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THE THEORY OF DISPERSION APPLIED TO ELECTRO-OPTICAL DISTANCE MEASUREMENT AND ANGLE MEASUREMENT

by

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CONTENTS

LIST OF UNITS AND SYMBOLS

unit of length:	metre	unless otherwise stated
unit of temperature:	centigrade	
unit of pressure:	torr	

Symbol	*meaning*	*page*
A	starting point of a light- or radio path	8, 13, 26
B	end point of a light- or radio path	8, 13, 26
C_e	humidity correction on refraction angle	27
c	light velocity in vacuum	10
c	index for carrier wave	8
D	$= \tilde{G}_1/\Delta_L\tilde{G}$, dispersion factor	17
$\bar{D}$	$= (\tilde{n}_1 - 1)/\Delta_L\tilde{n}$	18
e	humidity function of a particular place	12, 39
F_a, F_b	functions of a particular place	13
G	phase-dispersion function for dry air	12, 39
$\tilde{G}$	group-dispersion function for dry air	14
$\bar{G}$, $\bar{\tilde{G}}$	as G and $\tilde{G}$, but for the Lorentz-Lorenz equation	43
g	$= 1/\lambda$, the inverse of the wavelength in vacuum	10
g	index for using the group refraction index	44
H	geometric elevation of B in A	26
I_e, I_ϱ, I_{ee}, $I_{\varrho e}$, $I_{\varrho\varrho}$	integrals of meteorological conditions on the X-axis	14
I_{eZ}, $I_{\varrho Z}$	integrals of meteorological conditions on the X-axis	26
K_L, K_M	factors to calculate a distance from measurements on three wavelengths	20
k_L	$= R\,\partial n_L/\partial Z$, refraction coefficient for visible light	41
k_M	$= R\,\partial n_M/\partial Z$, refraction coefficient for microwaves	41
L	index for light	13, 17
M	index for microwaves	13, 17
N	denominater of K_L and K_M	48
n	(phase) refraction index	12, 31, 39
$\tilde{n}$	group refraction index	11, 31
p	total pressure of the air in torr	39
p_3	partial pressure of water vapour in torr	39
p_4	partial pressure of CO_2 in torr	24
R	($6.38 \cdot 10^6$ m), ray of curvature of the earth in m	15
S	geometrical (straight-line) distance between A and B	13
t	in chapter 2: time	8
t	other chapters: temperature	36, 39
t_e, t_ϱ	particular average temperatures introduced in θ_e and θ_ϱ	19, 23

Symbol	*meaning*	*page*
U_A, U_B	vibration function in A or in B	8
U_0	mean amplitude of U_A	8
X, Y, Z	carthesian coordinates; origin in A, X-axis through B, Y-axis parallel to horizontal plane in B	14, 26
β	extinction between A and B	8
Γ	phase-dispersion function for humidity correction	12, 31
$\tilde{\Gamma}$	group-dispersion function for humidity correction	14, 31
Δ_L	operator for a quantity on the wavelength λ_2 minus the quantity for λ_1 (both optical wavelengths)	17
Δ_M	operator for a quantity on a micro wavelength minus the quantity on the optical wavelength λ_1	17
δ	operator indicating a small variation	22, 36, 37, 47
δ_{LL}	correction if using the Lorentz-Lorenz form	43
ε	small quantity, independent of the position	13
η	observed elevation of B in A	26
θ_ϱ, θ_e	functions allowing for the difference between radio waves and light waves	19, 42
Φ	effect of dispersive extinction on travelling time	9
λ	wavelength in vacuum	10
μ	a function of the position ($\mu \approx 1$ or $\mu < 1$)	13
π	$= 3.14\ldots$	10
ϱ	function of the local temperature and total pressure (ϱ is roughly proportional to the density of the air)	12, 31, 39
σ	$= c \times \tau$, optical path length	10
σ_m	$= c \times \tau_m$, measured path length for a group	10
σ_g	path length calculated with $\tilde{n}$	44
τ, τ_m	retardation or travelling time	8, 10
Ω	second order terms	14
ω	angular frequency ($= 2\pi \times$ frequency)	8
ω_s	angular frequency of modulation	8
ω_c	angular frequency of carrier	8
1	index belonging to primary optical wavelength	17
2	index belonging to secondary optical wavelength	17

SUMMARY

A general expression is derived for the group propagation time in a dispersive inhomogeneous medium. This expression is more exact than the usual concept of the group refraction index.

The above mentioned expression is applied to an extension of the theory of MORITZ. MORITZ gives the propagation time and the refractive angle in an inhomogeneous medium as a power series expansion. So first order- and second order corrections or errors are derived for electromagnetic distance measurement on one, two and three wavelengths and for angle measurement on two wavelengths.

For the E.D.M. the following influences are considered: the first order dry air correction for the refraction index, the first order humidity correction, and the dry air-, humidity-, and mixed curvature corrections. For the three-wavelengths method the different temperature dependence for light- and for microwaves and the inaccuracies in the dispersion formulae for the air are also considered.

For angle measurements, where the dispersion effect is extremely small, only the first order corrections for dry air and for humidity are considered.

For E.D.M. an accuracy of a few parts in 10^9 seems to be obtainable provided that the refractive index of air as a function of the wavelength, the temperature, the pressure and the humidity are known with sufficient accuracy, and provided that the instrumental errors are sufficiently small.

For angle measurement on two optical wavelengths it will be very difficult to obtain a sufficient instrumental accuracy because of the small phase dispersion and because of the serious influence of very local effects in angle measurements.

Chapter 1

INTRODUCTION

The accuracy of electromagnetic distance measurement (E.D.M.) is seriously influenced by local variations of the refractive index of the air. To correct for these variations one should know the average group refraction index for E.D.M. or, for measuring angles, the average value of the variations of the phase refraction index perpendicular to the path. In a *dispersive* medium, i.e. a medium in which this refraction index depends on the wavelength, the average values can in first order approximation be eliminated by measuring the distance (or the angle) simultaneously on two wavelengths, provided the measurements can be executed with sufficient accuracy.

To correct also for the average composition of the air (humidity) it may be useful to measure on three wavelengths instead of on two.

The possibilities and the limitations of these methods are considered by extending the above mentioned first order theory.

With respect to E.D.M., a general expression is derived for the group travelling time (equation 5a). In order to derive detailed expressions for calculating the geometric distance from travelling time measurements and from meteorological measurements, the general expression is applied to a theory given by MORITZ in [1]. In this theory MORITZ expresses the geometric distance in a measured travelling time and in integrals of the refractive index and its derivates along a straight line. This theory may be used for E.D.M. in an inhomogeneous medium where the variations of the refraction index are small and gradual.

The theory derived in this paper is aimed at the estimation of errors caused by different physical sources in E.D.M. using one, two or three wavelengths. Numerical results are given in table 8.

Similar considerations lead to an estimation of the errors in agle measurements (chapter 7). Here only a first order approximation is used because the effects are small. Moreover the measurement of the refraction angle is extremely difficult.

For many years different investigators have tried to use the dispersion method to measure or to eliminate the refraction angle: [2], [3, pag. 81], [4], [20]. Up to now the success does not seem to be very great.

In the case of E.D.M. dispersion measurements are better applicable because the group dispersion is considerably larger than the phase dispersion (about three times as large). Moreover by using photo-electric methods, as is done in E.D.M., one can spread the wavelengths more than in the visual case. Promising experiments have recently been described by J. C. OWENS [5] and by M. T. PRILIPIN [6].

Chapter 2

THE GROUP PROPAGATION TIME OF AN AMPLITUDE-MODULATED SIGNAL

In electronic distance measurement the travelling time is measured, or more exactly the phase difference of a modulation of the carrier wave between starting and end point. If the medium is dispersive (i.e. the propagation velocity changes with the frequency) the travelling time of the modulation (group travelling time) differs from the travelling time of a strictly monochromatic wave (phase travelling time).

In this chapter an expression will be derived giving the group travelling time for an amplitude-modulated wave received by a quadratic detector (e.g. photo-multiplier).

Suppose in A (the starting point) there is an amplitude-modulated vibration with an angular frequency of the carrier ω_c and an amplitude $U_0 \cdot (1+m \sin \omega_s t)$, in which m is the degree of modulation and ω_s ($\ll \omega_c$) the angular frequency of the modulation. This vibration may be written as [7, section 7 par. 1]:

$$U_A(t) = U_0 \cdot (1+m \sin \omega_s t) \cdot \sin \omega_c t \quad \ldots \ldots \quad (1)$$

or as the sum of three harmonic vibrations:

$$U_A(t) = U_0 \sin \omega_c t + \frac{mU_0}{2} \cos(\omega_c - \omega_s)t - \frac{mU_0}{2} \cos(\omega_c + \omega_s)t$$

Suppose in the end point B we have a vibration $U_B(t)$ related to $U_A(t)$ by linear differential equations. This assumption will do for all cases of electronic distance measurement. Then $U_B(t)$ may be written as the sum of three components with the same frequencies as in $U_A(t)$, however with retardations $\tau(\omega)$ and losses $\beta(\omega)$ which may be considered as linear functions within the narrow frequency band $\omega_c - \omega_0 < \omega < \omega_c + \omega_0$. So one can state:

$$\tau(\omega_c - \omega_s) = \tau - \omega_s \frac{d\tau}{d\omega} \qquad \tau(\omega_c + \omega_s) = \tau + \omega_s \frac{d\tau}{d\omega}$$

$$\beta(\omega_c - \omega_s) = \beta - \omega_s \frac{d\beta}{d\omega} \qquad \beta(\omega_c + \omega_s) = \beta + \omega_s \frac{d\beta}{d\omega}$$

with:

$$\left.\begin{array}{lll} \tau = \tau(\omega_c) & \dfrac{d\tau}{d\omega} = \left(\dfrac{d\tau(\omega)}{d\omega}\right) & \text{for} \quad \omega = \omega_c \\ \beta = \beta(\omega_c) & \dfrac{d\beta}{d\omega} = \left(\dfrac{d\beta(\omega)}{d\omega}\right) & \text{for} \quad \omega = \omega_c \end{array}\right\} \quad \ldots \ldots \quad (2)$$

So the vibration in the end point becomes:

$$
\begin{aligned}
U_B(t) &= \beta(\omega_c) \cdot U_0 \sin\{\omega_c t - \omega_c \cdot \tau(\omega_c)\} + \\
&+ \beta(\omega_c - \omega_s) \times \frac{mU_0}{2} \cos\{(\omega_c - \omega_s)t - (\omega_c - \omega_s) \times \tau(\omega_c - \omega_s)\} - \\
&- \beta(\omega_c + \omega_s) \times \frac{mU_0}{2} \cos\{(\omega_c + \omega_s)t - (\omega_c + \omega_s) \times \tau(\omega_c + \omega_s)\}
\end{aligned}
$$

$$
\begin{aligned}
U_B(t) &= \beta U_0 \sin(\omega_c t - \omega_c \tau) + \\
&+ \left(\beta - \omega_s \frac{d\beta}{d\omega}\right) \frac{mU_0}{2} \cos\left\{(\omega_c - \omega_s)t - \omega_c \tau + \omega_s \tau + \omega_c \omega_s \frac{d\tau}{d\omega} - \omega_s^2 \frac{d\tau}{d\omega}\right\} - \\
&- \left(\beta + \omega_s \frac{d\beta}{d\omega}\right) \frac{mU_0}{2} \cos\left\{(\omega_c + \omega_s)t - \omega_c \tau - \omega_s \tau - \omega_c \omega_s \frac{d}{d\omega} - \omega_s^2 \frac{d\tau}{d\omega}\right\}
\end{aligned}
$$

For phase measurement of the modulation of a light signal, use is made of a squaring detector, for example a photo-multiplier. In such a detector the signal is filtered so, that only the low frequency components can be observed (ω of the order of magnitude of ω_s)*).

After some calculations one can write for the relevant terms of $U_B^2(t)$:

$$
\begin{aligned}
U_B^2(t) &= \ldots + \beta^2 m U_0^2 \cos\left\{\omega_s^2 \frac{d\tau}{d\omega}\right\} \sin\left\{\omega_s t - \omega_s \tau - \omega_c \omega_s \frac{d\tau}{d\omega}\right\} - \\
&- \beta \omega_s \frac{d\beta}{d\omega} m U_0^2 \sin\left\{\omega_s^2 \frac{d\tau}{d\omega}\right\} \cos\left\{\omega_s t - \omega_s \tau - \omega_c \omega_s \frac{d\tau}{d\omega}\right\} - \\
&- \left\{\beta^2 - \omega_s^2 \left(\frac{d\beta}{d\omega}\right)^2\right\} \frac{m^2 U_0^2}{4} \cos 2\left\{\omega_s t - \omega_s \tau - \omega_c \omega_s \frac{d\tau}{d\omega}\right\}
\end{aligned}
$$

or with

$$
\operatorname{tg} \Phi = \frac{\omega_s}{\beta} \frac{d\beta}{d\omega} \operatorname{tg}\left(\omega_s^2 \frac{d\tau}{d\omega}\right): \qquad \ldots \qquad (3)
$$

$$
\left.
\begin{aligned}
U_B^2(t) &= \ldots + \beta m U_0^2 \left[\beta^2 \cos^2\left\{\omega_s^2 \frac{d\tau}{d\omega}\right\} + \omega_s^2 \left(\frac{d\beta}{d\omega}\right)^2 \sin^2\left\{\omega_s^2 \frac{d\tau}{d\omega}\right\}\right]^{\frac{1}{2}} \times \\
&\times \sin\left\{\omega_s t - \omega_s \tau - \omega_c \omega_s \frac{d\tau}{d\omega} - \phi\right\} - \\
&- \left\{\beta^2 - \omega_s^2 \left(\frac{d\beta}{d\omega}\right)^2\right\} \frac{m^2 U_0^2}{4} \cos 2\left\{\omega_s t - \omega_s \tau - \omega_c \omega_s \frac{d\tau}{d\omega}\right\}
\end{aligned}
\right\} \quad \ldots \quad (4)
$$

The term Φ of the phase of the first component appears to be extremely small for electro-optical distance measurement. This is demonstrated in the following unfavourable case:

*) This is not the case with interference of carrier waves in the Väisälä base measurement method.

$\omega_c = 2\pi \cdot 10^{15}$ radians/sec (carrier wavelength $\approx 0.3\ \mu$m) $\omega_s = 2\pi \cdot 10^{9}$ radians/sec (modulation wavelength ≈ 30 cm) $\tau = 10^{-3}$ sec (distance ≈ 300 km) $-\frac{\omega_c}{\tau} \cdot \frac{d\tau}{d\omega} \approx -\omega_c \cdot \frac{dn}{d\omega} \approx +\lambda_c \frac{dn}{d\lambda_c} \leqslant 10^{-4}$ for atmospheric air if n is the (phase) refraction index at a circular frequency ω_c and λ_c the (vacuum) wavelength.
With these "unfavourable" values one finds: $\mathrm{tg}\, \Phi = 10^{-6} \frac{\omega_c}{\beta} \frac{d\beta}{d\omega} \mathrm{tg}\, (2\pi \cdot 10^{-4})$
So if the dispersion of the attenuation*) $\left(\frac{\omega_c}{\beta} \frac{d\beta}{d\omega}\right)$ is not very high, $\mathrm{tg}\, \Phi$ is extremely small indeed.

Comparing the phases of the right hand terms of (4) with the phase of the modulation of (1) and neglecting Φ one finds the retardation τ_m of the modulation, i.e. the measured travelling time: see equation (5a), where the index c for the carrier wave has been omitted. Equation (5) can also be written as (5b), (5c) and (5d), where use is made of:

- the optical path length σ or σ_m, defined by τ or τ_m multiplied with the light velocity in vacuum c;
- the wave number g (6c);
- the wavelength in vacuum λ (6d).

τ and σ are values for the (monochromatic) carrier wave, the index m is used for the measured values on the modulation.

(5a) $\tau_m = \tau + \omega \frac{\partial \tau}{\partial \omega}$	(5b) $\sigma_m = \sigma + \omega \frac{\partial \sigma}{\partial \omega}$	. . . . (5)
(5c) $\sigma_m = \sigma + g \frac{\partial \sigma}{\partial g}$	(5d) $\sigma_m = \sigma - \lambda \frac{\partial \sigma}{\partial \lambda}$	

Definitions:

(6a) $\sigma = \tau \cdot c$	(6b) $\sigma_m = \tau_m \cdot c$	. . . . (6)
(6c) $g = \frac{\omega}{2\pi c} (= 1/\lambda)$	(6d) $\lambda = 2\pi c/\omega$	

*) This dispersion may be caused by absorption in the atmosphere or in any optical or electronical filter, or in the directional sensitivity of the instrument.

The relation (5d) is very similar to the expression used in the resolution of the U.G.G.I. [8], giving the "group refraction index" $\tilde{n}$ as a function of λ and the phase refraction index n:

$$\tilde{n} = n - \lambda \frac{\partial n}{\partial \lambda} \left(\equiv n + g \frac{\partial n}{\partial g} \right) \quad \ldots \quad (7)$$

Both relations are physically identical if it is supposed that the electromagnetic (or other) waves follow a straight-line path between A and B. This is the case normally found in textbooks about group propagation.

Note: For radio waves in the lower atmosphere the dispersion is so small that phase propagation time may always be used in this case.

Chapter 3

THE INFLUENCE OF AN INHOMOGENEOUS MEDIUM ON ELECTROMAGNETIC DISTANCE MEASUREMENT USING ONE CARRIER WAVE

The general principle of the geometric optics is a relation between the phase refraction index n in a point and the optical path σ between some origin and that point.

$$(\text{grad}\, \sigma)^2 = n^2 \quad (8)$$

This relation, corresponding to the well-known principle of Fermat, may be derived from the field equations of Maxwell for a purely monochromatic wave in an isotropic inhomogeneous medium which does not change considerably over a distance of one wavelength [9, ch. III].

The refraction index n in the lower atmosphere may be written as a function of the place and of the wavelength, if time-effects are not considered. Owens [10] gives probably the most accurate expressions for n as a function of the wavelength, the meteorological conditions and the composition of the air for visual light in the atmosphere. In our paper however the much simpler formulae of Edlèn [11, eq. 22, 12 and 1] will be used for visual light. For radio waves the equation of Essen and Froome will be used [12, eq. 2]. Only when considering the errors introduced by these equations (chapter 6 and appendix II), the Owens' expressions are introduced.

When using the equations of Edlèn and of Essen and Froome, n may be written as:

$$n = 1 + G\varrho + \Gamma e \quad (9)$$

where: n is considered as a function of the position and of g,
g is the inverse of the wavelength in vacuum,
G and Γ are only functions of g (for light) or constants (for radio waves),
ϱ is a function of the local temperature and the total pressure of the air (ϱ is roughly proportional to the density of the air), and
e is a function of the local partial pressure of the water vapour and of the local temperature and total pressure (for dry air $e = 0$).

Hence for light waves, and also for radio waves, ϱ and e are only functions of the place. G, Γ, ϱ and e will always be chosen so, that ϱ and e never become much larger than unity.

Some values are calculated in appendix I. G appears to be about $300 \cdot 10^{-6}$ for light- and for radio waves. Γ_M for radio waves (microwaves) has about the same magnitude, but for light $\Gamma_L \approx -4 \cdot 10^{-6}$. The more complete values are mentioned in table 1.

In [1] MORITZ gives an approximation for calculating the optical path and the refraction angle from (8), if n is known as a function of a particular place. MORITZ states:

$$n^2 = 1 + \varepsilon\mu \qquad (10)$$

where ε (≈ 0.0006) is independent of the place, and μ (≈ 1 or < 1) is a function of the place only. (The dependence on the wavelength is not mentioned in [1]).

MORITZ writes the optical path σ from an origin A to the point B as a power series in ε:

$$\sigma = S + \varepsilon F_a + \varepsilon^2 F_b + \ldots$$

where S is the geometric distance from A to B, and F_a, F_b, ... are functions of the place of B.

This series is only useful if n does not change too irregularly with the place. In cases of mirage, duct or reflections – when more than one light- (or radio) ray exists between A and B – and in the case of strong turbulences, the series does not give always a useful model.

Using this series MORITZ derives an expression for the geometric distance between A and B. This expression contains the refraction index and its partial derivates in all points of a straight line between starting point and end point. If this line is running through extra disturbed regions (near the ground) or *a fortiori* through the ground one must take for n and its derivates an extrapolation of the n field nearer to the physical rays.

For the refraction angles the elevation and the bearing of the starting point A, seen from the end point B, are written as power series in ε. Here the same restrictions may be mentioned.

If in the whole relevant space n is multiplied by any constant factor, the refraction angles do not change, and the optical paths are multiplied with the same factor. So the theory is not only true if $n \approx 1$, but in *any* case provided the relative variations of n are small. In this paper the theory of MORITZ is accepted.

Our equation (9) is consistent with the statement of MORITZ (10) if:

$$\varepsilon\mu = 2G\varrho + 2\Gamma e + G^2\varrho^2 + 2G\Gamma\varrho e + \Gamma^2 e^2 \qquad (11)$$

Using the results of MORITZ [8, eq. (3), (4), (8), (15), (13′)], the optical path length of a monochromatic wave is obtained after some calculations (12a). Introduction of (11) gives (12b).

$$\sigma = \int_0^S n\,\mathrm{d}X - \tfrac{1}{2}\int_0^S \frac{\left(\int_0^X \frac{\partial n}{\partial Y} X\,\mathrm{d}X\right)^2 + \left(\int_0^X \frac{\partial n}{\partial Z} X\,\mathrm{d}X\right)^2}{X^2}\,\mathrm{d}X \qquad (12a)$$

$$\sigma = S + GI_\varrho + \Gamma I_e - \tfrac{1}{2}G^2 I_{\varrho\varrho} - G\Gamma I_{\varrho e} - \tfrac{1}{2}\Gamma^2 I_{ee} \qquad (12b)$$

where:

$$\left.\begin{aligned}
I_\varrho &= \int_0^S \varrho \mathrm{d}X \\
I_e &= \int_0^S e \mathrm{d}X \\
I_{\varrho\varrho} &= \int_0^S \frac{\left(\int_0^X \frac{\partial \varrho}{\partial Y} X\mathrm{d}X\right)^2 + \left(\int_0^X \frac{\partial \varrho}{\partial Z} X\mathrm{d}X\right)^2}{X^2}\,\mathrm{d}X \\
I_{\varrho e} &= \int_0^S \frac{\left(\int_0^X \frac{\partial \varrho}{\partial Y} X\mathrm{d}X\right)\left(\int_0^X \frac{\partial e}{\partial Y} X\mathrm{d}X\right) + \left(\int_0^X \frac{\partial \varrho}{\partial Z} X\mathrm{d}X\right)\left(\int_0^X \frac{\partial e}{\partial Z} X\mathrm{d}X\right)}{X^2}\,\mathrm{d}X \\
I_{ee} &= \int_0^S \frac{\left(\int_0^X \frac{\partial e}{\partial Y} X\mathrm{d}X\right)^2 + \left(\int_0^X \frac{\partial e}{\partial Z} X\mathrm{d}X\right)^2}{X^2}\,\mathrm{d}X
\end{aligned}\right\} \quad (13)$$

and where:

- S is the geometric straight-line distance between the starting point A and the end point B,
- X, Y, Z are carthesian coordinates with the origin in A and the X-axis through B, and the Y-axis parallel to the horizontal plane in B *),
- all integrals are taken along the X-axis, i.e. ϱ, e, and its derivates are taken for $Y = Z = 0$,
- all terms of third- and higher degree in G and Γ are neglected.

The equations (12a) and (12b) give only a monochromatic solution of the wave propagation. In electromagnetic distance measurement however the optical pathlength σ_m of a modulation is measured. This value may be found by introducing (5c). One finds:

$$S = \sigma_m - \tilde{G} I_\varrho - \tilde{\Gamma} I_e + \Omega \quad \ldots \quad (14)$$

with:

$$\Omega = \tfrac{1}{2}\left\{\tilde{G}^2 - g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2\right\} I_{\varrho\varrho} + \left\{\tilde{G}\tilde{\Gamma} - g^2 \frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right\} I_{\varrho e} + \tfrac{1}{2}\left\{\tilde{\Gamma}^2 - g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2\right\} I_{ee} \quad (15)$$

and:

$$\tilde{G} = G + g\frac{\mathrm{d}G}{\mathrm{d}g}; \qquad \tilde{\Gamma} = \Gamma + g\frac{\mathrm{d}\Gamma}{\mathrm{d}g} \quad \ldots \quad (16)$$

In the expression (14):

S is the wanted geometric distance between A and B,
σ_m follows directly from the measured travelling time by multiplying with c,
$(-\tilde{G} I_\varrho)$ is the correction for the (group) velocity in dry air,
$(-\tilde{\Gamma} I_e)$ is the correction on the term $(-\tilde{G} I_\varrho)$ for the humidity, and
Ω contains second order corrections.

*) With this choice of the axes the refraction in the Y-direction may be called lateral refraction, which normally will be much smaller than the (vertical) refraction in the Z-direction.

In order to find some quantitive statements on the different influences on the resulting distance S, a simplified model will be used:

simplified model

a The distance is "horizontal", i.e. starting- and end point are on one level surface.
b The values of ϱ and e are constant on this (spherical) surface.
c The lateral refraction is negligible, i.e. $\partial\varrho/\partial Y = 0$ and $\partial e/\partial Y = 0$.
d $\partial\varrho/\partial Z$ and $\partial e/\partial Z$ are constant in the relevant space. (17)

In this model the equation (18) for the integrals of (13) is easily found *).

$$\left.\begin{aligned} I_\varrho &= S\varrho - \frac{S^3}{12R}\frac{\partial\varrho}{\partial Z} \qquad I_e = Se - \frac{S_3^3}{12R}\frac{\partial e}{\partial Z} \\ I_{\varrho\varrho} &= \frac{S^3}{12}\left(\frac{\partial\varrho}{\partial Z}\right)^2 \qquad I_{\varrho e} = \frac{S^3}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z} \qquad I_{ee} = \frac{S^3}{12}\left(\frac{\partial e}{\partial Z}\right)^2 \end{aligned}\right\} \tag{18}$$

For the relative correction on the direct measured optical path length one finds:

$$\begin{aligned} \frac{S-\sigma_m}{S} = &- \tilde{G}\varrho - \tilde{\Gamma}e + \left\{\frac{\tilde{G}}{12R}\frac{\partial\varrho}{\partial Z}S^2 + \frac{\tilde{G}^2 - g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2 S^2\right\} + \\ &+ \frac{\tilde{G}\tilde{\Gamma} - g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}S^2 + \\ &+ \left\{\frac{\tilde{\Gamma}}{12R}\frac{\partial e}{\partial Z}S^2 + \frac{\tilde{\Gamma}^2 - g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2 S^2\right\} \end{aligned} \tag{19a}$$

---

*) *Derivation of the equation* (18)
The intersection of the (spherical) level surface through the end points A and B with the plane $Y = 0$ is described by the circle:

$$X^2+Z^2-XS+Z\sqrt{4R^2-S^2} = 0,$$

if $R \approx 6.38\cdot10^6$ m is the ray of curvature of the level surface. Since R is not much smaller than the usual rays of curvature of the light- or radio path, S^2 may be neglected in the approximations of (12a) and (12b). So one finds for the level curve (where ϱ and e have the constant values ϱ_0 and e_0):

$$Z = X(S-X)/2R$$

If $\partial\varrho/\partial Z$ and $\partial e/\partial Z$ are constant in the relevant field, one finds for the values of ϱ and e along the X-axis (that is along the straight line A–B):

$$\varrho = \varrho_0 - \frac{\partial\varrho}{\partial Z}\frac{X(S-X)}{2R} \quad \text{and} \quad e = e_0 - \frac{\partial e}{\partial Z}\frac{X(S-X)}{2R}$$

Substituting these values and the constant values $\partial\varrho/\partial Z$ and $\partial e/\partial Z$ in (13) and omitting the index 0 the equations (18) are obtained.

or with very good approximation (see appendix III):

$$\boxed{\frac{S-\sigma_m}{S} \approx 1-\tilde{n}+\frac{S^2}{12R}\frac{\partial \tilde{n}}{\partial Z}+\frac{S^2}{24}\left(\frac{\partial \tilde{n}}{\partial Z}\right)^2} \qquad \text{(19b)}$$

where: R is the radius of the level surface through A and B; and
$\tilde{n}$ is the group refraction index defined by (7).

Based on the literature and on own estimates some values of the air refractivity and of meteorological conditions are derived in appendix I and compiled in table 1. The meteorological values are indicated for "normal conditions" (usual in moderate climats) and for exceptional conditions (seldom occurring in different climates, but not strictly extremes). With these values of table 1 the different terms of (19a) have been calculated, see table 3. The physical meaning of the terms is indicated in table 2.

It may easily be seen that the second order terms can become important, particularly for the longer distances (tens of kilometres). It is also clear that the humidity may not always be neglected in electro-optical distance measurement.

Chapter 4

ELECTROMAGNETIC DISTANCE MEASUREMENT ON TWO OPTICAL WAVELENGTHS

By this method two independent measurements are executed:

1 The optical path σ_{m1} measured on the optical wavelength λ_1, and
2 The difference $\Delta_L\sigma_m = \sigma_{m2} - \sigma_{m1}$ between the optical paths on two wavelengths λ_2 and λ_1. This difference can often be measured with a much higher precision than the path σ_{m1} itself.

Owing to the dispersion of the air it is possible to calculate the distance S if all values in (14) are known or negligible except S and I_ϱ. Writing down (14) for λ_1 and for λ_2 one finds after some calculations:

$$S = \sigma_{m1} - D\Delta_L\sigma_m - (\tilde{\Gamma}_1 - D\Delta_L\tilde{\Gamma})I_{eL} + \Omega_1 - D\Delta_L\Omega \quad \text{. . . . .} \quad (20)$$

where: $D = \tilde{G}_1/\Delta_L\tilde{G}$ can be called "dispersion factor";
the operator Δ_L indicates the value of the next quantity for λ_2 minus the value for λ_1;
index L indicates an optical wavelength for which the equations (a) and (c) of appendix I hold;
the index M will be used in the next chapter to indicate microwaves (appendix I equations (b) and (d));
the indices 1 and 2 indicate the (optical) wavelengths λ_1 and λ_2.

σ_{m1} and $\Delta_L\sigma_m$ are directly measured. The values of $\tilde{G}_1$, $\tilde{\Gamma}_1$, etc. can be calculated with high precision from laboratory measurements [10] [11]. The dispersion factor D appears to be about 10 for two wide-spread visible wavelengths λ_1 and λ_2.

The different meteorological effects on the measurements will be estimated using the model (17). Substituting (15) and (18) into (20) one finds the relative difference between the distance S and the distance $(\sigma_{m1} - D\Delta_L\sigma_m)$ calculated from the measured path lengths σ_{m1} and $\Delta_L\sigma_m$:

$$\frac{S-(\sigma_{m1}-D\Delta_L\sigma_m)}{S} = -(\tilde{\Gamma}_1-D\Delta_L\tilde{\Gamma})e_L+$$
$$+\frac{\tilde{G}_1^2-g_1^2\left(\frac{\mathrm{d}G}{\partial g}\right)_1^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)_L^2 S^2-D\Delta_L\left\{\frac{\tilde{G}^2-g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2+$$
$$+\frac{\tilde{G}_1\tilde{\Gamma}_1-g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}}{12}\left(\frac{\partial\varrho}{\partial Z}\right)_L\left(\frac{\partial e}{\partial Z}\right)_L S^2-D\Delta_L\left\{\frac{\tilde{G}\tilde{\Gamma}-g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2+$$
$$+\frac{\tilde{\Gamma}_1^2-g_1^2\left(\frac{\partial\Gamma}{\partial g}\right)_1^2}{24}\left(\frac{\partial e}{\partial Z}\right)_L^2 S^2-D\Delta_L\left\{\frac{\tilde{\Gamma}^2-g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2+\frac{\tilde{\Gamma}_1-D\Delta_L\tilde{\Gamma}}{12R}\left(\frac{\partial e}{\partial Z}\right)_L S^2 \qquad (21a)$$

or with good approximation (see appendix III):

$$\frac{S-(\sigma_{m1}-\bar{D}\Delta_L\sigma_m)}{S} = \frac{\frac{\partial\tilde{n}_1}{\partial Z}-\bar{D}\Delta_L\frac{\partial\tilde{n}}{\partial Z}}{12R}S^2+\frac{\left(\frac{\partial\tilde{n}_1}{\partial Z}\right)^2-\bar{D}\Delta_L\left(\frac{\partial\tilde{n}}{\partial Z}\right)^2}{24}S^2 \qquad (21b)$$
$$\bar{D} = \frac{\tilde{n}_1-1}{\Delta_L\tilde{n}}$$

With the values of table 1 the terms of (21a) have been calculated, see table 4. The physical meaning of the terms is indicated in table 2. Comparing the case of two optical wavelengths (table 4) with the measurements on one optical wavelength (table 3) one sees:

1 The first order dry air correction is non-existent for the two wavelengths.
2 The humidity corrections are bigger for the two wavelengths than for one wavelength.
3 The second order dry air correction is smaller for the two wavelengths in most of the cases.

Chapter 5

ELECTROMAGNETIC DISTANCE MEASUREMENT ON TWO OPTICAL WAVELENGTHS AND ONE MICRO WAVELENGTH

For this method three independent measurements are executed:

1 The optical path σ_{m1} on the optical wavelength λ_1.
2 The difference $\Delta_L\sigma_m = \sigma_{m2} - \sigma_{m1}$ between the optical paths on two wavelengths λ_1 and λ_2, and
3 The difference $\Delta_M\sigma_m = \sigma_{mM} - \sigma_{m1}$ between the optical paths on a microwave λ_M and on the optical wavelength λ_1.

Like in chapter 4 the differences may often be measured with a much higher precision than the optical path σ_{m1} itself. Owing to the dispersion of dry air for optical waves and owing to the difference in velocity between optical waves and radio waves in water vapour, it is possible to correct the measurements for the mean refraction index of dry air and for the mean humidity. However some difficulties arise on account of the different temperature-humidity dependence of the refraction indices for optical- and for radio waves. Also the well-known reflections and other double path effects on radio waves might give much trouble, which may however, at least in principle, be eliminated by instrumentation and measuring methods.

To discuss the calculations for this three-wavelengths method equation (14) is written for λ_1, λ_2 and λ_M as:

$$\left.\begin{aligned} S+\tilde{G}_1 I_{\varrho L}+\tilde{\Gamma}_1 I_{eL} &= \sigma_{m1}+\Omega_1 \\ S+\tilde{G}_2 I_{\varrho L}+\tilde{\Gamma}_2 I_{eL} &= \sigma_{m2}+\Omega_2 \\ S+\tilde{G}_M I_{\varrho M}+\tilde{\Gamma}_M I_{eM} &= \sigma_{mM}+\Omega_M \end{aligned}\right\} \qquad (22)$$

The straight-line integrals I are not identical for light- and for microwaves because ϱ and particularly e are different functions for the two types of waves. For ϱ the difference is very small but for e it may be quite significant.

As the ratio's $\theta_\varrho = \varrho_M/\varrho_L$ and $\theta_e = e_M/e_L$ do not change very much with the meteorological conditions along the path, it is useful to state $I_{\varrho M}/I_{\varrho L} = \theta_\varrho$ and $I_{eM}/I_{eL} = \theta_e$ for some mean conditions. From appendix I, equation (I.1), follows with good approximation:

$$\theta_\varrho = 1 \qquad \theta_e = 1.36 - 0.01\,t_e \qquad (23)$$

where t_e is some average temperature along the path. For t_e an estimation of the mean temperature can be introduced. The approximation (23) will be used for accuracy considerations; for the calculations of the distance from measurements the more accurate first forms of (I.1) may be used (see appendix I).

With the values θ_ϱ and θ_e the equations (22) may be written explicitly in the measured quantities σ_m, $\Delta_L\sigma_m$ and $\Delta_M\sigma_m$. Equation (24) is then obtained, from which the wanted distance S may be solved (25). The integrals $I_{\varrho L}$ and I_{eL} may of course also be solved.

$$\left.\begin{aligned} S+\tilde{G}_1\cdot I_{\varrho L}+\tilde{\Gamma}_1\cdot I_{eL} &= \sigma_{m1}+\Omega_1 \\ (\Delta_L\tilde{G})\cdot I_{\varrho L}+(\Delta_L\tilde{\Gamma})\cdot I_{eL} &= \Delta_L\sigma_m+\Delta_L\Omega \\ (\theta_\varrho\tilde{G}_M-\tilde{G}_1)\cdot I_{\varrho L}+(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)\cdot I_{eL} &= \Delta_M\sigma_m+\Delta_M\Omega \end{aligned}\right\} \qquad (24)$$

$$S = \sigma_{m1}-K_L\Delta_L\sigma_m-K_M\Delta_M\sigma_m+\Omega_1-K_L\Delta_L\Omega-K_M\Delta_M\Omega \qquad (25)$$

with:

$$\left.\begin{aligned} K_L &= \frac{\tilde{G}_1\tilde{\Gamma}_M\theta_e-\tilde{\Gamma}_1\tilde{G}_M\theta_\varrho}{\Delta_L\tilde{G}\cdot(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)-\Delta_L\tilde{\Gamma}\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)} \approx \frac{\tilde{G}}{\Delta_L\tilde{G}} = D \approx 10 \\ K_M &= \frac{\tilde{\Gamma}_1\cdot\Delta_L\tilde{G}-\tilde{G}_1\cdot\Delta_L\tilde{\Gamma}}{\Delta_L\tilde{G}\cdot(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)-\Delta_L\tilde{\Gamma}\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)} \approx \frac{\tilde{\Gamma}_1\tilde{G}_2-\tilde{G}_1\tilde{\Gamma}_2}{\theta_e\tilde{\Gamma}_M\Delta_L G} \approx -0.02 \end{aligned}\right\} \qquad (26)$$

K_L and K_M are factors that depend – apart from the small influence of t_e and the very small influence of t_ϱ – only on the used optical wavelengths λ_1 and λ_2. The numerical values of K_L and K_M are calculated from table 1 for $\lambda_1 = 0.625$ µm and $\lambda_2 = 0.3636$ µm.

The different meteorological effects on the distance calculated from the measurements will be estimated using the model (17). Substituting (15) with (18) into (25) one finds the relative difference between the distance S and the distance $(\sigma_{m1}-K_L\Delta_L\sigma_m-K_M\Delta_M\sigma_m)$ calculated from the measured path lengths σ_{m1}, $\Delta_L\sigma_m$ and $\Delta_M\sigma_m$:

$$\begin{aligned} \frac{S-(\sigma_{m1}-K_L\Delta_L\sigma_m-K_M\Delta_M\sigma_m)}{S} &= \Bigg[\frac{\tilde{G}_1^2-g_1^2\left(\frac{dG}{dg}\right)_1^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)_L^2 S^2 - \\ &-K_L\Delta_L\left\{\frac{\tilde{G}^2-g^2\left(\frac{dG}{dg}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2-K_M\Delta_M\left\{\frac{\tilde{G}^2-g^2\left(\frac{dG}{dg}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2\Bigg]+ \\ &+\Bigg[\frac{\tilde{G}_1\tilde{\Gamma}_1-g_1^2\left(\frac{dG}{dg}\right)_1\left(\frac{d\Gamma}{dg}\right)_1}{12}\left(\frac{\partial\varrho}{\partial Z}\right)_L\left(\frac{\partial e}{\partial Z}\right)_L S^2-K_L\Delta_L\left\{\frac{\tilde{G}\tilde{\Gamma}-g^2\frac{dG}{dg}\frac{d\Gamma}{dg}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2- \\ &-K_M\Delta_M\left\{\frac{\tilde{G}\tilde{\Gamma}-g^2\frac{dG}{dg}\frac{d\Gamma}{dg}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2\Bigg]+\Bigg[\frac{\tilde{\Gamma}_1^2-g_1^2\left(\frac{d\Gamma}{dg}\right)_1^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2 S^2- \\ &-K_L\Delta_L\left\{\frac{\tilde{\Gamma}^2-g^2\left(\frac{d\Gamma}{dg}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2-K_M\Delta_M\left\{\frac{\tilde{\Gamma}^2-g^2\left(\frac{d\Gamma}{dg}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2\Bigg] \end{aligned}$$

(27a)

or with good approximation (see appendix III):

$$\frac{S-(\sigma_{m1}-K_L\Delta_L\sigma_m-K_M\Delta_M\sigma_m)}{S}=\frac{S^2}{24}\left(\frac{\partial\tilde{n}}{\partial Z}\right)_1^2-K_L\Delta_L\left(\frac{\partial\tilde{n}}{\partial Z}\right)^2-K_M\Delta_M\left(\frac{\partial\tilde{n}}{\partial Z}\right)^2 \qquad (27b)$$

Note: Since for microwaves $dG/dg = d\Gamma/dg = 0$ (no dispersion), the wave number g disappears in (27a) for these waves.

With the values of table 1 the different terms of (27a) have been calculated, see table 5. The physical meaning of the terms is indicated in table 2. Comparing the case of two optical- and one radio wavelength with that of two optical wavelengths only, one sees:

1 Only the higher order terms exist for the 3-λ-method.
2 The dry air second order term is essentially equal for both cases.
3 The *second order* humidity effects may be greater for the 3-λ-method.

Chapter 6

A SURVEY OF THE NON-INSTRUMENTAL INACCURACIES OF ELECTROMAGNETIC DISTANCE MEASUREMENTS

6.1 General

In this chapter the non-instrumental inaccuracies are estimated for the different methods of E.D.M. considered in the preceding chapter. The results are complied in table 8. These values have been calculated form the assumed inaccuracies in the meteorological measurements and in the refractivity of the air from the tables 6 and 7 respectively. For the "favourable" and "typical" cases in table 8 the "typical" values of table 3, 4 and 5 were taken, for the "unfavourable" cases high absolute values of these tables were chosen.

Most of the values of table 8 may more or less be interpreted as standard deviations; the influences of the formulae (row 7, 8 and 9) are given as tolerances. The "unfavourable" cases give rather exceptional, but not extreme values.

With regard to the measurements the following assumptions are made:

1 One measurement takes about 15 to 30 minutes measuring time.
2 Careful measurements of temperature, barometric pressure and humidity are executed near both ends.
3 Vertical angles are measured at both end points in order to find the vertical density gradient of dry air $\partial\varrho/\partial Z$. See SAASTEMOINEN [13].

6.2 Explanations to table 8

Row 1 *The influence of errors in the density function ϱ, occurring only in the case of one wavelength.*

Partial differentiation of (19a), neglecting the higher order terms, gives:

$$\frac{\partial S}{\partial \varrho} = \frac{S\tilde{G}}{1+\tilde{G}\varrho+\tilde{F}e} = S\tilde{G} \quad \text{for} \quad \tilde{n} \to 1.$$

So the relative variation of S becomes:

$$\frac{\delta S}{S} = \frac{\delta \varrho}{\varrho} \cdot \varrho\tilde{G}$$

Substitution of the figures from the tables 1 and 6 gives the values in table 8.

Row 2 *The first order influence of the humidity.*

Calculated analogously to row 1 but with the values of e and δe from the tables 1 and 6.

Row 3 *The effect of errors in the estimation of the density gradient $\partial\varrho/\partial Z$ for dry air. (The accuracy of the third term of table 3 or the second term of table 4 or the first term of table 5).*

The values are calculated assuming $\delta(\partial\varrho/\partial Z)/(\partial\varrho/\partial Z) = 0.1$ (table 6), taking the values for $\tilde{G}$, $\tilde{\Gamma}$, D, K, etc. from table 1 and with $\partial\varrho/\partial Z$ from table 6. For a distance of 100 km however the errors for the "unfavourable" cases are assumed to amount to only twice the errors for a distance of 30 km.

Row 4 *The second order humidity influence. (The total accuracy of the 4th and 5th term in table 3 or of the 3rd and 4th term in table 4 or of the 2nd and 3rd term in table 5).*

For $\delta(\partial e/\partial Z)/(\partial e/\partial Z)$ unity has been substituted because it is hardly possible to estimate $\partial e/\partial Z$. So for the values in table 8 for "typical" and "unfavourable" cases the sums of the relevant terms are taken from the tables 3, 4 or 5. For the "favourable" cases the errors are supposed to be 25% of the typical values. For a distance of 100 km the errors for the "unfavourable" cases are assumed to amount to only twice the errors for a distance of 30 km.

If $\partial e/\partial Z$ can be calculated from measurements of the humidity gradients, the values of row 4 may possibly be reduced by a factor 10.

Row 5 and 6 *The influence of errors in estimating the temperature t_e. This temperature t_e is necessary to account for the difference between the temperature dependence of the refractive index for light waves and radio waves (essentially for the water vapour).*

t_e is assumed to be known with the same accuracy as the mean air temperature (table 6). The temperature t_e may be calculated from temperature measurements at the end points or from $I_{\varrho L}$, calculated iteratively from the equations (24)*). The inaccuracies caused by errors in the accepted value for t_e are developed in appendix IV.

Row 7, 8 and 9 *The influence of errors in the formulae giving the refraction index of air as a function of pressure, temperature, humidity, wavelength, etc.*

The error δS in the calculated distance is found from the equations (19b), (21b) and (27a) or (27b). See appendix V. δS can explicitly be written as a function of the errors in the group refraction indices and in the group dispersion $\delta\tilde{n}_L$, $\delta\tilde{n}_M$ and $\delta\Delta_L\tilde{n}$. Numerical values for these errors are taken from literature (see table 7) assuming a moderate humidity ($e = 0.1$, i.e. $p_3 = 7.2$ torr) for "favourable" and "typical" conditions and a very high humidity ($e = 1$, i.e. $p_3 = 72$ torr) for "unfavourable" conditions.

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*) Note on the calculation of t_e from (24).
The temperature t_e is estimated as a first approximation. θ_e is calculated with the most accurate form of (I.1). $I_{\varrho L}$ is calculated from the 2nd and the 3rd equation of (24), neglecting the Ω-terms and assuming $t_\varrho = t_e$. Now ϱ_L is found with (18): $\varrho_L = I_{\varrho L}/S$. With appendix I, equation (c), t may be found if the air pressure p is known. Eventually iteration is possible.

This method seems to be more accurate than the direct measurement of t because p may be measured with a good accuracy. However there may be a significant discrepancy between the calculated t and the wanted t_e.

Row 10 *Errors caused by deviations of CO_2-contents of the air.*

Variations of the CO_2-contents of the air are described in [14] and roughly in [15]. The following values are assumed for the mean partial pressure p_4 and for its standard deviation:

mean partial pressure of CO_2	standard deviation	
	favourable or typical	unfavourable
$p_4 = 0.25$ torr	0.02 torr	0.08 torr

The influence on the refraction indices and on the dispersion is determined by:

$\partial \tilde{n}_L / \partial p_4$	$\partial \tilde{n}_M / \partial p_4$	$\partial \Delta_L \tilde{n} / \partial p_4$
$0.2 \cdot 10^{-6}$ torr^{-1}	$0.25 \cdot 10^{-6}$ torr^{-1}	$0.024 \cdot 10^{-6}$ torr^{-1}
[11]	[12]	[11]

Considering the influence of CO_2 as errors in the refraction index, the values of row 10 are found by introducing the figures from the above small tables into the equations (a), (b) and (c) of appendix V.

Row 11 *The uncertainty in the light velocity in vacuum, giving a constant deviation in the scale.*

If the distance is not expressed in "light seconds" but in internationally defined metres the uncertainty in the light velocity (table 7) may give a corresponding scaling error in the distance.

6.3 General remarks and conclusions on table 8

1 If the mean value is taken from a number of distances measured in different weather conditions, the meteorological effects on this mean value will tend to decrease from the "unfavourable" values to "typical-" and from "typical-" to "favourable" values. This, however, will hardly be the case for the errors in the formulae (row 7, 8 and 9). For methods more independent of visibility conditions (microwaves) the gain may be relatively high.

2 An error in the velocity c of light in vacuum means a constant error in the scale for all radar-type measurements. Descrepancies will only occur if the measurements are compared with very accurate distance measurements independent of c (invar wires, VÄISÄLÄ bases).

3 The inaccuracies from the optical formulae (row 7 and 8) are partly constant errors in the scale for electro-optical distance measurement. So the inaccuracies in the *shape* of geodetic configurations will often be a factor 2 better than the values suggested in the rows 7 and 8.

4 As already mentioned in the introduction of MORITZ' theory, ambiguous optical paths, such as reflections, ducts, etc., may give rise to errors. Although these errors may be quite important, even for optical wavelengths, table 8 does not account for these effects because a quantitative description is very difficult and because the effects can be limited by appropriate design of the instruments and the methods (narrow beams, diversity in place or in frequency, etc.).

5 The influence of the CO_2-contents and the errors in the formulae of ESSEN and FROOME (row 10 and 9) will nearly always be small compared with other effects. Also the influence of θ_e (row 5 and 6) will seldom be important.

6 For electro-optical distance measurement on one wavelength the effect of ϱ (row 1), that is essentially the accuracy of the temperature, is normally by far the most important source of errors. In "unfavourable" cases however on distances of more than 10 km the influence of the vertical refraction may become even more important (row 3 and to a less degree, row 4).

7 For microwave distance measurement (one λ) usually the mean humidity (row 2) is the dominating source of errors. For distances longer than 10 km with great humidity gradients the errors from row 4 ($\partial e/\partial Z$) may become significant.

8 For two optical wavelengths the errors in the formulae (row 7 and 8) are important and also the humidity of the air (row 2). For distances longer than 3 km also the vertical refraction may be significant (row 3 and to a less extent, row 4).

9 For the method with three wavelengths the influence of e (row 2) disappears but the other factors are not very different from the two-wavelengths method.

10 With one optical wavelength an accuracy of 1 ppm should normally be obtainable, with one micro wavelength about 4 ppm, with two optical wavelengths 0.15 ppm and with three wavelengths a somewhat better accuracy. The gain of the third wavelength seems very small. However the influence of the humidity (row 2) may increase considerably if only two wavelengths are used, and the gain will be much higher if the formulae for the refraction index (row 7 and 8) are better known.

11 The errors in the measurement of the optical paths (σ_m, $\Delta_L\sigma_m$ and $\Delta_M\sigma_m$) i.e. the instrumental- and the reading errors etc. are not considered in the above.

The influence of these errors is found by differentiating the calculated distance S partially in the equations (19b), (21b) and (25). Neglecting the higher order terms one finds with $\tilde{n} \rightarrow 1$:

Table 9

method	$\partial S/\partial\sigma_m$	$\partial S/\partial\Delta_L\sigma$	$\partial S/\partial\Delta_M\sigma$
one wavelength	1	–	–
two wavelengths	1	$-D \approx -10$	–
three wavelengths	1	$-K_L \approx -10$	$-K_M \approx 0.02$

So an error in the measured optical path σ_m enters directly in the result, an error in the optical difference $\Delta_L\sigma$ is multiplied with $(-D)$ and an error in $\Delta_M\sigma$ is decreased in the result to $(-K_M)$ times its value.

Chapter 7

THE DISPERSION FOR ANGLE MEASUREMENT

Roughly analogue to chapter 4 the theory of Moritz [1] may be used to calculate the angle of refraction from measurements of the directions on two wavelengths. For angle measurements however the group travelling time does not enter into the problem: only the phase refraction index is needed.

Because the effect of the dispersion on the direction is very small only the *first order* effect on *vertical* angles will be treated, although there is no real difficulty in deriving the theory for higher order terms and for bearings.

Suppose a light source B has in the point of observation A a geometric elevation H (see figure 1).

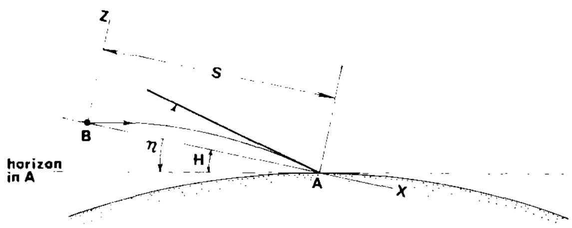

Figure 1 The elevation of B in A

B light source
A observer
H geometric elevation of B in A
η observed elevation of B in A

Rectangular Carthesian coordinate system:
Origin in B
X-axis through A
Y-axis perpendicular to the vertical in B

According to Moritz [1, equation 23, second] is then:

$$H = \eta + \frac{1}{S} \int_0^S \frac{\partial n}{\partial Z} X \mathrm{d}X \quad \ldots\ldots \quad (28)$$

where: the axes are taken in accordance to figure 1,
$\partial n / \partial Z$ is approximately a vertical component for a nearly horizontal direction BA,
the integral is taken along the X-axis.

Introducing (9) one finds:

$$H = \eta + G I_{\varrho Z} + \Gamma I_{eZ}$$

with:

$$I_{\varrho Z} = \frac{1}{S} \int_0^S \frac{\partial \varrho}{\partial Z} X \mathrm{d}X \quad \text{and} \quad I_{eZ} = \frac{1}{S} \int_0^S \frac{\partial e}{\partial Z} X \mathrm{d}X$$

In this equation only G and Γ and the measured elevation η depend on the wavelength.

So writing down the equation for two wavelengths and taking the difference one finds:

$$\left.\begin{aligned} H &= \eta_1+G_1 I_{\varrho Z}+\Gamma_1 I_{eZ} \\ 0 &= \Delta_L\eta+(\Delta_L G)I_{\varrho Z}+(\Delta_L\Gamma)I_{eZ} \\ \text{with: } \Delta_L G &= G_2-G_1, \quad \Delta_L\Gamma = \Gamma_2-\Gamma_1 \quad \text{and} \quad \Delta_L\eta = \eta_2-\eta_1 \end{aligned}\right\} \quad (29)$$

or eliminating $I_{\varrho Z}$:

$$H = \eta_1 - \frac{G_1}{\Delta G}\Delta\eta + C_e \quad (30)$$

with:

$$C_e = -\left(\frac{G_1}{\Delta G}\Delta_L\Gamma-\Gamma_1\right)I_{eZ}. \quad (31)$$

In equation (30) η_1 is measured directly, $\Delta_L\eta$ may be measured, although a sufficient precision is difficult to obtain. G_1, $\Delta_L G$, Γ_1 and $\Delta_L\Gamma$ can be calculated from formulae for the refraction index (e.g. appendix I equation (*c*). I_{eZ} however can at best roughly be estimated from the weather conditions. Since Γ is very small for (near) optical wavelengths the humidity correction will in general be neglected.

Table 10 shows that $G_1/\Delta_L G$ is rather high, particularly for visual observations*) (for example $\lambda_1 = 0.65$ µm and $\lambda_2 = 0.47$ µm). $G_1/\Delta_L G$ for phase velocity appears to amount to thrice the comparable quantity D for group velocity (see last column of table 10). So the angle difference $\Delta_L\eta$ should be measured with very high accuracy because errors in $\Delta\eta$ are multiplied with $G_1/\Delta_L G$.

Table 10. The coefficient of $\Delta_L\eta$ in equation (36)

λ_1 in µm	λ_2 in µm	$G_1/\Delta_L G$	$D = \tilde{G}_1/\Delta_L\tilde{G}$
0.65	0.47	80	26
0.625	0.3636	33	10.5
0.9	0.26	10	3.4

In order to estimate some numerical values for the refraction angle for the humidity correction, $\partial\varrho/\partial Z$ and $\partial e/\partial Z$ are supposed to be constant along the X-axis. In this case $I_{\varrho Z}$ and I_{eZ} can be written as:

$$I_{\varrho Z} = \tfrac{1}{2}\frac{\partial\varrho}{\partial Z}S \qquad \text{and} \qquad I_{eZ} = \tfrac{1}{2}\frac{\partial e}{\partial Z}S \quad (32)$$

From equation (30), (31) and (32) with the values from table 1 one finds the refraction angles in (sexagesimal) seconds of arc per kilometre in table 11.

*) It may be useful to measure η on a visual wavelength and to determine $\Delta\eta$ photo-electrically on two other wavelengths far apart.

Table 11. Refraction angles and humidity corrections

	typical	high	low
refraction angle $(\eta_1 - H)/S$	-4 ″/km	$+12$ ″/km	-50″/km
humidity correction C_e	$+0.02$″/km	$+0.8$ ″/km	$-$ 1″/km

If it becomes possible to measure the dispersion angle $\Delta\eta$ with sufficient accuracy, the refraction angle $(\eta_1 - H)$ can be calculated. The accuracy will then be limited by the refraction from humidity C_e or by inaccuracies of the dispersion formulae if the wavelengths are chosen too far from the visual region.

Chapter 8

CONCLUDING REMARKS

If the instrumental errors are kept sufficiently small and if two wavelengths are used, a gain in the precision of E.D.M. amounting to one order of magnitude should be possible. If additionally a radio wavelength is used to correct for the humidity, essential gain will only be reached in a very humid environment.

If the formulae for the refraction index of visual- and of infrared light should be better known, a somewhat better accuracy is probably possible with two optical wavelengths. With three wavelengths at least a gain of one order of magnitude should be possible.

The dispersion method may also be used to measure a refraction angle, but in this case it is extremely difficult to obtain a sufficient instrumental accuracy, because the angle measurement is very sensible to local changes in the refraction index, and because the dispersion of the air for the phase refraction index is only one third of the "group dispersion". Physically the accuracy is limited by the gradients of the humidity to 0.02″/km in normal conditions. This limit however may be decreased if the humidity gradient is measured or if the angles are measured from both ends.

Addendum

When this paper was ready in draft, I received the very interesting ESSA-report of Mr. THAYER [16], covering about the same subject. In this report Mr. THAYER considers distances between ground stations too, but more in detail, distances to satellites. I did not consider the latter cases although my approach may well be applied to distances to satellites.

The general conclusions of Mr. THAYER about terrestrial distances are in good agreement with my conclusions, in detail however there are some differences. In general my assumptions about meteorological circumstances are more "pessimistic" than Mr. THAYER's.

The error caused by the difference in the temperature-function of water vapour and dry air (my θ_e) is in the report of Mr. THAYER a factor 2.5 smaller than the value given in my paper. The cause must lay in the difference between his humidity formulae (32) and mine. I could not trace this difference in detail.

Mr. TENGSTRÖM finds in his article [4] a smaller dependence on the humidity than I do. He however does not account for variations of the humidity gradient, which variations are in my opinion more important than the effect mentioned by Mr. TENGSTRÖM.

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Table 1. Quantities about the refraction index

(dispersion) quantity	Functions of the wavelength			
	1	2	Δ_L	M
λ in μm	0.625	0.3636	-0.2614	$> 7 \cdot 10^{3}$
g in $(\mu m)^{-1}$	1.6	2.75	$+1.15$	$< 0.15 \cdot 10^{-3}$
G	$337 \cdot 10^{-6}$	$347 \cdot 10^{-6}$	$+ 10 \cdot 10^{-6}$	$332 \cdot 10^{-6}$
$\tilde{G} = G + g(dG/dg)$	$347 \cdot 10^{-6}$	$380 \cdot 10^{-6}$	$+ 33 \cdot 10^{-6}$	$332 \cdot 10^{-6}$
$\tilde{G}^2 - g^2(dG/dg)^2$	$120 \cdot 10^{-9}$	$143 \cdot 10^{-9}$	$+ 23 \cdot 10^{-9}$	$110 \cdot 10^{-9}$
$\tilde{G}\tilde{\Gamma} - g^2(dG/dg)(d\Gamma/dg)$	$-1.34 \cdot 10^{-9}$	$-1.30 \cdot 10^{-9}$	$+0.04 \cdot 10^{-9}$	$116 \cdot 10^{-9}$
Γ	$-4.04 \cdot 10^{-6}$	$-3.87 \cdot 10^{-6}$	$+0.17 \cdot 10^{-6}$	$348 \cdot 10^{-6}$
$\tilde{\Gamma} = \Gamma + g(d\Gamma/dg)$	$-3.87 \cdot 10^{-6}$	$-3.37 \cdot 10^{-6}$	$+0.50 \cdot 10^{-6}$	$348 \cdot 10^{-6}$
$\tilde{\Gamma}^2 - g^2(d\Gamma/dg)^2$	$14.9 \cdot 10^{-12}$	$11.1 \cdot 10^{-12}$	$-3.8 \cdot 10^{-12}$	$121 \cdot 10^{-9}$

curvature of the earth $R = 6.38 \cdot 10^{-6}$ m

(meteorological) quantity (lengths in metres)	Functions of meteorological conditions					
	light			microwaves		
	typical	high	low	typical	high	low
ϱ	$+0.85$	$+1$	$+0.5$	$+0.85$	$+1$	$+0.5$
e	$+0.1$	$+1$	0	$+0.12$	$+1$	0
$n-1\ (\lambda_1, \lambda_M)$	$280 \cdot 10^{-6}$	$340 \cdot 10^{-6}$	$170 \cdot 10^{-6}$	$320 \cdot 10^{-6}$	$600 \cdot 10^{-6}$	$170 \cdot 10^{-6}$
$\tilde{n}-1\ (\lambda_1, \lambda_M)$	$290 \cdot 10^{-6}$	$350 \cdot 10^{-6}$	$175 \cdot 10^{-6}$	$320 \cdot 10^{-6}$	$600 \cdot 10^{-6}$	$170 \cdot 10^{-6}$
$\partial\varrho/\partial Z$	$-0.11 \cdot 10^{-3}$	$+0.2 \cdot 10^{-3}$	$-1.1 \cdot 10^{-3}$	$-0.11 \cdot 10^{-3}$	$+0.2 \cdot 10^{-3}$	$-1.1 \cdot 10^{-3}$
$\partial e/\partial Z$	$-18 \cdot 10^{-6}$	$+0.8 \cdot 10^{-3}$	$-1.4 \cdot 10^{-3}$	$-30 \cdot 10^{-6}$	$+0.5 \cdot 10^{-3}$	$-1.1 \cdot 10^{-3}$
$\partial\tilde{n}/\partial Z\ (\lambda_1, \lambda_M)$	$-40 \cdot 10^{-9}$	$+80 \cdot 10^{-9}$	$-0.4 \cdot 10^{-6}$	$-50 \cdot 10^{-9}$	$+0.16 \cdot 10^{-6}$	$-0.5 \cdot 10^{-6}$
$(\partial\varrho/\partial Z)^2$	$+12 \cdot 10^{-9}$	$+1.2 \cdot 10^{-6}$	0	$12 \cdot 10^{-9}$	$+1.2 \cdot 10^{-6}$	0
$(\partial\varrho/\partial Z)(\partial e/\partial Z)$	$+2 \cdot 10^{-9}$	$+0.24 \cdot 10^{-6}$	$-0.8 \cdot 10^{-6}$	$+3 \cdot 10^{-9}$	$+0.4 \cdot 10^{-6}$	$-0.2 \cdot 10^{-6}$
$(\partial e/\partial Z)^2$	$+0.3 \cdot 10^{-9}$	$+2 \cdot 10^{-6}$	0	$+0.7 \cdot 10^{-9}$	$+1.2 \cdot 10^{-6}$	0
$(\partial\tilde{n}/\partial Z)^2$	$+1.5 \cdot 10^{-15}$	$+0.15 \cdot 10^{-12}$	0	$+2.2 \cdot 10^{-15}$	$+0.2 \cdot 10^{-12}$	0

Table 2. Explanation of the terms of equations (respectively table 3, 4, 5)

terms of equation (19a) (table 3)	
1st	first order correction for the refraction index of dry air
2nd	first order humidity correction
3rd	dry air curvature correction
4th	mixed curvature correction
5th	humidity curvature correction
terms of equation (21a) (table 4)	
1st	first order humidity correction
2nd	dry air curvature correction
3rd	mixed curvature correction
4th	humidity curvature correction
terms of equation (27a) (table 5)	
1st	dry air curvature correction
2nd	mixed curvature correction
3rd	humidity curvature correction

Table 3. Meteorological relative corrections on one wavelength

No.	term of equation (19a)	S in km	Light			Microwave		
			typical	high	low	typical	high	low
1st	$-\tilde{G}\varrho$	–	$-310\cdot 10^{-6}$	$-170\cdot 10^{-6}$	$-380\cdot 10^{-6}$	$-280\cdot 10^{-6}$	$-170\cdot 10^{-6}$	$-330\cdot 10^{-6}$
2nd	$-\tilde{\Gamma}e$	–	$+0.4\cdot 10^{-6}$	$+4\cdot 10^{-6}$	0	$-40\cdot 10^{-6}$	0	$-350\cdot 10^{-6}$
3rd	$\frac{\tilde{G}}{12R}\frac{\partial\varrho}{\partial Z}S^2+\frac{\tilde{G}^2-g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2 S^2$	1	$-0.5\cdot 10^{-9}$	$+1\cdot 10^{-9}$	$-1\cdot 10^{-9}$	$-0.5\cdot 10^{-9}$	$+1\cdot 10^{-9}$	$-1\cdot 10^{-9}$
		3	$-5\cdot 10^{-9}$	$+0.01\cdot 10^{-6}$	$-0.01\cdot 10^{-6}$	$-5\cdot 10^{-9}$	$+0.01\cdot 10^{-6}$	$-0.01\cdot 10^{-6}$
		10	$-0.05\cdot 10^{-6}$	$+0.1\cdot 10^{-6}$	$-0.1\cdot 10^{-6}$	$-0.05\cdot 10^{-6}$	$+0.1\cdot 10^{-6}$	$-0.1\cdot 10^{-6}$
		30	$-0.5\cdot 10^{-6}$	$+1\cdot 10^{-6}$	$-1\cdot 10^{-6}$	$-0.5\cdot 10^{-6}$	$+1\cdot 10^{-6}$	$-1\cdot 10^{-6}$
		100	$-5\cdot 10^{-6}$	–	–	$-5\cdot 10^{-6}$	–	–
		300	$-0.05\cdot 10^{-3}$	–	–	$-0.05\cdot 10^{-3}$	–	–
4th	$\frac{\tilde{G}\tilde{\Gamma}-g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}S^2$	1	$-0.2\cdot 10^{-12}$	$+0.1\cdot 10^{-9}$	$-0.03\cdot 10^{-9}$	$+0.03\cdot 10^{-9}$	$+4\cdot 10^{-9}$	$-2\cdot 10^{-9}$
		3	$-2\cdot 10^{-12}$	$+1\cdot 10^{-9}$	$-0.3\cdot 10^{-9}$	$+0.3\cdot 10^{-9}$	$+0.04\cdot 10^{-6}$	$-0.02\cdot 10^{-6}$
		10	$-0.02\cdot 10^{-9}$	$+0.01\cdot 10^{-6}$	$-3\cdot 10^{-9}$	$+3\cdot 10^{-9}$	$+0.4\cdot 10^{-6}$	$-0.2\cdot 10^{-6}$
		30	$-0.2\cdot 10^{-9}$	$+0.1\cdot 10^{-6}$	$-0.03\cdot 10^{-6}$	$+0.03-10^{-6}$	$+4\cdot 10^{-6}$	$-2\cdot 10^{-6}$
		100	$-2\cdot 10^{-9}$	–	–	$+0.3-10^{-6}$	–	–
		300	$-0.02\cdot 10^{-6}$	–	–	$+3\cdot 10^{-6}$	–	–
5th	$\frac{\tilde{\Gamma}}{12R}\frac{\partial e}{\partial Z}S^2+\frac{\tilde{\Gamma}^2-g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2 S^2$	1	$+0.9\cdot 10^{-12}$	$+0.07\cdot 10^{-9}$	$-0.04\cdot 10^{-9}$	$-0.1\cdot 10^{-9}$	$+3\cdot 10^{-9}$	$-1\cdot 10^{-9}$
		3	$+9\cdot 10^{-12}$	$+0.7\cdot 10^{-9}$	$-0.4\cdot 10^{-9}$	$-1\cdot 10^{-9}$	$+0.03\cdot 10^{-6}$	$-0.01\cdot 10^{-6}$
		10	$+0.09\cdot 10^{-9}$	$+7\cdot 10^{-9}$	$-4\cdot 10^{-9}$	$-0.01\cdot 10^{-6}$	$+0.3\cdot 10^{-6}$	$-0.1\cdot 10^{-6}$
		30	$+0.9\cdot 10^{-9}$	$+0.07\cdot 10^{-6}$	$-0.04\cdot 10^{-6}$	$-0.1\cdot 10^{-6}$	$+3\cdot 10^{-6}$	$-1\cdot 10^{-6}$
		100	$+9\cdot 10^{-9}$	–	–	$-1\cdot 10^{-6}$	–	–
		300	$+0.09\cdot 10^{-6}$	–	–	$-0.01\cdot 10^{-3}$	–	–

Table 4. Meteorological relative corrections on two optical wavelengths

No.	term of equation (21a)	S in km	typical	high	low
1st	$-(\tilde{\Gamma}_1 - D\Delta_L\tilde{\Gamma})e_L$	-	$+1 \cdot 10^{-6}$	$+10 \cdot 10^{-6}$	0
2nd	$\frac{\tilde{G}_1^2 - g_1^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)_1^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)_L^2 S^2 - D\Delta_L\left\{\frac{\tilde{G}^2 - g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2$	1	$-0.06 \cdot 10^{-9}$	0	$-6 \cdot 10^{-9}$
		3	$-0.6 \cdot 10^{-9}$	0	$-0.06 \cdot 10^{-6}$
		10	$-6 \cdot 10^{-9}$	0	$-0.6 \cdot 10^{-6}$
		30	$-0.06 \cdot 10^{-6}$	0	$-6 \cdot 10^{-6}$
		100	$-0.6 \cdot 10^{-6}$	0	–
		300	$-6 \cdot 10^{-6}$	0	–
3rd	$\frac{\tilde{G}_1\tilde{\Gamma}_1 - g_1^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)_1\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)_1}{12}\left(\frac{\partial\varrho}{\partial Z}\right)_L\left(\frac{\partial e}{\partial Z}\right)_L S^2 - D\Delta_L\left\{\frac{\tilde{G}\tilde{\Gamma} - g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2$	1	$-0.3 \cdot 10^{-12}$	$+0.1 \cdot 10^{-9}$	$-0.04 \cdot 10^{-9}$
		3	$-3 \cdot 10^{-12}$	$+1 \cdot 10^{-9}$	$-0.4 \cdot 10^{-9}$
		10	$-0.03 \cdot 10^{-9}$	$+0.01 \cdot 10^{-6}$	$-4 \cdot 10^{-9}$
		30	$-0.3 \cdot 10^{-9}$	$+0.1 \cdot 10^{-6}$	$-0.04 \cdot 10^{-6}$
		100	$-3 \cdot 10^{-9}$	–	–
		300	$-0.03 \cdot 10^{-6}$	–	–
4th	$\frac{\tilde{\Gamma}_1 - D\Delta_L\tilde{\Gamma}}{12R}\left(\frac{\partial e}{\partial Z}\right)_L S^2 + \frac{\tilde{\Gamma}_1^2 - g_1^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)_1^2}{24}\left(\frac{\partial e}{\partial Z}\right)_L^2 S^2 - D\Delta_L\left\{\frac{\tilde{\Gamma}^2 - g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2$	1	$+2 \cdot 10^{-12}$	$+0.2 \cdot 10^{-9}$	$-0.1 \cdot 10^{-9}$
		3	$+0.02 \cdot 10^{-9}$	$+2 \cdot 10^{-9}$	$-1 \cdot 10^{-9}$
		10	$+0.2 \cdot 10^{-9}$	$+0.02 \cdot 10^{-6}$	$-0.01 \cdot 10^{-6}$
		30	$+2 \cdot 10^{-9}$	$+0.2 \cdot 10^{-6}$	$-0.1 \cdot 10^{-6}$
		100	$+0.02 \cdot 10^{-6}$	–	–
		300	$+0.2 \cdot 10^{-6}$	–	–

Table 5. Meteorological relative corrections at two optical- and one micro wavelength

No.	term of equation (27a)	S in km	typical	high	low
1st	$\frac{\tilde{G}_1^2 - g_1^2\left(\frac{dG}{dg}\right)_1^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)_L^2 S^2 - K_L\Delta_L\left\{\frac{\tilde{G}^2 - g^2\left(\frac{dG}{dg}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2 - K_M\Delta_M\left\{\frac{\tilde{G}^2 - g^2\left(\frac{dG}{dg}\right)^2}{24}\left(\frac{\partial\varrho}{\partial Z}\right)^2\right\}S^2$	1 3 10 30 100 300	$-0.06\cdot10^{-9}$ $-0.6\cdot10^{-9}$ $-6\cdot10^{-9}$ $-0.06\cdot10^{-6}$ $-0.6\cdot10^{-6}$ $-6\cdot10^{-6}$	0 0 0 0 0 0	$-6\cdot10^{-9}$ $-0.06\cdot10^{-6}$ $-0.6\cdot10^{-6}$ $-6\cdot10^{-6}$ – –
2nd	$\frac{\tilde{G}_1\tilde{\Gamma}_1 - g_1^2\left(\frac{dG}{dg}\right)_1\left(\frac{d\Gamma}{dg}\right)_1}{12}\left(\frac{\partial\varrho}{\partial Z}\right)_L\left(\frac{\partial e}{\partial Z}\right)_L S^2 - K_L\Delta_L\left\{\frac{\tilde{G}\tilde{\Gamma} - g^2\frac{dG}{dg}\frac{d\Gamma}{dg}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2 - K_M\Delta_M\left\{\frac{\tilde{G}\tilde{\Gamma} - g^2\frac{dG}{dg}\frac{d\Gamma}{dg}}{12}\frac{\partial\varrho}{\partial Z}\frac{\partial e}{\partial Z}\right\}S^2$	1 3 10 30 100 300	$+0.3\cdot10^{-12}$ $+3\cdot10^{-12}$ $+0.03\cdot10^{-9}$ $+0.3\cdot10^{-9}$ $+3\cdot10^{-9}$ $+0.03\cdot10^{-6}$	$+0.2\cdot10^{-9}$ $+2\cdot10^{-9}$ $+0.02\cdot10^{-6}$ $+0.2\cdot10^{-6}$ – –	$-3\cdot10^{-12}$ $-0.03\cdot10^{-9}$ $-0.3\cdot10^{-9}$ $-3\cdot10^{-9}$ – –
3rd	$\frac{\tilde{\Gamma}_1^2 - g_1^2\left(\frac{d\Gamma}{dg}\right)_1^2}{24}\left(\frac{\partial e}{\partial Z}\right)_L^2 S^2 - K_L\Delta_L\left\{\frac{\tilde{\Gamma}^2 - g^2\left(\frac{d\Gamma}{dg}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2 - K_M\Delta_M\left\{\frac{\tilde{\Gamma}^2 - g^2\left(\frac{d\Gamma}{dg}\right)^2}{24}\left(\frac{\partial e}{\partial Z}\right)^2\right\}S^2$	1 3 10 30 100 300	$+0.08\cdot10^{-12}$ $+0.8\cdot10^{-12}$ $+8\cdot10^{-12}$ $+0.08\cdot10^{-9}$ $+0.8\cdot10^{-9}$ $+8\cdot10^{-9}$	$+0.1\cdot10^{-9}$ $+1\cdot10^{-9}$ $+0.01\cdot10^{-6}$ $+0.1\cdot10^{-6}$ – –	0 0 0 0 0 0

Table 6. Assumed values for the inaccuracies of meteorological measurements and estimations (to be interpreted as standard variations)

quantity or its variation		favourable	typical	unfavourable	source
temperature	δt	0.5 °C	1 °C	2 °C	[17] and own estimates
total pressure	δp	0.7 torr	0.7 torr	1.5 torr	[17] and own estimates
relative variation of dry air density	$\delta\varrho/\varrho$	$1.7 \cdot 10^{-3}$	$3.3 \cdot 10^{-3}$	$6.7 \cdot 10^{-3}$	calculated from δt and δp above with appendix I
maximum vapour pressure at air temperature	p_{3max}	6.5 torr	13 torr	72 torr	for 5 °C, 15 °C and 45 °C
vapour pressure	δp_3	$0.02 p_{3max}$	$0.05 p_{3max}$	$0.1 p_{3max}$	[5] [17]
humidity	δe_L	0.0018	0.009	0.1	calculated from δp_3 with appendix I
	δe_M	0.0025	0.011	0.1	
dry air refractivity gradient	$\partial\varrho/\partial Z$	$-0.06 \cdot 10^{-3}\ m^{-1}$	$-0.11 \cdot 10^{-3}\ m^{-1}$	$-1.3 \cdot 10^{-3}\ m^{-1}$	from table 1: 50% of the typical value, 100% of this value, or the most unfavourable value
	$\frac{\delta(\partial\varrho/\partial Z)}{\partial\varrho/\partial Z}$	0.1	0.1	0.2	own estimate
humidity gradient	$\frac{\delta(\partial e/\partial Z)}{\partial e/\partial Z}$	1	1	1	own estimate
temperature for the 3-λ-method	$\delta t_\varrho, \delta t_e$	0.5 °C	1 °C	2 °C	$\delta t_\varrho = \delta t_e = \delta t$ (see above)
CO_2 partial pressure	δp_{CO_2}	0.04 torr	0.04 torr	0.08 torr	[14]

Table 7. Inaccuracies of the refraction formulae for air and of the light velocity in vacuum

formulae	deviation	favourable- or typical conditions		unfavourable conditions	
		value	literature	value	literature
EDLEN *) [11, equation (1), (15) and (22)]	$\delta\tilde{n}_1$	$0.05 \cdot 10^{-6}$	[11, p. 73, first column]	$0.5 \cdot 10^{-6}$	[10, table V]
	$\delta\Delta_L\tilde{n}$	$0.005 \cdot 10^{-6}$	[11, p. 76, under (1), + p. 79, table 6, ×3 for group-effect]	$0.08 \cdot 10^{-6}$	[10, section V, p. 58] (0.25% of $\Delta_L\tilde{n}$)
OWENS	$\delta\tilde{n}_1$	$0.05 \cdot 10^{-6}$	[10, section I, p. 51], [11]	$0.05 \cdot 10^{-6}$	[10, section I, p. 51], [11]
[10, table III]	$\delta\Delta_L\tilde{n}$	$0.005 \cdot 10^{-6}$	[10, section I, p. 51], [11]	$0.005 \cdot 10^{-6}$	[10, section I, p. 51], [11]
ESSEN and FROOME	$\delta\tilde{n}_M$	$0.1 \cdot 10^{-6}$	[12, p. 4]	$0.8 \cdot 10^{-6}$	[12, p. 4]
light velocity in vacuum	$\delta c/c$	$0.3 \cdot 10^{-6}$	[18]	$0.3 \cdot 10^{-6}$	[18]

*) The difference between the formulae (12) and (15) of [11] appear to have no significant influence on the values of this table.

Table 8. Errors originating from different sources in electromagnetic distance measurement, expressed in ppm.

source of error	row No.	distance in km	$\lambda_1 = 0.625\ \mu m$ one optical wavelength fav.	typ.	unfav.	one micro wavelength fav.	typ.	unfav.	$\lambda_1 = 0.625\ \mu m$, $\lambda_2 = 0.3636\ \mu m$ two optical wavelengths fav.	typ.	unfav.	$\lambda_1 = 0.625\ \mu m$, $\lambda_2 = 0.3636\ \mu m$ two optical- and one micro wavelength fav.	typ.	unfav.
1st order dry air (ϱ)	1	–	0.5	1	2	0.5	1	2	–	–	–	–	–	–
1st order humidity (e)	2	–	0.007	0.04	0.4	0.9	4	30	0.02	0.09	1	–	–	–
2nd order dry air ($\partial\varrho/\partial Z$)	3	3	<0.001	<0.001	0.04	<0.001	<0.001	0.04	<0.001	<0.001	0.02	<0.001	<0.001	0.02
		10	0.003	0.006	0.4	0.003	0.006	0.4	<0.001	0.001	0.2	<0.001	0.001	0.2
		30	0.03	0.06	4	0.03	0.06	4	0.003	0.01	2	0.003	0.01	2
		100	0.3	0.6	8	0.3	0.6	8	0.03	0.1	5	0.03	0.1	5
2nd order humidity ($\partial e/\partial Z$) and eventually ($\partial\varrho/\partial Z$)	4	3	<0.001	<0.001	0.002	<0.001	0.001	0.07	<0.001	<0.001	0.003	<0.001	<0.001	0.003
		10	<0.001	<0.001	0.02	0.003	0.015	0.7	<0.001	<0.001	0.03	<0.001	<0.001	0.03
		30	<0.001	0.001	0.2	0.03	0.15	7	0.001	0.002	0.3	<0.001	<0.001	0.3
		100	0.002	0.007	2	0.3	1.5	15	0.005	0.02	0.6	0.001	0.004	0.6
θ_e Edlen	5	–	–	–	–	–	–	–	–	–	–	0.004	0.007	0.15
θ_e Owens	6	–	–	–	–	–	–	–	–	–	–	0.002	0.004	0.07
Edlen's eq.	7	–	0.05[1]	0.05[1]	0.5	–	–	–	0.1[1]	0.1[1]	1.3	0.1[1]	0.1[1]	1.3
Owens' eq.	8	–	0.05[1]	0.05[1]	0.05	–	–	–	0.1[1]	0.1[1]	0.1	0.1[1]	0.1[1]	0.1
Essen and Fr.'s eq.	9	–	–	–	–	0.1	0.1	0.8	–	–	–	0.002	0.002	0.02
CO_2	10	–	0.004	0.004	0.02	0.005	0.005	0.02	0.007	0.007	0.03	0.007	0.007	0.03
vacuum velocity	11	–	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]	0.3[2]

[1] Possibly about 50% of this error is constant for all the measurements of the same type, provided the difference in level is not too large.
[2] A constant deviation in the scale for E.D.M.

Appendix I

SOME VALUES USED IN THIS PAPER

1 General

For the refraction index of atmospheric air two sets of formulae are used. For visual light the results of EDLÈN [11, equations (12), (1) and (22)] are written in the form (I.a) and (I.c). For radio waves the formula of ESSEN and FROOME [12, equation (1)] is used in the form (I.b) and (I.d):

$$n_L = 1 + G_L \varrho_L + \Gamma_L e_L \qquad \text{(I.a)}$$

$$n_M = 1 + G_M \varrho_M + \Gamma_M e_M \qquad \text{(I.b)}$$

$$\left.\begin{aligned} G_L &= 1.2168\,\{83.42 + 24060.30(130 - g^2)^{-1} + \\ &\quad + 159.97(38.7 - g^2)^{-1}\}\cdot 10^{-6} \\ \varrho_L &= \frac{0.0013874p}{1.2168}\cdot\frac{1 + p(0.817 - 0.0133t)\cdot 10^{-6}}{1 + 0.0036610t} \\ \Gamma_L &= -\frac{72}{100}(5.722 - 0.0457g^2)\cdot 10^{-6} \\ e_L &= p_3/72 \end{aligned}\right\} \qquad \text{(I.c)}$$

$$\left.\begin{aligned} G_M &= 332\cdot 10^{-6} \\ \varrho_M &= \frac{1.1419\cdot 10^{-3}p}{1 + 0.003663t} \\ \Gamma_M &= 348\cdot 10^{-6} \\ e_M &= \left\{\frac{19.117}{(1 + 0.003663t)^2} - \frac{0.1814}{1 + 0.003663t}\right\}\frac{p_3}{1000} \end{aligned}\right\} \qquad \text{(I.d)}$$

where:
- n_L and n_M are the (phase) refraction indices for visual light and for radio waves,
- g is the wave number (= 1/wavelength) in $(\mu m)^{-1}$
- p is the total pressure in torr,
- p_3 is the partial pressure of the water vapour in torr, and
- t is the temperature in centigrades.

The arbitrary factors "1.2168" and "72" in (I.c), and the numerical values of G_M and Γ_M in (I.d) are so chosen that the quantities ϱ and e will hardly become larger than unity.

2 The factors independent of the meteorological conditions (G, Γ, etc.)

G_L and Γ_L have been calculated for $g_1 = 2.75\,(\mu m)^{-1}$ and for $g_2 = 1.6\,(\mu m)^{-1}$ (near the limits of validity of the EDLÈN expressions). G_M and Γ_M are directly given by (I.d). See table 1 of this paper.

$\tilde{G}_L$, $\tilde{\Gamma}_L$ and the other functions of g are also calculated from (I.c). The analogue functions for microwaves are easily found because there is no dispersion in this case.

These values are also assembled in table 1 of this paper.

3 The factors depending only on local meteorological conditions (ϱ_L, ϱ_M, e_L, e_M)

Assuming the meteorological conditions *) of table I.1 typical, very high and very low values of the quantities ϱ and e have been calculated for light and for microwaves. The results are assembled in table 1 of this paper.

Table I.1. Assumptions for meteorological conditions

for the quantity:	typical			high			low		
	t	p	rel. hum.	t	p	rel. hum.	t	p	rel. hum.
ϱ_L, ϱ_M	+14 °C	760 torr	–	−30 °C	780 torr	–	10 °C	450 torr	–
e_L, e_M	+14 °C	760 torr	60%	+45 °C	–	100%	–	–	0%
	typical			high or low					
$\partial/\partial Z$ (dry)	+14 °C	760 torr	–	+14 °C	760 torr	–			
$\partial/\partial Z$ (hum. or mixed)	+14 °C	760 torr	60%	+30 °C	760 torr	100%			

4 The factors depending on the vertical variations of the local meteorological conditions (functions of $\partial\varrho/\partial Z$, $\partial e_L/\partial Z$ and $\partial e_M/\partial Z$)

The vertical derivates of ϱ and e are found by differentiating the third and fourth expressions of (I.c) and (I.d). With good approximation the following expressions are found:

$$\frac{\partial\varrho}{\partial Z} = \frac{\partial\varrho_L}{\partial Z} = \frac{\partial\varrho_M}{\partial Z} = \frac{1.14\cdot 10^{-3}}{1+0.0037t}\frac{\partial p}{\partial Z} - \frac{4.19\cdot 10^{-6}p}{(1+0.0037t)^2}\frac{\partial t}{\partial Z}$$

$$\frac{\partial e_L}{\partial Z} = 0.0139\frac{\partial p_3}{\partial Z} \qquad \frac{\partial e_M}{\partial Z} = \frac{0.0189}{(1+0.0037t)^2}\frac{\partial p_3}{\partial Z} - \frac{0.14\cdot 10^{-3}p_3}{(1+0.0037t)^3}\frac{\partial t}{\partial Z} \tag{I.e}$$

*) The assumptions are made for terrestrial measurements. For much greater heights (aeroplanes, satellites) the values of ϱ and e tend to become zero, provided ionospheric effects are unimportant.

In order to obtain numerical values for $\partial\varrho/\partial Z$, $\partial e_L/\partial Z$ and $\partial e_M/\partial Z$ use is made of the results of HÖPCKE [19] who gives a great number of values for the refraction coefficients k_L and k_M which have been calculated from meteorological measurements of $\partial t/\partial Z$ and $\partial p_3/\partial Z$. HÖPCKE used the formulae (I.f):

$$\left.\begin{aligned} k_L &= 6.38\frac{\partial t}{\partial Z} - 2.48\frac{\partial p}{\partial Z} + 0.32\frac{\partial p_3}{\partial Z} \\ k_M &= 8.61\frac{\partial t}{\partial Z} - 2.30\frac{\partial p}{\partial Z} - 37.8\frac{\partial p_3}{\partial Z} \end{aligned}\right\} \quad \text{(I.f)}$$

assuming:

$$\frac{\partial p}{\partial Z} = -0.089 \text{ torr/m} \quad \text{(I.g)}$$

With (I.f) and (I.g) the basic meteorological values of HÖPCKE are here expressed in k_L and k_M:

$$\frac{\partial t}{\partial Z} = 0.155k_L + 0.001k_M - 0.034 \quad \text{(I.h)}$$

and:

$$\frac{\partial p_3}{\partial Z} = 0.036k_L - 0.026k_M - 0.002 \quad \text{(I.i)}$$

Numerical values of the terms of the equations (19a), (19b), (21a), (21b), (27a) and (27b) are found by using the equations (I.e), (I.g), (I.h) and (I.i) with the meteorological conditions of table I.1. So one finds:

for all values of the dry air terms and for the typical values of the humid- and mixed terms
$\frac{\partial\varrho}{\partial Z} = -0.45\cdot10^{-3}k_L$

for high- or low values of the humid- and mixed terms
$\frac{\partial\varrho}{\partial Z} = -0.39\cdot10^{-3}k_L$
$\frac{\partial e_L}{\partial Z} = 0.49\cdot10^{-3}k_L - 0.36\cdot10^{-3}k_M - 0.03\cdot10^{-3}$ $\frac{\partial e_M}{\partial Z} = -0.39\cdot10^{-3}k_M + 0.08\cdot10^{-3}$

(I.j)

Values for k_L and k_M were estimated from the graphs of HÖPCKE [15]. For the typical values was chosen:

$$k_L = +0.25 \quad \text{and} \quad k_M = +0.30 \qquad \text{(I.k)}$$

For high and low values, combinations of k_L and k_M were chosen for each individual term. A few examples are indicated in table I.2.

Table I.2

k_L	k_M	some combinations of k_L and k_M from the graphs of HÖPCKE [19], giving high- or low values of different terms of (19a), (19b), (21a), (21b), (27a) and (27b)
−0.5	+3	
+1	−1	
+2.5	+1	
+1	+3	

5 The relations between the meteorological factors for radio waves and for light waves (θ_ϱ and θ_e)

In chapter 5 of this paper a comparison is made between measurements on optical- and radio wavelengths. In this case the ratio's $\theta_\varrho = \varrho_M/\varrho_L$ and $\theta_e = e_M/e_L$ are wanted in some approximation. These values are easily found from (I.c) and (I.d):

$\theta_\varrho = 1.0015-(2t+0.817p-0.013tp)\cdot 10^{-6}$	$\theta_e = 1.3634-0.010037t+55\cdot 10^{-6}t^2$	(I.1)
$\theta_\varrho \approx 1$	$\theta_e \approx 1.36-0.01\,t$	

Appendix II

THE LORENTZ-LORENZ EQUATION FOR THE REFRACTION INDEX

H. A. LORENTZ and L. LORENZ have derived a formula for the refraction index of a medium [9, chapter II]. For one component this equation may be written as:

$$\frac{n^2-1}{n^2+2} = \tfrac{2}{3}\bar{G}\bar{\varrho}, \qquad \text{(II.a)}$$

where $\bar{G}$ is only dependent on the (vacuum) wavelength and $\bar{\varrho}$ on the density, i.e. on temperature and pressure. If $n-1 \ll 1$ this expression does not differ much from an expression such as (9) for dry air:

$$n-1 = G\varrho \qquad \text{(II.b)}$$

Neglecting higher powers of $G\varrho$ one finds for dry air from (II.a) and (II.b) and with (16):

$$\left.\begin{array}{ll} G\varrho = \bar{G}\bar{\varrho}+\tfrac{1}{6}(\bar{G}\bar{\varrho})^2 & \tilde{G}\varrho = \tilde{\bar{G}}\bar{\varrho}+\tfrac{1}{6}\left\{\tilde{\bar{G}}^2\bar{\varrho}^2-g^2\left(\dfrac{\mathrm{d}\bar{G}}{\mathrm{d}g}\right)^2\bar{\varrho}^2\right\} \\ \text{with:} & \tilde{\bar{G}} = \bar{G}+g\dfrac{\mathrm{d}\bar{G}}{\mathrm{d}g} \end{array}\right\} \qquad \text{(II.c)}$$

In this paper the formula for the refraction index of dry air was supposed to be of the form (II.b), which form is often used, for example by EDLÈN [11]. If the refraction index is calculated with the probably more accurate Lorentz-Lorenz expression (II.a), (OWENS [10]), one should apply some correction to the equations to calculate the distance S. Since the influence of the humidity is normally only small and difficult to introduce into the calculations, only the dry air correction is given here.

The corrections are found by substituting $G\varrho$ and $\tilde{G}\varrho$ with (II.c) into (19a), (21a) and (27a). Also in the quantities D, K_L and K_M this substitution should be made.

So one finds forms analogue to (19a), (21a) and (27a), but with $\bar{G}\bar{\varrho}$ and $\tilde{\bar{G}}\bar{\varrho}$ instead of $G\varrho$ and $\tilde{G}\varrho$ (also in $\Delta_L G$, D, K_L and K_M), and with an additional correction term δ_{LL} to the right-hand member of (19a), (21a) and (27a):

$$\delta_{LL} = \tfrac{1}{6}\left\{\tilde{\bar{G}}^2-g^2\left(\frac{\mathrm{d}\bar{G}}{\mathrm{d}g}\right)^2\right\}\bar{\varrho}^2 \qquad \text{(II.d)}$$

Appendix III

THE ERROR INTRODUCED IF AN ELECTROMAGNETIC DISTANCE MEASUREMENT IS CALCULATED WITH THE GROUP REFRACTION INDEX

The usual way of deriving a distance S from the measured optical path σ_m is principally different from the theory in this paper: normally one substitutes the group refraction index $\tilde{n}$ (defined by (7)) in the monochromatic solution of the Maxwell equations instead of substituting the phase refraction index n, and introducing the group effect later on. This usual method however is not more than an approximation because the Fermat principle (8) does not hold generally for the group refraction index. In order to demonstrate the (very small) error, the "group path" σ_g will be calculated. This quantity is defined by the substitution of $\tilde{n}$ instead of n in the monochromatic solution for σ.

In the used approximation one finds this σ_g by substituting $\tilde{n}$, $\partial\tilde{n}/\partial Y$ and $\partial\tilde{n}/\partial Z$ in (12a) for the corresponding n-values. From (7), (9) and (16) one finds:

$$\tilde{n} = 1+\tilde{G}\varrho+\tilde{\Gamma}e \qquad \frac{\partial\tilde{n}}{\partial Y} = \tilde{G}\frac{\partial\varrho}{\partial Y}+\tilde{\Gamma}\frac{\partial e}{\partial Y} \qquad \frac{\partial\tilde{n}}{\partial Z} = \tilde{G}\frac{\partial\varrho}{\partial Z}+\tilde{\Gamma}\frac{\partial e}{\partial Z} \quad \ldots \quad \text{(III.a)}$$

Substitution in (12a) and (13) gives:

$$\sigma_g = S+\tilde{G}I_\varrho+\tilde{\Gamma}I_e-\tfrac{1}{2}\tilde{G}^2I_{\varrho\varrho}-\tilde{G}\tilde{\Gamma}I_{\varrho e}-\tfrac{1}{2}\tilde{\Gamma}^2I_{ee}$$

Comparing this form with (14) and (15) one finds for the error of the usual method of calculation:

$$\boxed{\sigma_g-\sigma_m = -\tfrac{1}{2}g^2\left(\frac{\mathrm{d}G}{\mathrm{d}g}\right)^2I_{\varrho\varrho}-g^2\frac{\mathrm{d}G}{\mathrm{d}g}\frac{\mathrm{d}\Gamma}{\mathrm{d}g}I_{\varrho e}-\tfrac{1}{2}g^2\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}g}\right)^2I_{ee}} \quad \ldots\ldots \quad \text{(III.b)}$$

With the values of table 1 this error is found to be smaller than 0.03 Ω. Since the Ω-terms can only very roughly be measured or estimated the error $(\sigma_g-\sigma_m)$ can be neglected in all practical cases.

In the formulae (19b), (21b) and (27b) of this paper $(\sigma_g-\sigma_m)$ is neglected. The forms become so much simpler, but the influence of the humidity does not appear explicitly.

Appendix IV

THE INFLUENCE OF ERRORS IN ESTIMATING THE FICTITIOUS TEMPERATURE t_e

(See chapter 5 under equation (23b) and section 6.2, "row 5 and 6")

To find the influence of t_e the equations (25) and (26) are partially differentiated, neglecting the higher order terms. After some calculations one finds:

$$\frac{\partial S}{\partial t_e} = -\left\{\Delta_L\sigma_m\cdot\frac{\partial K_L}{\partial \theta_e} + \Delta_M\sigma_m\cdot\frac{\partial K_M}{\partial \theta_e}\right\}\frac{d\theta_e}{dt_e}$$

$$\frac{\partial K_L}{\partial \theta_e} = \frac{K_M\tilde{\Gamma}_M\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)}{\Delta_L\tilde{G}\cdot(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)-\Delta_L\tilde{\Gamma}\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)}$$

$$\frac{\partial K_M}{\partial \theta_e} = \frac{-K_M\tilde{\Gamma}_M\cdot(\Delta_L\tilde{G})}{\Delta_L\tilde{G}\cdot(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)-\Delta_L\tilde{\Gamma}\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)}$$

Using the model (17) of chapter 3 one can substitute (18) in the second and third equation of (24). So one finds with neglection of the higher order terms:

$$\frac{\Delta_L\sigma_m}{S} = \varrho_L\cdot(\Delta_L\tilde{G})+e_L\cdot(\Delta_L\tilde{\Gamma}) \qquad \frac{\Delta_M\sigma_m}{S} = \varrho_L\cdot(\theta_\varrho\tilde{G}_M-\tilde{G}_1)+e_L\cdot(\theta_e\tilde{\Gamma}_M-\tilde{\Gamma}_1)$$

From the above expressions one easily finds:

$$\frac{\partial S}{S\partial t_e} = K_M\tilde{\Gamma}_M e_L\frac{d\theta_e}{dt_e}$$

In the same way:

$$\frac{\partial S}{S\partial t_\varrho} = K_M\tilde{G}_M\varrho_L\frac{d\theta_\varrho}{dt_\varrho}$$

Using the approximation (23b) for θ_ϱ and for θ_e, the influence of t_ϱ should be neglected, and for t_e one finds with the values of table 1:

	typical conditions	high humidity
$\frac{\partial S}{S\partial t_e} = -0.01K_M\tilde{\Gamma}_M$	$+7\cdot10^{-9}$ (°C)$^{-1}$	$+70\cdot10^{-9}$ (°C)$^{-1}$

With the inaccuracies for t_e stated in table 6 the values of row 5 in table 8 are found.

The above mentioned results are derived from formula (23b) of this paper which is based on the formulae of EDLÈN [11]. In these formulae the humidity correction is independent of the temperature. However elaborating the Owens' formulae [10] numerically, one finds there a humidity correction essentially inversely proportional to the absolute temperature. For a wavelength of 0.625 μm one finds for e_L from OWENS:

$$e_L \approx p_3/67\,(1+0.0037t)$$

So one finds instead of the second equation (23b):

$$\theta_e = 1.28-0.0047t$$

The influence of t_e for the Owens' formulae appears to be about half the values given in row 5 of table 8.

Appendix V

THE INFLUENCE OF ERRORS IN THE GROUP REFRACTION INDEX ($\delta\tilde{n}_L$, $\delta\tilde{n}_M$) AND IN THE DISPERSION ($\delta\Delta_L\tilde{n}$)

1 For one wavelength

For this case (19b) will be applied neglecting the higher order terms:

$$S = \sigma_m + S(1-\tilde{n})$$

The differential form gives the error δS caused by an error $\delta\tilde{n}$ in the group refraction index:

$$\delta S = (1-\tilde{n})\delta S - S\delta\tilde{n}$$

or: $$\frac{\delta S}{S} = -\frac{\delta\tilde{n}}{\tilde{n}}$$

or: $$\boxed{\left|\frac{\delta S}{S}\right| \approx |\delta\tilde{n}|} \qquad \text{(V.a)}$$

2 For two optical wavelengths

Neglecting the higher order terms in (21b) one finds:

$$S = \sigma_{m1} - D\Delta_L\sigma_m - (\tilde{n}_1 - 1)S + D\Delta_L\tilde{n}S$$

The differential form gives the error δS caused by the errors $\delta\tilde{n}_1$ and $\delta\Delta_L\tilde{n}$:

$$\delta S = -\Delta_L\sigma_m\delta D - (\tilde{n}_1 - 1)\delta S + D\Delta_L\tilde{n}\delta S - S\delta\tilde{n}_1 + DS\Delta_L\tilde{n} + \Delta_L\tilde{n}S\delta D,$$

where δD is the variation in D caused by variations in $\tilde{n}_1$, and in $\Delta_L\tilde{n}$, from which follows that the coefficient of δD becomes approximately zero.

It may easily be seen that:

$$\Delta_L\sigma_m \approx S\Delta_L\tilde{n} \qquad \text{(following from (14), (III.a) and (18)}$$

So one finds:

$$\frac{\delta S}{S} = -\frac{\delta\tilde{n} - D\delta\Delta_L\tilde{n}}{\tilde{n} - D\Delta_L\tilde{n}}$$

or with good approximation:

$$\boxed{\left|\frac{\delta S}{S}\right| \leqslant |\delta \tilde{n}| + |D\delta \Delta_L \tilde{n}|} \qquad \text{(V.b)}$$

3 For two optical- and one radio wavelength

Neglecting the higher order terms in (25) and multiplying its two members with:

$$\varrho_L e_L N \equiv \varrho_L e_L \cdot \{\Delta_L \tilde{G} \cdot (\theta_e \tilde{\Gamma}_M - \tilde{\Gamma}_1) - \Delta_L \tilde{\Gamma} \cdot (\theta_\varrho \tilde{G}_M - \tilde{G}_1)\}$$

one finds:

$$e_L \varrho_L N S = e_L \varrho_L N \sigma_{m1} - (\tilde{G}_1 \varrho_L e_M \tilde{\Gamma}_M - \tilde{\Gamma}_1 e_L \varrho_M \tilde{G}_M) \Delta_L \sigma_m - (\tilde{\Gamma}_1 e_L \varrho_L \Delta_L \tilde{G} - \tilde{G}_1 \varrho_L e_L \Delta_L \tilde{\Gamma}) \Delta_M \sigma_m$$

In the differential form of this equation for constant σ_{m1}, $\Delta_L \sigma_m$ and $\Delta_M \sigma_m$ one substitutes:

$$\begin{aligned} \sigma_{m1} &= S \cdot (1 + \tilde{G}_1 \varrho_L + \tilde{\Gamma}_1 e_L) \\ \Delta_L \sigma_m &= S \cdot (\Delta_L \tilde{G} \cdot \varrho_L + \Delta_L \tilde{\Gamma} \cdot e_L) \\ \Delta_M \sigma_m &= S \cdot (\tilde{G}_M \varrho_M - \tilde{G}_1 \varrho_L + \tilde{\Gamma}_M e_L - \tilde{\Gamma}_1 e_L) \end{aligned}$$

(following from (14) and (18), neglecting the higher order terms)

So one get after some elaborations:

$$\frac{\delta S}{S} = -(1 + K_M)\delta \tilde{n}_1 + K_L \delta \Delta_L \tilde{n} + K_M \delta \tilde{n}_M$$

or:

$$\boxed{\left|\frac{\delta S}{S}\right| \leqslant |(1 + K_M)\delta \tilde{n}_1| + |K_L \delta \Delta_L \tilde{n}| + |K_M \delta \tilde{n}_M|} \qquad \text{(V.c)}$$