rotterdam no observations were made. As Gouda is already an excellent central point in triangle *LUD* (though it is a pity that angle *UGD* has not been measured) there is little need of the station Oudewater. The visit to his mother at Oudewater is the reason that he nevertheless included it in his triangulation. The distance *OG* is only about 11 km. It is by far the smallest distance of the network. Moreover the combination of angles used for its determination is unfortunate in such a way, that in the condition equation z (No. 26) corrections to 13 angles occur.

Haarlem (*Hl*) is determined very badly in spite of the small corrections p_{26} and p_{25} to the angles *Hg* and *L* in triangle *HgLHl* (see table 26 No. 10). Neither the sharp angle *Hl* of about 12° in this triangle, nor the angle *U* in triangle *LHlU* (No. 12 of the table), however, have been measured. They would have improved highly the position of Haarlem though I don't underestimate at all the difficulties concerning the measurement of this latter angle. The distance *UL* is almost 44 km, *UHl* more than 46 km, the longest of the whole network. And these distances had to be overlooked with the naked eye as SNELLIUS' instruments were not yet fitted with optics. In my opinion only the early morning hours of the summer of 1615, in which the triangulation network was measured, would have given a chance for succes (point with the sun in the back). Observing the angle *LHlU*, which can be best executed in the afternoon, was somewhat easier, also because of the very high sighting point of the tower of the cathedral in Utrecht. Nowadays such an achievement would have been impossible as the atmosphere has become too polluted by industrial smoke and other defilements.

The bad determination of Haarlem manifests itself by the large v's, $-6.69'$ and $+8.19'$, in the angles *Hl* and *U* in No. 12 of the table.

The determination of Amsterdam is insufficient in the printed edition of *E.B.* as angle $44 = 43°18'$ was not measured (see No. 13). There is therefore not any check on anlge 47. If it had been wrong for an arbitrary amount, SNELLIUS would not have found it as he made no use of the check that in triangle *LHlAm* angle $Hl = 360° - (45+49) = 109°54'$. Apparently he considered it sufficient to compute $Am = 42°46'$ and $Hl = 110°03'$ (italic numbers in column 6) from two sides (*LHl* and *LAm*) and the angle contained.

With these data he computed the length *HlAm*, the base of triangle *HlAmAl* (No. 14 of the table). It has no ideal form as the top angle in Alkmaar (about 34°) is rather sharp. The very large amounts v in that triangle show that, if no identification error(s) has (have) been made, the small closing error of 2′ must be ascribed to chance.

It is clear that by the badly determined points *Hl* and *Am* and the badly checked triangle *HlAmAl* the position of Alkmaar in the triangulation network is very poor.

In the centre of the network the agreement between the adjusted angles and the *R.D.*-results is excellent for the very large triangle *LDU* (No. 7 of the table). The side Dordrecht-Utrecht of this triangle has a length of more than 44 km. According to annotations in the Brussels' copy of his book SNELLIUS changed later on the excellent observations $L = 63°26'$ and $U = 54°08'$ into 63°03′ and 54°29′ respectively. They must be imputed to identification errors. In his *De Magnitudine Terrae* VAN MUSSCHENBROEK alters them again into 63°23′ and 54°25′. The angle in Dordrecht is fixed at 62°12′. Because of this change the angle in Leiden is reduced to about the same amount as the original observation. The angles in Utrecht and Dordrecht, however, deviate about 21′ and 19′ respectively from the data of *R.D.*

In 1960 [106] and after more than two centuries, I could rehabilitate SNELLIUS, restore his original observations and signalize VAN MUSSCHENBROEK's unreliable revision of SNELLIUS' work. I discussed this unreliability already fully in the paragraphs 36–40. As it concerned there the revision of the base line nets I will give still one other example which relates to the computation of the triangulation network. I could add several others. It concerns the angles round the central point Dordrecht. I have mentioned them in column 1 of table 27 with the numbers as indicated in fig. 48.

Nº	Erat. Batavus			R.D.	Magn. terrae			Differences v	
angle	observation	corr. p	adjusted		„adjusted"	prob.	page	5 – 6	5 – 4
1	2	3	4	5	6	7	8	9	10
	° ′	′	° ′	° ′	° ′			′	′
36	54 12	+ 2.00	54 14.00	54 15.35	54 00	XXXVI	414	+ 15.35	+ 1.35
5	25 50	+ 1.94	25 51.94	25 51.95	25 49	XVI	405	+ 2.95	+ 0.01
36-5	28 22	+ 0.06	28 22.06	28 23.40	28 11			+ 12.40	+ 1.34
18	62 28	– 1.78	62 26.22	62 31.28	62 12	XXI	407	+ 19.28	+ 5.06
32	44 20	– 1.13	44 18.87	44 24.67	44 04	XXXII	412	+ 20.67	+ 5.80
34	72 15	+ 3.42	72 18.42	72 15.09	72 10	XXXV	413	+ 5.09	– 3.33
40	66 11	+ 1.72	66 12.72	66 08.87	66 13	XXXVIII	415	– 4.13	– 3.85
38	86 19	+ 2.72	86 21.72	86 16.69	86 19	XXXVII	414	– 2.31	– 5.03
	359 55	+ 5.01	360 00.01	360 00.00	359 09			+ 51.00	– 0.01

Table 27

The columns 2–4 give the amounts relating to SNELLIUS' observations. With the *R.D.*-amounts they are copied from table 26. In columns 6–8 are the "adjusted" angles according to VAN MUSSCHENBROEK's *M.T.* and the references to this book. One sees that VAN MUSSCHENBROEK's sum of the angles is 51′ (!) too small (that of SNELLIUS 5′). A comparison between the v's in the columns 9 and 10 speaks also volumes. It will be clear that every further discussion of VAN MUSSCHENBROEK's work can be omitted.

In the triangles *DUZ*, *DZB*, *DWR* and *DWB* (Nos. 16, 17, 19 and 20 of table 26) the agreement between the columns 8 and 9 is less good than in triangle *LDU*, in spite of the small corrections p to the observations. The rather poor construction of the network by which – apart from identification errors – large errors in the measured angles don't find expression in the corrections p, gives rise to this bad agreement.

All four triangles are concerned in condition equation w (No. 23) in table 22. It is a side equation with Dordrecht as central point and radii to U, Z, B, W, R and G. In order to form this equation, 12 angles along the perimeter of the hexagon are necessary. Five of them, however, have not been measured (2 in R, and 1 in each of the stations G, U and Z). Therefore they must be expressed into angles which have been observed. The chance will play too great a part and may result in less good a mutual position of e.g. W and B. To a greater extent it will find expression in the position of Bergen op Zoom, extrapolated on the side WB by means of triangle $WBBz$ (No. 21 of table 26).

44 Computation of the lengths of the sides in Snellius' adjusted triangulation, the lengths of the sides in the R.D.-co-ordinate system and the transformation of Snellius' network to the identical points of the R.D.

The lengths of the sides of the network, expressed in roods, are also given in table 26 (columns 12–14). Column 12 gives the amounts mentioned by SNELLIUS. It was my intention to give the results of my check on these computations in a special column, in the same way as in the computation of his base extension nets. This intention had to be rejected, the number of mistakes in his calculations being too big. Already in No. 2 of the table $LD = 10633.1$ is not correct; it must be 10618.1. The agreement with the correctly computed $LD = 10634.7$ in No. 3 is therefore less good than it seems. In No. 10, where the two sides $LHl = 7040.4$ and $HgHl = 10725.7$ have been computed from the base $LHg = 4103.3$ roods, even both results are wrong. The former amount should be 7030.7, the latter 10715.9. Perhaps they are printer's errors for 7030.4 and 10715.7. Fortunately they do not influence the final result of the triangulation.

Column 14 of the table gives the lengths of the sides on the conformal sphere, computed from the *R.D.*-data. To the length l_{PQ} between two points P and Q in the plane of projection a correction Δl_{PQ} (mm per 100 m) $= (\Delta l_P + \Delta l_Q)/2$ must be given in order to find the length k_{PQ} of the chord between P and Q on the sphere. In this formula Δl_P e.g. is

$$\Delta l_P = 9.22 - \frac{X_P'^2 + Y_P'^2}{1629.4} \quad [107]$$

X_P' and Y_P' are the co-ordinates in km (see table 28).

The correction c_{PQ} from the chord k_{PQ} to the arc b_{PQ} on the sphere is

$$c_{PQ} = \frac{k_{PQ}^{\ 3}}{24r^2} \approx \frac{l_{PQ}^{\ 3}}{24r^2} \approx \frac{l_{PQ}^{\ 3}}{9777.2} \quad [108]$$

In this formula $k_{PQ}(l_{PQ})$ is expressed in km, c_{PQ} in cm.

In order to give an impression of the size of the corrections, I give underneath the computation of the side LD.

As $\Delta l_L = 6.91$ mm per 100 m and $\Delta l_D = 6.80$ mm per 100 m,

$\Delta l_{LD} = 6.86$ mm per 100 m.

As $l_{LD} = 40005.94$ m, $\Delta l_{LD} = 2.74$ m and $c_{LD} = 0.07$ m. Therefore

$b_{LD} = 40005.94 + 2.74 + 0.07 = 40008.75$ m $= 10623.7$ roods (see Nos. 2, 3 and 7 of table 26).

As one sees, the correction c_{PQ} can almost be neglected even for a distance $l_{PQ} \approx 40$ km. As c_{PQ}, however, is directly proportional to the third power of l_{PQ}, c_{PQ} increases very rapidly if l_{PQ} increases. For Alkmaar–Bergen op Zoom e.g. ($l_{AlBz} \approx 130.43$ km), $c_{AlBz} = 2.27$ m and $b_{AlBz} = 130430.89+5.66+2.27 = 130438.82$ m $= 34635.9$ roods (see No. 31 of table 26).

Finally I have mentioned in column 13 the lengths of the sides if one uses the adjusted angles from column 8 and the length $LHg = 4107.92$ roods, the mean of the amounts 4107.98 and 4107.87 roods in the numbers 15 and 16 of table 16. As the network of plane triangles has been adjusted, there are – in contrast with column 12 – no differences if a side can be computed in two different ways ($LD = 10608.1$ roods in 2 and 3).

The way in which I computed the lengths in column 13 is similar to that discussed in § 25. Starting from the co-ordinates $X'Y'$ in the *R.D.*-system of Leiden and Utrecht (columns 2 and 3 in table 28) and with the adjusted angles 1, ..., 54, I computed by intersection the co-ordinates of the 12 other points of the meridian chain. The checks carried out, are also checks on the correctness of the condition equations and the normal equations.

Points	System $X'Y'$ R. D.		System XY (Snellius)		System XY brought into sympathy with system $X'Y'$		Differences	
							V_i	W_i
i	X'_i	Y'_i	X_i	Y_i	X'_i	Y'_i	(6 - 2)	(7 - 3)
1	2	3	4	5	6	7	8	9
ALKMAAR	- 43551.02	+ 53310.25	- 43647.69	+ 53421.30	- 43587.46	+ 53393.06	- 36.44	+ 82.81
HAARLEM	- 51065.62	+ 25399.10	- 51120.24	+ 25298.04	- 51073.28	+ 25297.28	- 7.66	- 101.82
AMSTERDAM	- 33343.44	+ 24499.86	- 33410.34	+ 24413.65	- 33378.07	+ 24401.50	- 34.63	- 98.36
LEIDEN	- 61342.49	+ 725.08	- 61342.49	+ 725.08	- 61304.17	+ 750.86	+ 38.32	+ 25.78
UTRECHT	- 18222.58	- 7145.44	- 18222.58	- 7145.44	- 18223.95	- 7142.85	- 1.37	+ 2.59
GOUDA	- 46450.83	- 15854.87	- 46433.49	- 15850.98	- 46418.36	- 15822.19	+ 32.47	+ 32.68
OUDEWATER (south)	- 35507.86	- 14653.88	- 35466.23	- 14654.19	- 35459.01	- 14633.85	+ 48.85	+ 20.03
THE HAGUE	- 74077.22	- 8106.07	- 74090.59	- 8115.78	- 74048.17	- 8074.26	+ 29.05	+ 31.81
ROTTERDAM	- 62066.00	- 25615.64	- 62083.10	- 25641.58	- 62062.20	- 25594.31	+ 3.80	+ 21.33
ZALTBOMMEL	- 9317.10	- 38171.70	- 9327.31	- 38127.55	- 9356.92	- 38106.38	- 39.82	+ 65.32
BREDA	- 42438.97	- 62806.85	- 42375.99	- 62951.68	- 42396.26	- 62888.18	+ 42.71	- 81.33
WILLEMSTAD	- 65636.93	- 51105.36	- 65625.18	- 51162.39	- 65618.90	- 51092.40	+ 18.03	+ 12.96
DORDRECHT	- 50150.39	- 37683.38	- 50157.16	- 37743.55	- 50154.02	- 37694.80	- 3.63	- 11.42
BERGEN OP ZOOM	- 76335.44	- 72933.16	- 76425.06	- 73030.31	- 76425.12	- 72935.54	- 89.68	- 2.38

Table 28

In columns 4 and 5 of table 28 we find these co-ordinates from which one computes $LHg = 15515.12$ m. As, according to SNELLIUS' measurement, $LHg = 4107.92$ roods, all distances computed from the co-ordinates in these columns must therefore be multiplied by $4107.92 : 15515.12 = 0.264769$ in order to find the distances in column 13. The method used has the advantage that it facilitates the computation of the final result, the length of the side Alkmaar–Bergen op Zoom. We will discuss presently the long-winded method which SNELLIUS used in order to find this distance.

The way of computing has also the advantage that the co-ordinates can easily be con-

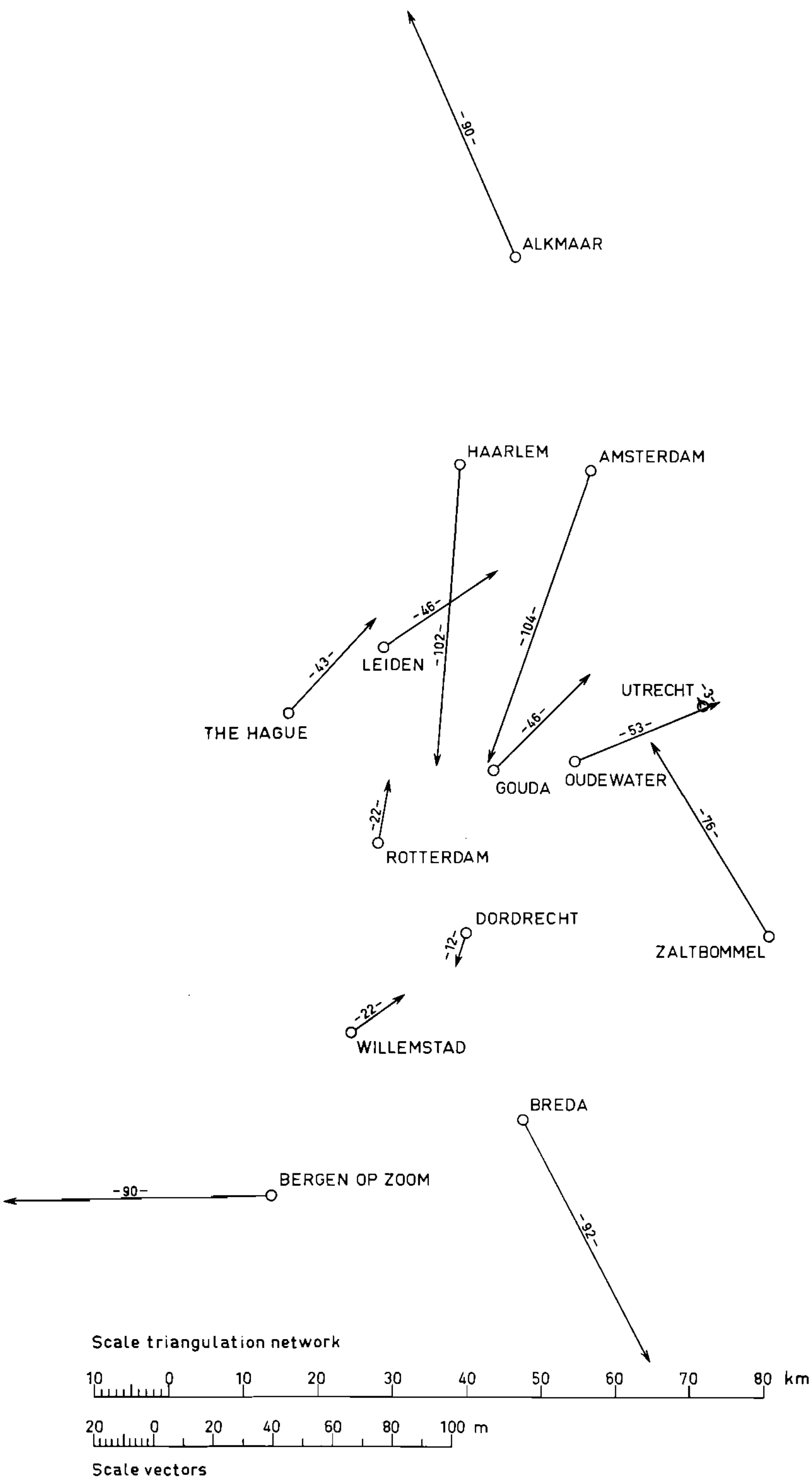

-90-
ALKMAAR
HAARLEM
AMSTERDAM
-46-
-102-
-104-
-43-
LEIDEN
THE HAGUE
UTRECHT
-3-
-46-
-53-
GOUDA
OUDEWATER
-76-
-22-
ROTTERDAM
DORDRECHT
-12-
ZALTBOMMEL
-22-
WILLEMSTAD
BREDA
-90-
BERGEN OP ZOOM
-92-
Scale triangulation network
10 0 10 20 30 40 50 60 70 80 km
20 0 20 40 60 80 100 m
Scale vectors

Fig. 49

nected with those in columns 2 and 3 by a similarity transformation. It is similar to the transformation of TYCHO BRAHE's triangulation in table 9. The results are mentioned in table 28 (columns 6 and 7). The remaining differences v_i and w_i in the columns 8 and 9 of the table are represented as vectors in fig. 49. It is analogous to fig. 29.

The length of the vectors is often very large, even larger than in TYCHO BRAHE's triangulation. It must be said, however, that the area of the latter is very much smaller. The influence of the fiction that SNELLIUS carried out his measurements in the plane of projection becomes insignificant in comparison with the inaccuracy of the observations. The very large vectors in Haarlem (102 m), Amsterdam (104 m), Alkmaar (90 m), Breda (92 m) and Bergen op Zoom (90 m) do confirm my criticism in § 43 on the strength of the construction of the triangulation.

45 Computation of the length Alkmaar-Bergen op Zoom

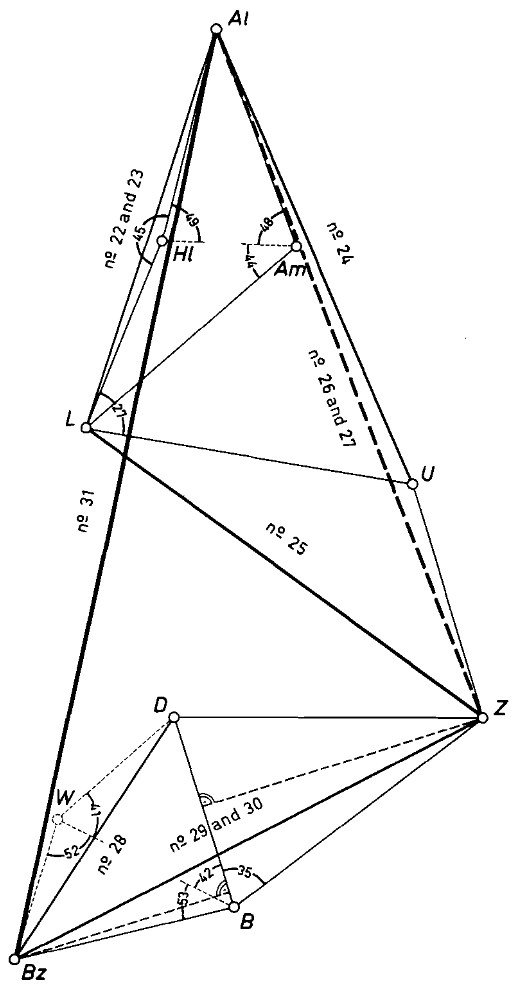


Fig. 50

The long-winded method with which SNELLIUS computed the distance *AlBz* is illustrated in fig. 50. The numbers along the sides of this figure refer to the reference numbers in table 26. *AlL* has been computed twice (in No. 22 and No. 23). The computation of *AlU* in triangle *LUAl* can be found in No. 24 and that of *LZ* in No. 25. The computation of *AlZ* has also been checked. SNELLIUS computed this distance once from two sides and the angle contained in triangle *AlLZ* (No. 26) and once from the same data in triangle *AlUZ* (No. 27). The results of these computations differ 32.4 roods (about 122 m). Apparently SNELLIUS preferred the latter result of 25963.6 roods to the former 25996.0 for in the computation of *AlBz* from *AlZ*, *BzZ* and the angle contained in triangle *BzZAl*, he uses *AlZ* = 25966.0 roods (No. 31). *BzZ* in this triangle is also checked: it has been computed

a from two sides and the angle contained in triangle *BZBz* (No. 29),

b in a geometrical manner (the computation of the lengths of the perpendiculars from *Z* and *Bz* upon *BD*) from the four sides and the diagonal *BD* of quadrangle *DZBBz* (No. 30).

DBz of this quadrangle was found from two sides and the angle contained in triangle

DBBz (No. 28). Angle *B* in this triangle is the sum of the angles 42 and 53 (89°23′). In No. 28 as well as in No. 29 of his computation, however, SNELLIUS uses 42+53 = 90°12′. As I stated in § 41 this amount had to be rejected. In No. 28, *DBz* = 11751.7 is therefore wrong. This mistake he could have found if he had computed the same distance from the three data in triangle *DWBz* (see the dotted lines in fig. 50). He would have found then *DBz* = 11667.6 roods. The results *ZBz* = 20076.4 (No. 29) and *ZBz* = 20076.8 (No. 30), though a good check on his computation, are both wrong because the wrong angle 90°12′ demonstrates its influence in both computations. With the more correct amount of 42+53 = 89°23′ he would have found *ZBz* = 20051.4 in No. 29 and *ZBz* = 20051.3 in No. 30. In the adjusted triangulation the distance is 20027.0 roods, according to the data of the *R.D.* 20048.2 roods.

In spite of the error SNELLIUS' final result *AlBz* = 34710.6 roods differs but about $-2^0/_{00}$ from the *R.D.*-amount of 34635.9 roods, if we take for the length of the rood 3.766 m. The distance 34590.1 roods in the adjusted net deviates $+1.3^0/_{00}$ from the *R.D.* The difference 45.8 roods ≈ 172.5 m at a distance of more than 130 km is remarkable small in my opinion. It is the best obtainable result in those days, also thanks to the eminent determination of the base line *LHg* and its excellent checks and in spite of the rather poor construction of the northern part of the meridian chain and the many errors in the calculation.

46 Determination of latitudes and determination of the azimuth Leiden-The Hague

It will be clear that the excellent relative accuracy of at the most 0.002 in the length Alkmaar–Bergen op Zoom cannot be maintained in the determination of the difference in latitude between these two places, also necessary for the computation of the earth's circumference. As this difference is about 68.3′ this relative accuracy would mean a deviation of about 8″ in the difference between the heights of the pole in the terminal points of the meridian chain. As I quoted already in § 32 SNELLIUS used for the astronomical part of his triangulation an iron quadrant "mounted with bronze and larger than 5½ feet" (radius about 1.75 m). 1′ on the limb of this instrument represented about 0.5 mm. Though it was no doubt the best instrument that could be made in those days it was of course impossible to make readings on it with the accuracy required, apart from the difficulties of pointing with the naked eye.

I shall not discuss the many other causes which may have influenced the accuracy of the astronomical observations – the determination of the latitudes of Alkmaar, Bergen op Zoom and his house in Leiden and the azimuth from his house to the Jacobstoren in The Hague – as it is not known how they were carried out. SNELLIUS only states that "in Alkmaar we have measured the height of the pole with diligence and with care" and that "for the height of the pole in Leiden has been found 52°10½′, again and again and in different manners" [109].

The determination of the latitude of Alkmaar was carried out on a private building, about 55 roods (207 m) south of the tower and in Bergen op Zoom also on a private building about 33 roods (124 m) north of the tower. For the latitude of his astronomical station in Alkmaar SNELLIUS finds 52°40½′, for that in Bergen op Zoom 51°29′. As 1″ in latitude represents about 31 m, the latitude of his triangulation point in Alkmaar would be about 6.7″ more than the amount mentioned above and the latitude of the tower in Bergen op Zoom about 4.0″ less.

Though SNELLIUS works these differences into his computations – I come presently to the details – they are far below the accuracy of the observations. I suppose that for that reason he crossed out the passage concerning these astronomical stations on page 197 (lines 3–9 from top) of the Brussels' copy of his *Eratosthenes Batavus.*

According to the *R.D.*-data the latitude of the triangulation point Alkmaar is 52°38′00.97″ and that of Bergen op Zoom 51°29′43.30″ [110]. The first amount deviates −2′35.7″ from SNELLIUS' observation, the latter +47.3″. As the amounts have different signs, SNELLIUS makes an "error" in the difference in latitude of −3′23.0″ on a true difference of 1°08′17.67″, that is −5 percent. If all his other work would have been faultless he would have found a circumference of the earth which was also 5 percent too small.

I just mentioned the word "error" in inverted comma's. There is, however, no question of an error in the usual meaning of the word. The latitude of Bergen op Zoom is almost ideal if one takes into account the imperfection of both the instrument used and the observation with the naked eye. The "error" in the latitude of Alkmaar, though somewhat larger, is also quite acceptable. At the end of § 26 I said already that even TYCHO BRAHE attained no better results. A rather large relative error in the earth's circumference was inevitable in SNELLIUS' days unless a much greater part of the meridian would have been measured.

The measurements for the determination of the latitude in Leiden and the azimuth of the side Leiden (townhall) – The Hague (Jacobstoren) (*LHg*) of the network have been carried out on the roof of SNELLIUS' house (see § 28 and *O* at the north side of the map fig. 40). In order to connect these measurements to his triangulation he measured also the angles *POL* between the spires of the Pieterskerk (*P*) and the Leiden townhall (*L*) and *POHo* between the Pieterskerk and the Hooglandse kerk (*Ho*) (see fig. 40).

The mutual location of these towers was already determined in the numbers 8, 9 and 10 of table 17. In § 49 I shall discuss the results of this first resection in history of geodesy. Suffice it here to state that I could compute from it the *R.D.*-co-ordinates $X' = -61426.3$, $Y' = +367.4$ of SNELLIUS' station and its latitude $\phi = 52°09'21.8''$. As I stated before SNELLIUS found $\phi = 52°10\frac{1}{2}'$. It differs but 1′08.2″ from the *R.D.*

According to SNELLIUS his house lies 95 roods south of the townhall. From the *R.D.*-co-ordinates of this point and those of his house and from the convergence of meridians $\gamma_O = -42'29''$ in *O*, I could easily verify this amount. I find 358.7 m = 95.2 roods, corresponding with a difference in latitude of 11.6″. One finds of course the same difference between the geographical latitudes of the townhall ($\phi = 52°09'33.38''$) and his house.

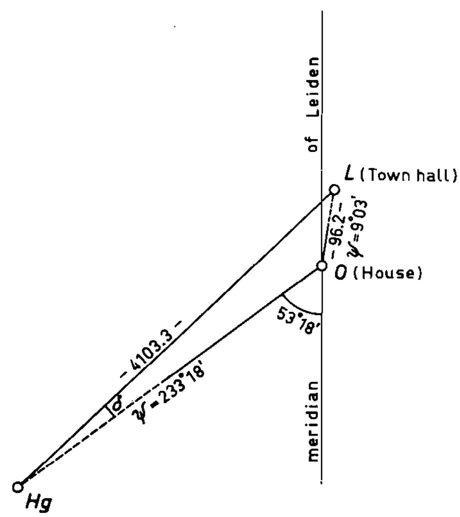


Fig. 51

On page 207 of his *E.B.* SNELLIUS mentions the result of the astronomical orientation. He finds that the azimuth to *L* is 9°03′ (east of the north) and to *Hg* (180°+) 53°18′ = 233°18′ (see fig. 51). According to SNELLIUS the distance *OL* is 96.2 roods. He borrows it from the resection. According to the *R.D.* it is 367.4 m = 97.6 roods. As the length *LHg* = 4103.3 roods (table 16 Nos. 15 and 16 column 9), the angle δ can be computed from three data in triangle *LOHg*. From this angle and the azimuth *OHg* follows the azimuth *LHg* = 232°21′44″. According to the *R.D.* it is 234°33′07.26″ [111]. As SNELLIUS made no error in the computation of δ the very large error of almost 2°12′ in the azimuth must be ascribed to a mistake. I am afraid that the cause of the error will remain unexplained because it is not known how SNELLIUS determined the azimuth.

47 Computation of the azimuth Alkmaar-Bergen op Zoom and of the length of one degree upon the meridian of Alkmaar

In fig. 52, see page 196 of his *E.B.*, I have indicated how SNELLIUS computes the azimuths Leiden–Alkmaar and Alkmaar–Bergen op Zoom. He reduces the (wrong) azimuth *LHg* = 232°21′44″ with the sum of the angles 1, 13 and 46, measured in Leiden and with the angle 31°21¾′, found in his problem XVII (table 26 No. 22 column 6) [112]. The result is the azimuth *LAl* = 15°28′.

The azimuth *AlBz* is determined in a similar way. Starting from *AlL* = 180°+*LAl* and with the aid of the angles 39°02¼′ and 35°02′ (table 26 Nos. 26 and 31) in Alkmaar one finds the azimuth *AlBz* = 191°27¾′. SNELLIUS finds 191°26′ as, for a reason which I can not explain, the amount of 74°32′ in the figure has been changed into 74°33¾′. The difference of 1¾′ means of course nothing compared with the capital error in neglecting the difference of the convergence of the meridians in Leiden (−42′28.94″) and Alkmaar (−30′34.99″). This difference is almost 12′. The correct azimuth *LAl* is 17°59′00.11″ and the azimuth *AlL* = 198°11′02.26″. The azimuth *AlBz* is 194°03′01.87″. The difference 4°08′00.39″ between the latter amounts is, apart from the small angle between arc and chord in the map projection, comparable with the angle *LAlBz* = 4°00¼′ in fig. 52 because the error in the orientation as well as the difference in the convergence of the meridians in Leiden and Alkmaar is of no influ-

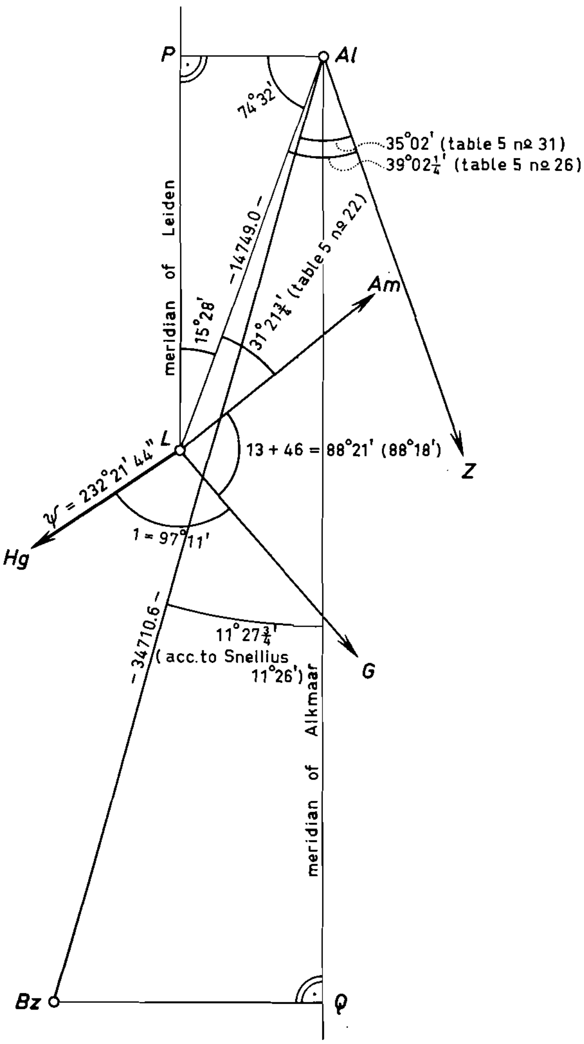


Fig. 52

ence on this angle. In the adjusted net the angle is 4°01′47″.

SNELLIUS computes the "differences in latitude" LP and AlQ between Leiden and Alkmaar and between Alkmaar and Bergen op Zoom respectively in the two right-angled triangles in fig. 52. According to him LP = 14214.9 roods and AlQ = 34018.2 roods. The first amount should represent an angle:

$$(52°40'30''+6.7'')-(52°10'30''+11.6'') = 29'55.1'',$$

the latter:

$$(52°40'30''+6.7'')-(51°29'-4.0'') = 1°11'40.7''.$$

From these data one finds that one degree on the meridian is 28507 (28510) and 28476 (28473) roods respectively. The amounts in brackets have been computed by SNELLIUS in a somewhat different way on the pages 198 and 197 of his book. They prove that he made no mistakes in these calculations. He is satisfied with the result: "Both computations have given the same amount, as near as possible". The mean, rounded-off at 28500 roods, is 3.65 percent too small; the correct amount is r/ϱ = 6382650/57.29578 = 111398.3 metres = = 29580.0 roods.

With the lengths AlL and $AlBz$ in the adjusted net and with the adjusted angles but with the "wrong" latitudes and the (very) wrong orientation one finds that one degree on the meridian is 28423 and 28359 roods respectively. The rounded-off mean of 28400 roods is about 4 percent too small.

48 Comparison between Snellius' results in § 47 and the R.D.-data

At the end of § 47 I placed the words "differences in latitude" in inverted comma's as SNELLIUS made an essential error in the determination of these differences. For – I confine my self now to the difference in latitude between Alkmaar and Bergen op Zoom – it was his intention to determine the length of the arc of the meridian between Alkmaar and the parallel circle of Bergen op Zoom. He computes this length from the formula

$$AlQ = AlBz \cos \overline{AlBz}$$

in which $\overline{AlBz}$ is the (astronomical) azimuth of $AlBz$ (see fig. 53).

In spherical trigonometry which had to be applied, this formula is analogous to

$$\tan AlQ = \tan AlBz \cos \overline{AlBz}.$$

In this formula AlQ is the distance from Alkmaar to the intersection point of the *great* circle through Bz perpendicular to the meridian of Alkmaar. The error made by SNELLIUS is therefore the arc QV.

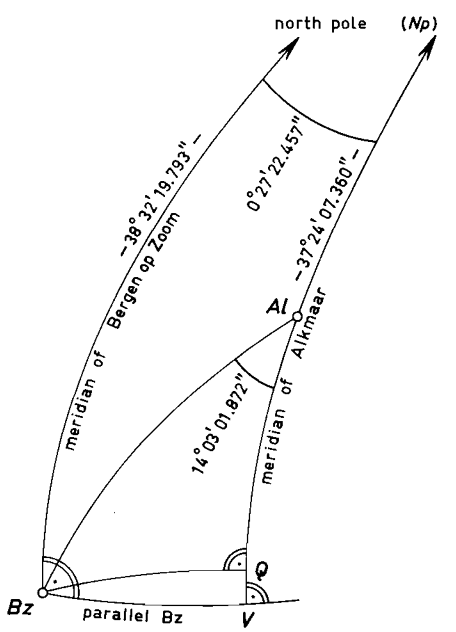


Fig. 53

In order to determine this distance I converted the geographical co-ordinates ϕ (latitude) and λ (longitude) of *Al* and *Bz* into the corresponding amounts ψ and l on the conformal sphere.

For the conversion of the latitudes one has

$$\psi - \psi_0 = 0.998738(\phi - \phi_0) + 0.000000024(\phi - \phi_0)^2$$

with $\psi_0 = 52°07'15.950''$ and $\phi_0 = 52°09'22.178''$.

ψ_0 and ϕ_0 in this formula are expressed in seconds [113].

For the conversion of the longitudes holds:

$$l = 1.004753\lambda$$

in which l and λ on sphere and ellipsoid are the longitudes with respect to the centre Amersfoort of the Dutch map projection [114].

In table 29 the co-ordinates of *Al*, *Bz* and *V* have been mentioned in both systems and table 30 shows the result of the computation with spherical trigonometry in the triangles *NpAlBz* and *NpQBz*. The data for the computations in table 30 are borrowed from the columns 4 and 5 of table 29.

Points	Geographical co-ordinates		Co-ordinates upon conf. sphere	
	φ	λ	ψ	l
1	2	3	4	5
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Al	52 38 00.966	-0 38 36.195	52 35 52.640	-0 38 37.296
Bz	51 29 43.297	-1 05 57.872	51 27 40.207	-1 05 59.753
V	51 29 43.297	-0 38 36.195	51 27 40.207	-0 38 37.296

Table 29

Sides	R.D.- lengths			Lengths E.B. (in roods)
	in sec of arc	in meters	in roods	
1	2	3	4	5
Al Q	4089.244	126537.4	33600.0	34018.2
Q V	3.189	98.7	26.2	—
Al Bz	4215.324	130438.8	34635.9	34710.6
Bz Q	1023.318	31665.5	8408.3	6898.4
Bz V	1642.457	31665.7	8408.3	—

Table 30

The lengths of the sides on the conformal sphere are given in seconds of arc and in metres and roods. The distance *BzV* along the parallel of Bergen op Zoom is also mentioned. It has a radius $r \cos \psi_{Bz}$.

One second of arc of a great circle on the sphere amounts to 30.943960 m in length. The length *AlBz* corresponds exactly with the amount found in a different way in § 44. The distance *QV* is 98.7 m (26.2 roods). It is clear that the very great difference between the amounts *BzQ* in the columns 4 and 5 of the table must be imputed to wrong orientation.

It is interesting that on page 196 of the Brussels' copy of *E.B.* SNELLIUS finds *AlBz* = 34626.2 roods. This amount happens to correspond almost exactly with the correct value of 34635.9 roods. But this is a coincidence because the many changes he made in the angles of the network are often deteriorations. As an example of such deterioration I mentioned already in § 43 the angles of triangle *LDU*. As he makes neither alterations in the dominating determination of the latitudes nor in the less important determination of the orientation, the results of the triangulation are rather poor. They are, however, due to the imperfection of the instruments used for the determination of latitudes. It seems therefore ridiculous that, on page 212 of his book, the circumference, the area and the content of the earth are mentioned in 20 (!) figures behind the decimal point [115].

49 Snellius' solution of the resection problem

Had SNELLIUS carried out his astronomical measurements in Leiden on the tower of the townhall *L* instead of on the roof of his house *O* he would, most probably, not have been the first geodesist who determined a point by resection. As I wrote in § 46 (fig. 51) the distance *OL* had to be determined in order to compute the latitude of *L* and the azimuth *LHg*. As the mutual position of the spires *P*(ieterskerk), *L*(townhall) and *Ho*(oglandse kerk) was known, there are still two other independent data necessary in order to compute *OL* in quadrangle *PLHoO*. These data for SNELLIUS were *POL* = 32° 57′ and *POHo* = 64° 40′.

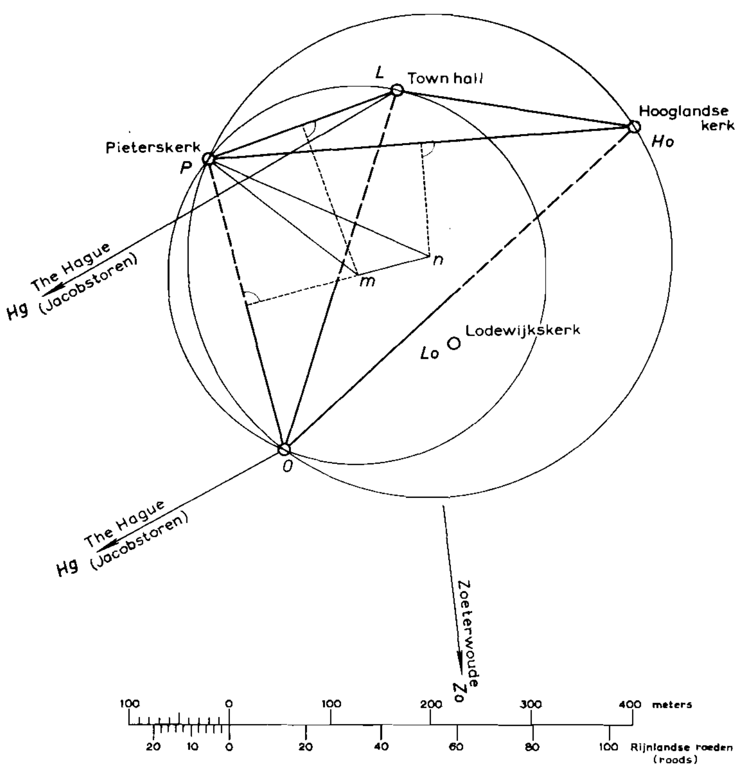


Fig. 54

With the measurement of these angles SNELLIUS solved the problem which we know nowadays as the resection problem. One finds the computation on the pages 204–206 of his *Eratosthenes Batavus*. It runs as follows.

In fig. 54 n and m are the centres of the circumscribed circles of the triangles $OPHo$ and OPL respectively. The line connecting these two points is perpendicular to OP and divides this line into two equal parts. As the sides of triangle $PLHo$ are known one can compute

$$Pn = PHo : 2 \sin POHo \text{ and } Pm = PL : 2 \sin POL.$$

As $LPm = 90° - POL$ and $HoPn = 90° - POHo$, the angle nPm is known:

$$\begin{aligned} nPm &= LPm - HoPn - LPHo \\ &= (90° - POL) - (90° - POHo) - LPHo \\ &= POHo - POL - LPHo \end{aligned}$$

Angle $LPHo$ in this formula can be computed from the three sides of triangle $PLHo$.

The angles m and n can be computed now from the three data in triangle Pmn. OP follows then from:

$$OP = 2Pn \sin n = 2Pm \sin m$$

Unfortunately SNELLIUS made no use of this check. Finally the demanded length OL follows from the sine rule in triangle OPL; OHo, if required, from the sine rule in triangle $OPHo$.

Table 31 gives the results of SNELLIUS' computations (column 3), my verifications of these computations (column 4) and the amounts found from the *R.D.*-data. The lengths in the table are expressed in roods.

One sees (column 3) that the sum of the angles in triangle Pnm is not 180 °, that the computation of OP (OL) has not been checked and that a check on the accuracy of his measurements has been omitted. He could have verified them e.g. by the measurement of the angle $LOLo$ (see fig. 54) as not only the position of P, L and Ho with respect to the base points a and e is known (table 17 Nos. 7, 3 and 6 respectively) but also the position of Lo (table 17 No. 5). SNELLIUS' shoddy way of calculating contrasts, once again, highly with that of his teacher VAN CEULEN.

Of course this criticism detracts nothing from his great merit that he applied a geometrical problem in such an excellent way in practical geodesy. During a long time LAURENT POTHENOT has been considered as the man who solved the problem of resection for the first time. He did not publish it, however, until December 31st, 1692 in a meeting of the *Académie des Sciences* in Paris. SNELLIUS' priority is therefore an established fact. But the wrong opinion held very long. Even the 1877-edition of JORDAN's *Handbuch der Vermessungskunde* I, page 314 [116] still mentions POTHENOT as the author. In 1879, however, VON BAUERNFEIND gives SNELLIUS the credit of being the first who solved the problem [117]. JORDAN's opinion of course has also been reconsidered, as I assume also on account of an abstract of VAN DER PLAATS' excellent paper [7] in *Zeitschrift für Vermessungswesen* [118].

Fortunately SNELLIUS used for his resection the three towers, L, P and Ho. As they are known in the *R.D.*-co-ordinate system, the co-ordinates $X' = -61426.3$, $Y' = +367.4$ of the point O could be computed. In 1960 I set out these co-ordinates on the terrain with the

Angles and sides 1	Table 17 nos 2	Snellius 3	Checked 4	R.D. 5
POL		32°57′	——	——
POHo		64 40	——	——
PL	9	52.0	52.12	53.06
LHo	8	62.6	62.61	63.27
HoP	10	110.9	110.91	112.63
Pn		61.35	61.36	62.31
Pm		47.80	47.91	48.78
LPHo		15°56′	16°12.3′	15°50.9′
nPm		15 47	15 30.5	15 52.1
m		123 57.5	124 21.4	123 13.1
n		40 50.7	40 08.1	40 54.8
OP		79.30	79.10	81.61
OL		96.2	95.80	97.55
OHo		118.2	118.64	120.04

Table 31

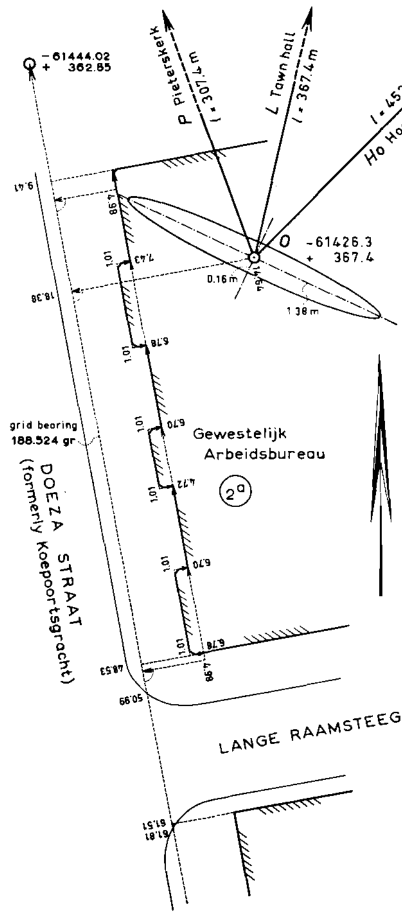


Fig. 55

aid of a point $X' = -61444.02$, $Y' = +382.85$ which could be determined by resection and by means of a line with a grid bearing 188.524 grades, parallel to the front of the "*Gewestelijk arbeidsbureau*" (Provincial labour bureau) in which the point lies (fig. 55).

In memory of the place where this first resection had been carried out the *Landmeetkundig Gezelschap "Snellius"* – I mentioned this society already in § 15 – placed there a brass memorial tablet which was unveiled on December 2, 1960 in the presence of among others, the president curator and the rector of Leiden university [119]. It is over a door of a room on the groundfloor of the said bureau in the present Doezastraat, the former Koepoortsgracht, about perpendicular under SNELLIUS' station in 1615. The tablet is reproduced in fig. 56. The English translation of the text runs

"Here lived Willebrord Snel van Royen (Snellius 1580–1626). On this place he determined about 1615, as the first in history of geodesy, a point by resection. Presented by the Geodetic Society "Snellius". Delft, on December 2nd, 1960".

The accuracy of the position of O is of course dependent on the standard deviation μ in

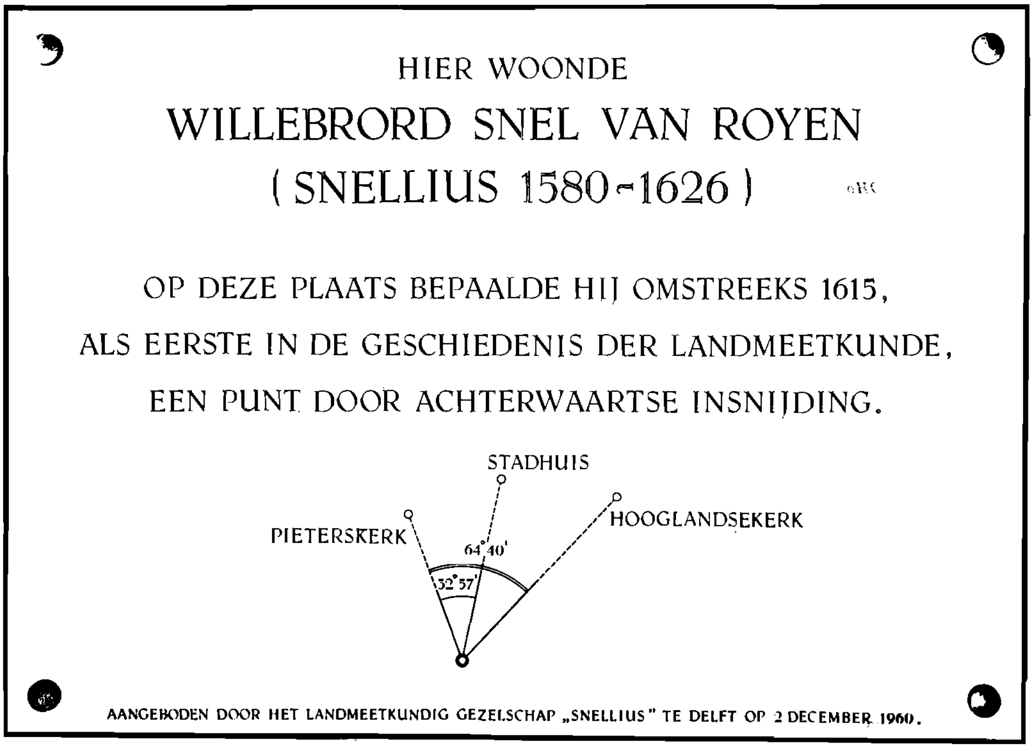


Fig. 56

the measurement of the two angles *POL* and *POHo* and on the geometrical position of the points *P*, *L* and *Ho*. It can be described by the following formulae:

$$\cot 2\psi = \frac{m_{X'}^2 - m_{Y'}^2}{-2m_{X'Y'}},$$

$$a^2 = \frac{m_{X'}^2 + m_{Y'}^2}{2} + \frac{m_{X'Y'}}{\sin 2\psi} \quad \text{and}$$

$$b^2 = \frac{m_{X'}^2 + m_{Y'}^2}{2} - \frac{m_{X'Y'}}{\sin 2\psi}$$

In these formulae a is the semi long axis, b the semi short axis of the standard ellipse, ψ the grid bearing of the long axis,

$$m_{X'}^2 = \mu^2 Q_{xx} = \mu^2 \frac{[BB]}{D},$$

$$m_{Y'}^2 = \mu^2 Q_{yy} = \mu^2 \frac{[AA]}{D} \quad \text{and}$$

$$m_{X'Y'} = \mu^2 Q_{xy} = -\mu^2 \frac{[AB]}{D} \quad \text{with}$$

$$D = [AA][BB] - [AB]^2.$$

As, for $\varrho = 3437.75'$ and l in metres,

$$A_{POL} = a_{OL} - a_{OP} = \frac{-\varrho \cos \psi_{OL}}{l_{OL}} + \frac{\varrho \cos \psi_{OP}}{l_{OP}} = +1.42$$

$$A_{POHo} = a_{OHo} - a_{OP} = \frac{-\varrho \cos \psi_{OHo}}{l_{OHo}} + \frac{\varrho \cos \psi_{OP}}{l_{OP}} = +5.14$$

$$B_{POL} = b_{OL} - b_{OP} = \frac{\varrho \sin \psi_{OL}}{l_{OL}} - \frac{\varrho \sin \psi_{OP}}{l_{OP}} = +5.92$$

and

$$B_{POHo} = b_{OHo} - b_{OP} = \frac{\varrho \sin \psi_{OHo}}{l_{OHo}} - \frac{\varrho \sin \psi_{OP}}{l_{OP}} = +9.15,$$

one finds

$$[AA] = 28.44, \quad [BB] = 118.77, \quad [AB] = 55.44 \quad \text{and} \quad D = 304.23.$$

Therefore:

$$m_{X'}^2 = 0.3904\mu^2, \quad m_{Y'}^2 = 0.0935\mu^2 \quad \text{and} \quad m_{X'Y'} = -0.1822\mu^2$$

and

$$2\psi = 256.48 \text{ gr} \quad (\psi = 128.24 \text{ gr}),$$

$a = 0.69\mu$ and $b = 0.08\mu$ (a and b in metres, μ in minutes).

For an estimated value $\mu = 2'$

$$a = 1.38 \text{ m} \quad \text{and} \quad b = 0.16 \text{ m}$$

The very flat standard ellipse is indicated in fig. 55.

50 Final speculations; Snellius' death

Of course SNELLIUS' work must be considered in the light of the time in which it was published, a time – I remarked it already – in which the telescope on the large, heavy and unhandy instruments was not invented yet and all computations had to be done by laborious ciphering, even without the use of logarithms.

Though SNELLIUS was a shoddy calculator the construction of his base line nets was excellent and the measurement of his triangulation as good as could be expected. He made only a serious mistake in the determination of his azimuth. He cannot be blamed for the deviations in the determinations of his latitudes and the error in the earth's circumference caused by these deviations. They were inherent to the instruments of his time.

Not only the scientifically justified plan of his triangulation has struck me, but above all the conscientious manner with which he, again and again, tried to improve his work, in spite of the human tragedies – the death of 15 of his 18 children – that have been his portion to such a great extent.

Atque ulterius fecit nihil (and then he did nothing more) says VAN MUSSCHENBROEK [120].

After a long illness SNELLIUS died on October 30th, 1626. His death shook the then scientific world in such a serious manner that CASPAR VAN BAERLE [121] wrote in a letter "Which

Hercules will succeed this Atlas" [122] and, alluding to SNELLIUS' *Eratosthenes Batavus*, discussed in detail in this paper, as well as to his *Tiphys Batavus* [123]: "He started for his Eratosthenes, no Tiphys brings him back" [120].

SNELLIUS is buried in the Pieterskerk in Leiden. His wife survived him only a year. She died on November 11th, 1627 and is buried next to him. The grave is still present in the church. On their tomb-stone it says [124]:

> *Hier leggen begraven Mr. Willebrordus Snellius, in sijn leven professor matheseos, sterf op den 30 Octobris 1626 ende Maria de Lange, sijn huisvrouwe, sterf op den 11 Novembris 1627.*

The English translation of the text runs:

> Here are buried Mr. Willebrordus Snellius, in his life professor in mathematics, died on October 30th, 1626 and Maria de Lange, his wife, died on November 11th, 1627.

Over the tomb-stone is a memorial, raised there by the children. It has a Latin text [125]. The translation runs:

> Dedicated to God, the Highest and the Greatest and to the posterity. For the most famous and learned man Mr. Willebrord Snel van Royen, the apple of the eye of the mathematicians among the Dutch and of the Academy which is here the most famous, in all respects the most famous, the most clever, the most worthy and the most deserving professor in mathematics, as well as for the most excellent pure spouse Maria de Lange, his beloved wife, the sorrowful children have erected this monument as a proof and as an undoubted token of their respect for their parents.
>
> Died October 30th, 1626 Died November 11th, 1627

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[2] N. D. HAASBROEK – Bij de herdenking van de meting van het eerste snelliuspunt in 1615 (Tijdschrift voor Kadaster en Landmeetkunde 1965, pages 383–393).
[3] N. D. HAASBROEK – Een analyse van Snellius' basesnetten in de omgeving van Leiden uit de jaren 1615 en 1622 (Tijdschrift voor Kadaster en Landmeetkunde 1966, pages 49–73).
[4] N. D. HAASBROEK – Een analyse van het driehoeksnet van Snellius tussen Alkmaar en Bergen op Zoom (Tijdschrift voor Kadaster en Landmeetkunde 1967, pages 3–36).
[5] Kurze Geschichte der Geodäsie (Bamberg 1950), page 31.
[6] Report of the Baltic Geodetic Commission 1930 (Helsinki 1931).
[7] J. D. VAN DER PLAATS – Overzicht van de graadmetingen in Nederland (Tijdschrift voor Kadaster en Landmeetkunde 1889, page 28).
[8] FERNAND VAN ORTROY – Bio-bibliographie de Gemma Frisius, de son fils Corneille et de ses neveux les Arsenius (Mémoires de l'Académie royale de Belgique, Classe des lettres IIe série XI, 1920, pages 167–189).
[9] F. SCHMIDT – Geschichte der geodätischen Instrumente und Verfahren im Alterum und Mittelalter (Neustadt an der Haardt 1935), page 364.
[10] W. KOOPMANS – Gemma Frisius (Geodesia No. 2 February 1967, pages 29–30).
[11] VAN DER PLAATS shares this opinion ([7], page 3).
[12] [8] page 11.
[13] [8] pages 13–14.
[14] [8] page 16.
[15] At the mediaeval university the possessor of the degree which followed on the baccalaureate; it gave him the right to lecture.
[16] [8] page 19.
[17] [8] pages 117–119.
[18] [8] page 28.
[19] Information by the rector of the university in his letter dated March 13th, 1967.
[20] [8] page 25.
[21] JOANNES DANTISCUS (1485–1548), bishop of Culm and Ermland. See HENRY DE VOCHT – JOHN DANTISCUS and his Netherlandish friends (Louvain 1961).
[22] [8] page 29.
[23] [8] page 35.
[24] [8] page 43.
[25] [8] pages 12–13.
[26] I borrow these figures from [8], pages 165–316.
[27] [8] pages 167–170.
[28] This copy is in the library of the Geodesy Department of the Delft Technological University.
[29] [8] pages 58–59.
[30] [21] page 223.
[31] The original text is to such an extent long-winded that I don't give a translation. Suffice it to give the train of thoughts. Only here and there I give, indicated by " ", a very free translation.
[32] [28] page 105.
[33] I think that GEMMA intends that the distance can be calculated by multiplying the number of steps between the towers with the steplength.
[34] Note 2 on page 223 of [21].
[35] [9] page 364.
[36] [10] page 30.
[37] TYCHO BRAHE's description of his instruments and scientific work (Copenhagen 1946), pages 96–97. The book is a translation of "Astronomiae instauratae mechanica" (Wandesburgi 1598).
[38] [8] page 99.
[39] [8] page 100.
[40] J. L. E. DREYER – Tycho Brahe (Karlsruhe 1894), page 12. The book was originally published in English (1890). I had the German translation at my disposal.
[41] [40] page 21.
[42] [40] page 28.
[43] [40] page 47.

[44] [40] pages 73–74.
[45] [40] page 91.
[46] [37] page 126.
[47] [37] page 29.
[48] N. E. NØRLUND – Danmarks Kortlaegning, part I (Copenhagen 1943), pages 30–43.
[49] [48] page 42.
[50] [37] page 9. According to a note in TYCHO BRAHE's Opera Omnia II (Copenhagen 1915, [88] page 455) the length of the cubit was supposed to be 400 mm.
[51] [37] page 29.
[52] [40] page 121.
[53] [40] page 133.
[54] [40] page 160.
[55] Before it was wrecked the globe's diameter has been measured by PICARD, RØMER and HORREBOW. They found about 1.492 m. It agrees with NØRLUND's computation of the foot's length. See: N. E. NØRLUND – De gamle danske Laengdeenheder, Geodaetisk Instituts Skrifter, 3. Raekke Bind III, København 1944, page 32 and [37] page 9. The globe can be found in fig. 6 behind the pillar between the two tables.
[56] [37] page 105.
[57] [40] pages 207–208.
[58] [40] page 248.
[59] [40] page 217.
[60] [40] page 401.
[61] [37] page 83.
[62] "Later on in the year 1564, I secretly had a wooden astronomical radius made according to the direction of Gemma Frisius. This instrument was provided with an accurate division utilizing transversal points" ([37] page 108).
[63] [37] page 142.
[64] [1] page 30.
[65] [9] page 328.
[66] [9] page 341.
[67] [9] page 339.
[68] [9] page 363.
[69] [9] page 327.
[70] [9] page 357.
[71] [48] pages 40–41.
[72] [48] page 41.
[73] [37] page 81.
[74] [37] page 82.
[75] It is unknown whether has been pointed at the northern or at the southern tower of the cathedral or midway between the two.
[76] [48] page 39.
[77] The church exists no more. The uncertainty in its reconstruction ([48] page 38) is so great that its co-ordinates had to be rounded-off at metres.
[78] On account of the inaccuracy of the measurements the correction for map projection has been neglected.
[79] JORDAN – Handbuch der Vermessungskunde III (Stuttgart 1916), pages 134–135.
[80] See the detailed dissertation F. HARKINK – "Les conditions du canevas" in Bulletin géodésique No. 17 (September 1950). The paper was also published in the Dutch language under the title "De voorwaardevergelijkingen in een veelhoeksverband" in Tijdschrift voor Kadaster en Landmeetkunde 1951, pages 149–170.
[81] Table 10 (§ 25) column 5 Nos. 6, 7, 24, 25, 26.
[82] [37] page 40.
[83] [37] pages 41–42.
[84] The moments of eastern and western elongation are determined by the formula:
$\cos t = \tan \phi \cot \delta$
in which ϕ is the latitude of the station ($\phi_U \approx 55°54.5'$), δ the declination of the pole star and t the hour angle ($t \approx \pm 86° \approx \pm 5\frac{3}{4}$ hour). The height h at that moment can then be computed from $\cos h = \cos \phi \sin t$ ($h \approx 56°$). The time elapsed between a successive eastern and western elongation, $2 \times 5\frac{3}{4} = 11\frac{1}{2}$ hour, is much too long for accurate work. Moreover it is difficult to find a suitable period for the observations as both must be made by darkness.

[85] "The longitude (Opera Omnia V page 309) we have estimated at 36°45′ at which we have of course considered the difference between the meridians used by Ptolemy and Copernicus, and, as far as possible the thorough calculations of the latter" ([37] page 139).

[86] Translation Danish-Dutch by Mrs SKAT RØRDAM-GERLING, Virum, Denmark.

[87] This method of determination of differences in latitude was also applied by SNELLIUS. In § 48 I shall minutely go into the theoretical objections against this method. For the small area of TYCHO BRAHE's triangulation, however, they can be neglected.

[88] Annotated by J. L. E. DREYER (Copenhagen 1915), page 383 line 1 and 2; see also the note on page 457.

[89] R. A. GRAY – The life and work of TYCHO BRAHE (Journal of the Royal Astronomical Society of Canada, Vol. XVII 1923, page 105).

[90] „Waarboek" EE 1 folio 141 recto in Municipal Archives Leiden.

[91] The map is a copy of map XVII a in W. PLEYTE – Leiden vóór 300 jaar en thans. The original map is in the Municipal Archives Leiden.

[92] "Kohier van het schoorsteengeld" folio 213 in Municipal Archives Leiden.

[93] "Waarboek" NN folio 219 verso in Municipal Archives Leiden.

[94] A reproduction of the text can be found under No. 320 in Mr. K. J. F. C. KNEPPELHOUT VAN STERKENBURG – De gedenkteekenen in de Pieterskerk te Leyden (Leiden 1864).

[95] See e.g. LLOYD A. BROWN – The story of maps (Boston 1949) pages 28–29.

[96] Translation Latin-Dutch and Greek-Dutch by dr. J .A. VEERING of the Delft Technological University.

[97] [7] pages 7 and 8.

[98] JORDAN – Handbuch der Vermessungskunde, erster Band (Stuttgart 1920), page 501.

[99] For a description of the rood's length see also Trigonometrical Survey, special publication No. 2 South African Units of length and area by D. R. HENDRIKS (Pretoria 1955), page 14.

[100] The pages 176–178. The translation Latin-Dutch is by dr. VEERING ([96]).

[101] In ancient times one computed and drew in sand.

[102] On August 6 and 7, 1575.

[103] The distance from Oudewater to Enkhuizen is about 81 km. As the influence of the terrestrial curvature for this distance is about 0.5 km, the reflection of the blaze against the clouds might have been seen in Enkhuizen.

[104] A detailed description of the computation has been given in [3], pages 69 and 70.

[105] JORDAN – Handbuch der Vermessungskunde erster Band (Stuttgart 1920), pages 501–503.

[106] [1] page 26.

[107] On account of the large lengths of the sides the formula is somewhat more accurate than that on page 6 of the Dutch "Handleiding voor de Technische Werkzaamheden van het Kadaster (H.T.W. 1956)". The change in sign means that the correction is from projection plane to sphere.

[108] One can also compute the relation between l_{PQ} and b_{PQ} with formula (16) on page 23 of HK. J. HEUVELINK – De stereografische kaartprojectie in hare toepassing bij de Rijksdriehoeksmeting (Delft 1918).

[109] Eratosthenes Batavus, pages 196 and 197.

[110] The geographical co-ordinates ϕ and λ and the convergence of meridians γ have been computed with DE GROOT's formulae from the co-ordinates $X'Y'$ in the plane of projection ([107], page 6).

[111] For the computation of the astronomical azimuths in this paper I used:

a. the ten-place trigonometric tables in the sexagesimal and the decimal system by D. DE GROOT (Publication of the Netherlands Geodetic Commission, Delft 1961) for the computation of the bearings of the chords in the plane of projection,

b. formula (17*) on page 25 of [108] for the computation of the angle Δ between arc and chord. One can write this formula as:

$$\Delta''_1 = \frac{0.00126591}{m_1}(X_1'Y_2' - X_2'Y_1');$$

X' and Y' are expressed in km; m_1 is the scale of projection in point 1; Δ_1 (in seconds) is the correction which must be given to the bearing of the chord 1–2 in order to find the bearing of the arc,

c. DE GROOT's formulae for the determination of the convergence of meridians ([110]).

[112] It is not quite clear how SNELLIUS comes to the amount of 13+46 = 88°21′. It is possible that he measured it separately and found a value somewhat larger than 88°18′. The *R.D.*-amount of 88°24.42′ points to this supposition.

[113] Formula (2) on page 11 of [108]. The formula is suited for computation with a calculating machine.

[114] Formula (1) on page 6 of [108].
[115] He borrows the number π, in 35 figures behind the decimal point, from VAN CEULEN (see § 29).
[116] [7] page 38.
[117] CARL MAX VON BAUERNFEIND – Elemente der Vermessungskunde, Band II (1879), page 166.
[118] Year 1892, pages 298–302.
[119] N. D. HAASBROEK – Bij de onthulling van een gedenkplaat (Tijdschrift voor Kadaster en Landmeetkunde 1960, pages 339–343).
[120] HANS FREUDENTHAL – Van sterren tot inlegzolen (Arnhem 1954), page 48.
[121] CASPAR VAN BAERLE (Barlaeus) (1584–1648), Dutch scientist and poet.
[122] "Oud Holland" IV, year 1886, page 182.
[123] "Tiphys Batavus", a book on navigation (Leiden 1624). In this book SNELLIUS issues a detailed study on the curve which intersects all the meridians at the same angle. PEDRO NUNEZ (NONIUS, 1492–1577) had given the name rhumbus to this curve. SNELLIUS calls it loxodrome, still the present name (Nieuw Nederlands Biografisch Woordenboek, part VII, column 1160). The title "Tiphys Batavus" of his book has been borrowed from the name of the steersman of the Argo, the ship with which, according to Greek mythology, the Argonauts under the command of JASON brought the Golden Fleece to Greece after many adventures.
[124] No. 83 in [94].
[125] No. 84 in [94]; translation Latin-Dutch by dr. VEERING (see [96]).