PUBLICATION OF THE NETHERLANDS GEODETIC COMMISSION
(UITGEGEVEN DOOR DE RIJKSCOMMISSIE VOOR GEODESIE)

THEORY AND PRACTICE

OF

PENDULUM OBSERVATIONS AT SEA

PART II

SECOND ORDER CORRECTIONS, TERMS OF BROWNE AND MISCELLANEOUS SUBJECTS

BY

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PREFACE.

In this preface the writer wishes to pay a sincere tribute to Mr. B. C. BROWNE, who discovered several effects of the second order of the ship's movements in the pendulum observations at sea, for which the results have to be corrected. He wants especially to thank him for his kind and helpful cooperation for the further investigation of this subject of which this publication gives a report. Mr. BROWNE has given a contribution of fundamental importance to the problem of the determination of gravity at sea and the writer feels deeply indebted to him.

The writer wishes further to express his thanks to the Netherlands Navy, to the Authorities as well as to the Commanders, the Officers and Men of the submarines on board of which the research has been made, for their great cooperation in making the scientific observations possible and for their assistance during the investigations.

CONTENTS.

INTRODUCTION.

This sequel to the first publication in 1929 about "Theory and Practice of Pendulum Observations at Sea" deals mainly with a research about the second order corrections for the ship's movements, i.e. the corrections proportional to the square of the accelerations of the apparatus. In the course of 1937 Mr. B. C. BROWNE, Cambridge, published an important study on this subject in the Geophysical Supplement of the Monthly Notices of the R.A.S., Sept. 1937, in which he investigated their effect in case of the general problem of gravity determinations at sea and also in case pendulums are used for these determinations. He pointed out that these effects are not always negligible and that in case of strong wave-movements their effect on the gravity result may attain values of 10 milligal and more even if the observations are made in submerged submarines. This valuable study led to a narrow cooperation with Mr. BROWNE for the further research of these matters, for which the writer feels deeply indebted. In the investigations treated in this publication much is due to this cooperation.

The research led to a method for determining the horizontal accelerations, which, together with the data for the vertical accelerations that can be derived from the records of the main pendulums of the apparatus, provide us with a base for computing the corrections for these terms — the terms of BROWNE — to be applied to the gravity results. It appeared even possible to correct also the old observations at sea for these effects although, as will be mentioned in chapter III, these corrections depend on a somewhat uncertain supposition about the wave and ship's movements. Their mean error is, therefore, questionable and in some cases might considerably increase the mean error of the gravity result; we may, however, trust that future observations will lead to a lower estimate of these mean errors.

The Netherlands Navy has again given its full cooperation to the research by allowing the writer to accompany a submarine, Hr. Ms. O 12, on its return voyage from the West Indies in November and December 1937, in order to make observations about the wave-movements and to try the new apparatus for the measuring of the horizontal accelerations which had been constructed in the work-shop of the Meteorological Institute at De Bilt; this apparatus consisted in principle of a pendulum of very long period. As a fuller account of this trip will be given in a shortly to be published report about all recent gravity expeditions at sea of the Netherlands Geodetic Commission, the writer wishes here only to mention the helpful assistance obtained on board from the Captain, Lieut.-Commander H. C. W. MOORMAN, from his Officers and from a sergeant torpedoist PLAS who had been assigned by the Captain to help him for the instrumental work. Although the apparatus was not unsuccessful and gave important results for the wave-movements, it had to be reconstructed. This was done in such a way that it would fit in the existing pendulum apparatus; it in fact now fills up the space between the pendulum apparatus and the recording apparatus and it is recorded by the latter on the same strip of paper as the main pendulums. The gimbals of the whole apparatus had to be altered. A picture of the new apparatus is given in Plate I and of the whole apparatus in Plate II. In May 1938 the Navy again afforded an opportunity to put this new apparatus to the test by allotting Hr. Ms. Submarine O 13 for a trip towards the Atlantic near the end of the Channel. The writer was then accompanied by Dr. W. NIEUWENKAMP who henceforth will undertake the gravity work at sea for the Netherlands Geodetic Commission. The Captain Lieut-Commander C. B. M. VAN ERKEL and his Officers gave their full assistance. The results this time were fully satisfactory and so the new shape of the apparatus was adopted as final. The writer obtained again valuable material for the wave-movements.

After this trip the Netherlands Geodetic Commission has lent the pendulum apparatus combined with the new apparatus to the Department of Geodesy and Geophysics of the University of Cambridge in behalf of the gravity expedition planned and organized by Sir GERALD LENOX CONYNGHAM and Mr. B. C. BROWNE, who himself had also constructed a long period pendulum apparatus for the determination of the horizontal accelerations; this apparatus is meant for more general use than that constructed in Holland which was made exclusively for the pendulum work at sea. In September 1938 Mr. BROWNE left in a submarine allotted for this expedition by the British Navy, but as bad luck would have it, the ship was recalled after occupying the first two gravity stations because of the mobilization of the British fleet. Towards the end of the year the apparatus was returned to Holland together with an important new asset, the crystal controlled chronometer of the Bell Telephone Laboratories that had kindly been lent by the American Geophysical Union for use in Great Britain and Holland. This time-piece has a remarkably steady rate, far exceeding all other time-keepers in its regularity. EWING, HESS and HOSKINSON used it already for the gravity expedition of the U.S. Submarine BARRACUDA, which was organized in 1936 by Dr. RICHARD M. FIELD, and during this expedition it had kept its regularity of rate notwithstanding the ship's movements.

In Holland a great expedition was in preparation; in the beginning of 1939 Dr. NIEUWENKAMP would accompany a Netherlands Submarine to the East Indies via the Cape of Good Hope. One of the objects was to make new observations about the wave-movements and the second order corrections. The great accuracy of the new time-keeper offered a valuable new opportunity in this regard. Making gravity observations at the same spot at different depths below sea-level, where, therefore, the wave-movements might be expected to differ considerably, a direct check could be obtained whether all the effects of the ship's movements were indeed in the right way taken into account because otherwise the results could not be expected to agree. To make this check really efficient, it was necessary that no other disturbing effects should interfere and so it could not well be made with the ordinary type of precise time-keeper of which the rate is not steady enough for this purpose.

The political tension, however, prevented the expedition to come off. First came a delay and a change of the projected route and afterwards the plans had to be given up. Here again the writer wishes to express his gratitude to the authorities for the way they still tried to make the expedition possible, but the circumstances proved too strong. In July one station could be made during a trial trip of Hr. Ms. O 19 in the North Sea and here Dr. NIEUWENKAMP could at least for once try the above check about the effect of the ship's movements. The circumstances were, however, unfavourable and so the check could not be made with extreme accuracy; as far as it goes it confirms the theoretical deductions.

Besides the fundamental paper by Mr. BROWNE mentioned above, the writer has already in two previous papers [1]) given a short and provisional summary of the problems and the results of the observations concerning the BROWNE terms. This publication is a fuller report about the question as it stands to-day. Future observations may change some of the view-points and he should have liked to await the results of some more observations of the wave-movements before making this report, but it is not likely that such material will be obtained in the near future. So he decided not to wait any longer before publishing the present material.

The writer uses this opportunity to publish in Chapter IV a few other questions concerning pendulum observations at sea. They concern the adjustment of the pendulum apparatus and the determination of the difference of the pendulum-periods and of the correction for sway when using the apparatus on land. About both questions some new view-points have arisen since the publication of "Theory and Practice of Pendulum Observations at Sea" in 1929.

Special attention may be called to the assumptions mentioned at the end of the summary on page 5.

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1) Second order disturbance terms (BROWNE terms) in pendulum observations at sea, Proceedings Acad. of Sc. of Amsterdam, XL.8 (1937) and XLI.6 (1938).

Summary of the contents.

Chapter I contains the theoretical investigations about the effect of the accelerations of the pendulum apparatus on the gravity results. § 1 treats of the effect of the horizontal component perpendicular to the swinging-plane of the main pendulums, § 2 of that of the second horizontal component, § 3 of that of the vertical component and § 4 of the rotational effects. In all cases the effect has been determined up to the terms proportional to the square of the accelerations; there is no doubt that the terms proportional to the third power are negligible. In § 1 we shall also have to deal with the equations of motion of the damped pendulums of the old pendulum apparatus which have not been published in the paper of 1929; we need them for the investigation of the effect of the accelerations on these pendulums. In § 5 we have separately discussed the disturbing effects for observations made in harbours at the surface of the water.

Chapter II deals with the determination of the vertical and horizontal components of the accelerations. After an introduction in § 6, § 7 shows the way the vertical accelerations may be derived from the records of the main pendulums. These accelerations can, therefore, be determined for all the old observations. In § 8 a method is developed for determining the horizontal accelerations; this can be done by means of pendulums of very long period. At the end of this paragraph some general considerations follow about the determination of the total second order correction and § 9 gives a summary of the formulas of the preceding two paragraphs. § 10 treats of the new apparatus constructed for the determination of the horizontal accelerations; it contains two long period pendulums swinging in planes perpendicular to each other, one for each component.

Chapter III deals with the problem of determining the corrections for the old observations. For doing this we have to make suppositions about the way the horizontal and the vertical component of the acceleration are related. By assuming the wave-theory of GERSTNER or STOKES to be valid and by supposing that the apparatus describes circles in a vertical plane in the same way as the water-particles, we find such a relation and this makes it possible to deduce the complete effect of both components from the data about the vertical component alone; these can be derived from the old records. The study of the ship's and wave-movements is dealt with in § 11, and § 12 gives the results for these movements derived from the experiments during the voyages of Hr. Ms. O 12, O 13 and O 19. Besides getting thus a base for the computation of the corrections for the old pendulum results, we likewise obtained data regarding the theories about the wave-movement. § 13 gives the results for the corrections of the old pendulum observations of the stations nos 33—486 published in "Gravity Observations at Sea, 1923—1932", Netherlands Geodetic Commission 1934. These corrections have been derived according to the methods described in this publication.

Chapter IV deals with other problems concerning pendulum observations at sea; § 14 treats of the adjustment of the old pendulum apparatus and § 15 of methods for determining the difference of the pendulum-periods and of the correction for sway.

In this paper we shall refer to the previous publication: "Theory and Practice of Pendulum Observations at Sea" as the "publication of 1929". Contrary to what has been adopted there, and to what is usually done, we have indicated by T in this paper the complete period of the pendulums; i.e. the time of a double swinging instead of that of a single swinging. For the three components $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ of the acceleration we adopted the sign of the inertia forces of D'ALEMBERT working on the masses of the apparatus because of the translational movement of the system of coordinates carried along with the middle knife-edge of the apparatus. So the components of the acceleration of this knife-edge with regard to a fixed system of coordinates are $-\ddot{x}$, $-\ddot{y}$ and $-\ddot{z}$.

CHARTER I.

Theoretical Investigation of the effect of accelerations.

§ 1. *The effect of the horizontal component of the acceleration perpendicular to the swinging-plane of the main pendulums.*

Using the same coordinates as in the publication of 1929, we take the X axis vertically downwards, the Y axis in the swinging-plane of the main pendulum and the Z axis perpendicular to the X and Y axis. So we have to investigate here the effect of $\ddot{z}$ *) i.e. the Z component of the acceleration. This component combined with the acceleration of gravity g gives a resultant acceleration OA (fig. 1). OB represents the position of the swinging-plane of the main pendulums, determined by the movement of the apparatus in its gimbals and β is the angle AOB. If this situation were stable, the acceleration working in the swinging-plane would be OB instead of g and the effect δg on the result of the pendulum observations for which it has to be corrected, would be given by

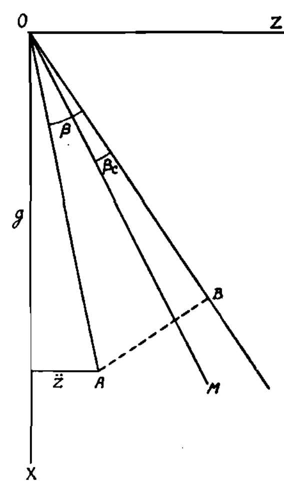

Fig. 1.

$$\frac{\delta g}{g} = \cos \beta \sec AOX - 1,$$

and if we neglect the fourth and higher powers of the angles β and AOX

$$\frac{\delta g}{g} = -\tfrac{1}{2}\beta^2 + \tfrac{1}{2}\left(\frac{\ddot{z}}{g}\right)^2 \quad . \; . \; . \; . \; . \; . \; . \quad (1\,A)$$

The angles β and AOX, however, are not constant as β and $\ddot{z}$ vary and so we have not only to consider this difference δg between g and the acceleration OB working in the swinging-plane as a varying quantity, but we have also to take into account the acceleration in the swinging-plane caused by the rotation of this plane round the Y axis.

We shall now assume that neither the variations of β nor those of $\ddot{z}$ have a term of the same or of half the period of the main pendulums. This is doubtless the case for observations made in submerged submarines; the period of the gimbals is many times larger and the waves in that case neither can have such a period. According to GERSTNER's wave-theory (or STOKES's modification of it) which as we shall see in Chapter III has been confirmed by the obser-

*) About the sign of $\ddot{z}$, see remark on page 5.

vations, the amplitudes of the waves are halved for every increase of depth of $^1/_9$ of the wave-length and as the length of a wave of a period of one second is only about 1.60 m, those waves must practically have disappeared at a depth of a few metres. This is confirmed by the experience obtained during the gravity determinations in submarines; the ship's movements have never shown a period of about one second. This was important as otherwise the writer would have met with considerable difficulties in making observations with pendulums of that period.

Basing our further deductions on this assumption, we may say that according to the investigation on pages 10 and 11 of the publication of 1929 pendulum observations made in case of fluctuating vertical accelerations of a period not coinciding with the period or half of the period of the pendulum, will give as result a value equalling the mean value of this acceleration during the observation. So for the effect δg we have to take the mean value of the above formula.

The rotation of the swinging-plane round the Y axis causes a centrifugal acceleration in the swinging-plane and, therefore, also contributes to the effect δg. The angular speed of this rotation is $\dot{\beta} + \frac{\dddot{z}}{g}$. When we introduce an X' axis in the swinging-plane and perpendicular to the Y axis, the contribution because of this centrifugal acceleration of an arbitrary mass-element dm of the pendulum to the momentum round the knife-edge of the forces working on the pendulum is

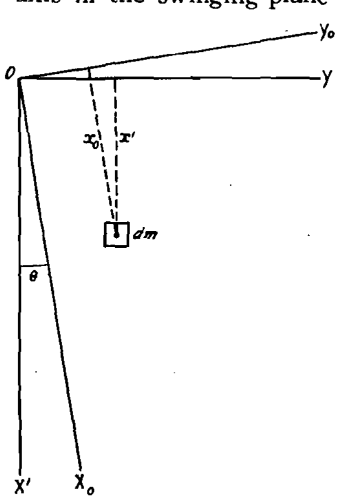

Fig. 2.

$$\left(\dot{\beta} + \frac{\dddot{z}}{g}\right)^2 x'y\,dm$$

If the deviation of the pendulum from its equilibrium position is θ and if we introduce an X_0 and Y_0 axis fixed to the pendulum in such a way that they coincide with the X' and Y axis for $\theta = 0$, we can write this as follows

$$\left(\dot{\beta} + \frac{\dddot{z}}{g}\right)^2 [(x_0^2 - y_0^2) \sin\theta \cos\theta + \\ + x_0 y_0 (\cos^2\theta - \sin^2\theta)]\,dm.$$

Integrated over the whole pendulum and writing

$\int (x_0^2 + y_0^2)\,dm = I$ (moment of inertia round the knife-edge),

$\int y_0^2\,dm = I_y$ (moment of inertia with regard to a plane through the X_0 axis and the knife-edge),

$\int x_0 y_0\,dm = 0$ (because of the symmetry of the pendulum with regard to the plane through the X_0 axis and the knife-edge),

we find, when neglecting the factor $\cos\theta$

$$\left(\dot{\beta} + \frac{\dddot{z}}{g}\right)^2 (I - 2\,I_y) \sin\theta.$$

The momentum of gravity itself round the knife-edge is

$$I \frac{g}{l} \sin \theta,$$

and so the effect of the centrifugal acceleration comes to the same as the effect of a δg given by

$$\frac{\delta g}{g} = \left(\dot{\beta} + \frac{\ddot{z}}{g}\right)^2 \left(1 - 2\,\frac{I_y}{I}\right) \frac{l}{g} = \frac{T^2}{4\pi^2} \left(1 - 2\,\frac{I_y}{I}\right) \left(\dot{\beta} + \frac{\ddot{z}}{g}\right)^2 \quad (1\,B)$$

in which T is the period of the main pendulum [1]).

Normally I_y is small with regard to I and so we may usually omit the factor $1 - 2\; I_y/I$.

Also for this effect δg we have to take the mean value over the whole observation. Together with the first effect we obtain (the dashes indicating mean values over the time of observation)

$$\frac{\delta g}{g} = -\tfrac{1}{2}\overline{\beta^2} + \tfrac{1}{2}\overline{\left(\frac{\ddot{z}}{g}\right)^2} + \frac{T^2}{4\pi^2}\overline{\left(\dot{\beta}\right)^2} + \frac{T^2}{2\pi^2}\overline{\dot{\beta}\,\frac{\ddot{z}}{g}} + \frac{T^2}{4\pi^2}\overline{\left(\frac{\ddot{z}}{g}\right)^2} \quad (1\,C)$$

For the determination of this correction we dispose in the first place of the indications of the damped pendulum introduced in the pendulum apparatus for the measurement of the tilt of the swinging-plane. In the second place we have the new apparatus for measuring $\ddot{z}$ which will be treated of in Chapter II.

We shall show presently, in a special investigation of the equation of motion of the damped pendulum, that this pendulum practically adopts a position coinciding with OA (fig. 1) and so its record, which represents the angle between this pendulum and a fixed plane of the pendulum apparatus, directly gives the angle β. Taken together with the data given by the new apparatus we obtain the two quantities β and $\ddot{z}$ necessary for computing the correction δg.

The following considerations may lead to a method for the practical execution of these computations. The angle β is composed of a constant part β_c brought about by an imperfect levelling of the apparatus in the gimbals (fig. 1), of a part $p_b \cos n_b t$ caused by the free swinging of the gimbals, $2\pi/n_b = T_b$ being the period of this swinging, and lastly a part $q_b \cos n_w t$ because of the forced swinging of the gimbals brought about by the waves of a period $2\pi/n_\omega = T_\omega$. If the ship's movement has more than one period T_ω we have correspondingly to add more terms of this type. So

$$\beta = \beta_c + p_b \cos n_b t + q_b \cos n_\omega t. \quad \ldots\ldots \quad (2)$$

Suppose we may represent $\ddot{z}$ by

$$\ddot{z} = b \cos n_w t \quad \ldots\ldots\ldots \quad (3)$$

The amplitude q_b depends on b and we may derive this relation from the

1) In this publication T indicates the complete period of the pendulums, i.e. the double time of swing. In previous publications the writer has used T for the time of a single swing, as is usually done.

equation of motion of the gimbals in the plane XOZ. If θ_b is the deviation of the gimbals from their equilibrium position ($\angle\ XOM$ in fig 1), we have

$$\theta_b = \frac{\ddot{z}}{g} + \beta - \beta_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the equation for θ_b is

$$\ddot{\theta}_b + n_b^2\ \theta_b - n_b^2\ \frac{\ddot{z}}{g} = 0.$$

Introducing (3) we find θ_b to have the following term of the period of the waves *)

$$\frac{n_b^2}{n_b^2 - n_\omega^2}\ \frac{b}{g}\ \cos\ n_\omega\, t = \frac{1}{1 - \varepsilon_b^2}\ \frac{b}{g}\ \cos\ n_\omega\, t,$$

where
$$\varepsilon_b = \frac{n_\omega}{n_b} = \frac{T_b}{T_\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So the amplitude q_b of the wave-period term of β is, according to (3) and (4)

$$q_b = \frac{1}{1 - \varepsilon_b^2}\ \frac{b}{g} - \frac{b}{g} = \frac{\varepsilon_b^2}{1 - \varepsilon_b^2}\ \frac{b}{g} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the wave-period is large, as is usually the case in submerged submarines. ε_b is small and therefore q_b as well.

We shall use this formula for the computation of the fourth term of ($1C$).

Introducing (2) and (3) in ($1C$) and making use of the following mean values

$$\overline{\cos^2 n_b\, t} = \overline{\cos^2 n_\omega\, t} = \overline{\sin^2 n_b\, t} = \overline{\sin^2 n_\omega\, t} = \tfrac{1}{2},$$

$$\overline{\cos\ n_b\, t} = \overline{\cos\ n_\omega t} = \overline{\sin\ n_b\, t} = \overline{\sin\ n_\omega t} = \overline{\sin n_b\, t\,.\,\sin n_\omega\, t} = 0,$$

we obtain

$$\frac{\delta g}{g} = -\tfrac{1}{2}\,\beta_c^2 - \tfrac{1}{4}\,p_b^2 - \tfrac{1}{4}\,q_b^2 + \tfrac{1}{4}\,\frac{b^2}{g^2} + \tfrac{1}{2}\,\frac{T^2}{T_b^2}\,p_b^2 + \tfrac{1}{2}\,\frac{T^2}{T_\omega^2}\,q_b^2 +$$
$$+ \frac{T^2}{T_\omega^2}\,\frac{\varepsilon_b^2}{1 - \varepsilon_b^2}\,\frac{b^2}{g^2} + \tfrac{1}{2}\,\frac{T^2}{T_\omega^2}\,\frac{b^2}{g^2}$$

or

$$\frac{\delta g}{g} = -\tfrac{1}{2}\,\beta_c^2 - \tfrac{1}{4}\,p_b^2\left(1 - 2\,\frac{T^2}{T_b^2}\right) - \tfrac{1}{4}\,q_b^2\left(1 - 2\,\frac{T^2}{T_\omega^2}\right) +$$
$$+ \tfrac{1}{4}\,\frac{b^2}{g^2}\left[1 + 2\,\frac{T^2}{T_\omega^2}\,\frac{(1 + \varepsilon_b^2)}{(1 - \varepsilon_b^2)}\right] \quad . \quad . \quad . \quad (7)$$

We may use this formula for the computation of δg. The first three terms may be derived from the curve for β, given by the damped pendulum. β_c is the deviation of the axis of the curve from the zero-line, p_b and q_b are the amplitudes of the fluctuations of the curve and T_b and T^ω their periods. These first three terms of formula (7) are the same as those given by formulas $17A$ and $17B$ of the publication of 1929, p. 16. When the ship's movement

*) In § 8, p. 34 e.s., we have made an investigation of the question whether the vertical accelerations of the same period would likewise give terms of this period in Θ_b. We found, as might be expected, that such terms are much smaller than those brought about by the horizontal accelerations. They may be neglected here.

has more than one period we have correspondingly to add terms to formula (3) and, as it is easy to see, this leads to further terms of formula (7) of the same shape as the third and fourth terms.

For the fourth term we need the record of the new apparatus that allows the determination of b/g. This term of the correction is one of the terms of BROWNE which have not been applied to former observations at sea. As T is small with regard to T_w, its main part is

$$\frac{\delta g}{g} = \tfrac{1}{4} \frac{b^2}{g^2} \quad \ldots \ldots \ldots \quad (8\,A)$$

with eventually more terms of the same shape if there are more than one wave-period; each quantity b then represents the amplitude of $\ddot{z}$ for the corresponding wave-period. We can also write this in the shape

$$\frac{\delta g}{g} = \tfrac{1}{2} \overline{\left(\frac{\ddot{z}}{g}\right)^2} \quad \ldots \ldots \ldots \quad (8\,B)$$

which comprises the case of more than one wave-period in one term.

The damped pendulum.

We have still to show that the damped pendulum, under the effect of the horizontal acceleration $\ddot{z}$, approximately adopts the position OA (fig. 1). For this purpose we shall first study its equation of motion in case there are no disturbing accelerations. This equation of motion has not been given in the publication of 1929.

As page 51 of this publication describes in more detail, the damped pendulum, introduced for recording the direction of the vertical with regard to the apparatus, consists of a system of two pendulums of different periods, swinging round knife-edges in line with each other and linked together by an oil-damping; they are not damped with regard to the apparatus. The pendulum with the largest period has a small oil-tank at its lower end and a second pendulum inside the other swings with its tail in the oil. So the pendulums are damped in themselves. The writer has adopted this device for preventing the pendulum being drawn out of its equilibrium by a damping attached to the apparatus.

Distinguishing the two pendulums by the indices 1 and 2, T_1 being the largest period, and indicating the moments of inertia round the knife-edges by I_1 and I_2 and the momentum of the damping round the knife-edges by C, the equations of motion are

$$\ddot{\theta}_1 + n_1^2\,\theta_1 + \frac{C}{I_1}\,(\dot{\theta}_1 - \dot{\theta}_2) = 0 \qquad n_1 = 2\,\pi/T_1 \ldots \quad (9\,A)$$

$$\ddot{\theta}_2 + n_2^2\,\theta_2 - \frac{C}{I_2}\,(\dot{\theta}_1 - \dot{\theta}_2) = 0 \qquad n_2 = 2\,\pi/T_2 \ldots \quad (9\,B)$$

and eliminating θ_2 from the first by means of the second equation we get

$$\ddot{\ddot{\theta}}_1 + C\left(\frac{1}{I_1} + \frac{1}{I_2}\right)\dddot{\theta}_1 + (n_1^2 + n_2^2)\,\ddot{\theta}_1 + C\left(\frac{n_2^2}{I_1} + \frac{n_1^2}{I_2}\right)\dot{\theta}_1 + n_1^2\,n_2^2\,\theta_1 = 0 \quad \ldots \ldots \quad (10)$$

The solution for θ_1 consists of four terms of the shape $A\,e^{xt}$ in which the x are the roots of the equation of the fourth degree in x having the same coefficients as the differential equation. In the normal case the roots are complex quantities $-k+int$, $-k-int$, $-k'+in't$ and $-k'-in't$, and the four terms combine in two, each a product of a damping-factor and a periodic term [1])

$$\theta_1 = A\,e^{-kt}\cos(nt+\varphi) + A'e^{-k't}\cos(n't+\varphi'),$$

A, A', φ and φ' integration-constants.

The solution for θ_2 has the same shape with the same values for k, k', n and n'.

It is clear that the most satisfactory damping would occur if the periods and damping of both terms of θ_1 and θ_2 would be equal, i.e. if $n = n'$ and $k = k'$; they would in that case combine into one term — the factor A taking the shape $A'' + B''\,t$ — and θ_1 as well as θ_2 would have a regular damping. We can easily derive that this case of two pairs of equal roots of the equation in x is realized when the following conditions are fulfilled

$$\frac{n_1}{n_2} = \frac{I_2}{I_1} \qquad (12) \qquad\qquad C = 2\,n_1 I_1\,\frac{(n_2-n_1)}{(n_2+n_1)} \qquad (13)$$

If l_1 and l_2 are the mathematical lengths of the pendulums,
m_1 and m_2 their masses,
h_1 and h_2 the distances from the centres of gravity to the knife-edges,
we may write the first condition in the following shapes

$$\frac{l_2}{l_1} = \frac{I_2^2}{I_1^2} \qquad (12\,A) \qquad\text{or}\qquad \frac{m_1 h_1}{m_2 h_2} = \frac{I_2}{I_1} \qquad (12\,B)$$

It is not difficult to realize this condition by bringing part of the mass of the first pendulum above the knife-edge and thereby reducing h_1 till $(12\,B)$ is fulfilled.

Formula (13) shows that for obtaining a good damping we cannot take $n_2 - n_1$ too small. So the pendulum-periods must differ sufficiently.

Introducing (12) and (13) in the equation for x and solving x we find n and the period T of the combination of the two pendulums. We obtain

$$T^2 = \frac{T_1 T_2}{1 - \dfrac{(T_1-T_2)^2}{4\,T_1 T_2}} \quad \ldots\ldots\ldots \quad (14\,A)$$

and neglecting $\dfrac{(T_1-T_2)^4}{16\,T_1^2\,T_2^2}$ with regard to 1

$$T^2 = \left[1 + \frac{(T_1-T_2)^2}{4\,T_1 T_2}\right] T_1 T_2 \quad\text{or}\quad T = \tfrac{1}{2}(T_1+T_2) \qquad (14\,B)$$

So T is approximately the mean of T_1 and T_2.

We can now undertake to determine the effect of an acceleration $\ddot{z} = b\cos n_w t$ on the system. The equations (9) become, when taking (12) and (13) into account

1) Real roots of (10) would mean an aperiodic damping and this is more difficult to realize.

$$\ddot{\theta}_1 + n_1^2\,\theta_1 + 2\,n_1\,\frac{(n_2 - n_1)}{(n_2 + n_1)}\,(\dot{\theta}_1 - \dot{\theta}_2) - n_1^2\,\frac{b}{g}\cos n_w\,t = 0 \quad (15\,A)$$

$$\ddot{\theta}_2 + n_2^2\,\theta_2 - 2\,n_2\,\frac{(n_2 - n_1)}{(n_2 + n_1)}\,(\dot{\theta}_1 - \dot{\theta}_2) - n_2^2\,\frac{b}{g}\cos n_w\,t = 0 \quad (15\,B)$$

By introducing

$$\theta_1 = d_1 \cos\,(n_\omega\;t + \varphi_1),$$
$$\theta_2 = d_2 \cos\,(n_\omega\;t + \varphi_2),$$

we may find the amplitudes d_1 and d_2 of the forced swinging in the period of the waves, as well as the phase-shifts φ_1 and φ_2 brought about by the damping terms in the equations (15). As only the first pendulum is recorded, d_1 and φ_1 are the quantities found in the record.

The result for d_1 is a rather complicated formula, which for small damping tends to the value

$$d_1 = \frac{n_1^2}{n_1^2 - n_\omega^2}\,\frac{b}{g} = \frac{1}{1 - \varepsilon_1^2}\,\frac{b}{g} \quad . \; . \; . \; . \quad (16\,A)$$

where

$$\varepsilon_1 = \frac{T_1}{T_\omega} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17\,A)$$

This is the amplitude which the first pendulum apart would adopt under the effect of $\ddot{z}$.

For a large damping the formula for d_1 tends to

$$d_1 = \frac{n_1\,n_2}{n_1\,n_2 - n_\omega^2}\,\frac{b}{g} = \frac{1}{1 - \varepsilon_1\,\varepsilon_2}\,\frac{b}{g} \quad . \; . \; . \; . \quad (16\,B)$$

where

$$\varepsilon_2 = \frac{T_2}{T_\omega} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17\,B)$$

which is the effect on a pendulum of a period equal to the geometric mean of T_1 and T_2.

For an arbitrary damping d_1 lies between these two values.

In the pendulum apparatus T_1 is slightly more than 1 sec and the geometric mean of T_1 and T_2 slightly less than 1 sec. The period of the ship's movements in submerged submarines is seldom smaller than 7 sec; according to GERSTNER's or STOKES's wave-theory waves of this period are halved for every increase of depth of 8½ m and so these waves are reduced to 1/16 th of their amplitude at a depth of 34 meters, while waves of shorter period are still more reduced. Introducing these values for T_ω, T_1 and T_2 in the formulas 16 and 17 we see that ε_1 and ε_2 are small and that the amplitude d_1 differs only by 2 % from the value b/g i.e. from the amplitude of $\ddot{z}/g$; for greater values of T_w this difference is still smaller. We may, therefore, conclude that with this approximation the damped pendulum adopts the position OA (fig. 1).

§ 2. *The effect of the horizontal component of the acceleration in the direction of the swinging-plane of the main pendulums.*

Keeping to the system of coordinates introduced in § 1 we have here to investigate the effect of $\ddot{y}$. Besides the principal effect of the acceleration

$\ddot{y}$ of the whole system we may ask the question whether the swinging of the gimbals round an axis parallel to the knife-edges could perhaps also have an effect on the pendulums. This swinging does not bring about a horizontal translation of the knife-edges because the knife-edges are supposed to be in the same horizontal plane as the axis of rotation of the gimbals, but it brings about a vertical acceleration; we shall determine its effect in the second half of this paragraph. It further causes a tilt of the agate planes supporting the knife-edges and the question is whether this has an appreciable effect on the pendulum-periods because of the finite dimensions of the contact-surface of the knife-edges. We shall assume that we may neglect this effect because we suppose the tilt to keep within narrow limits and even if there would be a slight effect, the resulting momentum working on the pendulums will probably be about the same for all three, if at least the knife-edges do not differ too much. Thus the effect will disappear when taking the difference of the equations of motion of the two pendulums of a pair for deriving the equation of motion of the fictitious pendulum of this pair.

We shall begin by investigating the effect of the acceleration $\ddot{y}$ of the system. Distinguishing the pendulums of a pair by the sub-indices 1 and 2 and assuming them for this problem to be exactly isochronous, the equations of motion are

$$\ddot{\theta}_1 + \frac{g}{l} \sin\theta_1 - \frac{\ddot{y}}{l} \cos\theta_1 = 0, \quad \ldots \ldots \quad (18A)$$

$$\ddot{\theta}_2 + \frac{g}{l} \sin\theta_2 - \frac{\ddot{y}}{l} \cos\theta_2 = 0. \quad \ldots \ldots \quad (18B)$$

We shall develop the goniometrical functions in series, which, as the maximum value of θ is of the order of 0.01, we may break off after the third power of θ,

$$\ddot{\theta}_1 + \frac{g}{l} \theta_1 (1 - \frac{1}{6} \theta_1^2) - \frac{\ddot{y}}{l} (1 - \frac{1}{2} \theta_1^2) = 0, \quad . \quad (19A)$$

$$\ddot{\theta}_2 + \frac{g}{l} \theta_2 (1 - \frac{1}{6} \theta_2^2) - \frac{\ddot{y}}{l} (1 - \frac{1}{2} \theta_2^2) = 0. \quad . \quad (19B)$$

Taking the difference of these equations and putting

$$\theta_1 - \theta_2 = \theta, \quad \frac{1}{2}(\theta_1 + \theta_2) = \vartheta \, . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

we get the equation of motion of the fictitious pendulum

$$\ddot{\theta} + \frac{g}{l} \theta [1 - \frac{1}{24} \theta^2 - \frac{1}{2} \vartheta^2 + \frac{\ddot{y}}{g} \vartheta] = 0. \; . \; . \; . \; . \; . \quad (21)$$

The three last terms between the brackets correspond to the corrections δg which we have to apply to the result for g given by the normal pendulum formula. They comprise the reduction to infinitely small amplitude which has already been derived on p 19 of the publication of 1929, and the effect of the acceleration $\ddot{y}$ which is one of the correction terms of Browne. As formula 21 shows, both are effects of the second order.

For determining these effects we need an approximate formula for the mean elongation ϑ which we may derive from the mean of the equations (19); we may neglect the terms of the order θ_1^3 and θ_2^3. Putting $g/l = n^2$ and so $n = 2\pi/T$ we get

$$\ddot{\vartheta} + n^2 \vartheta - n^2 \frac{\ddot{y}}{g} = 0. \quad \ldots \ldots \ldots \quad (22)$$

For its solution we need an assumption regarding $\ddot{y}$. Suppose, according to what was assumed in formula (3), page 8, for the component $\ddot{z}$

$$\ddot{y} = a \cos n_w t \quad \ldots \ldots \ldots \ldots \quad (23)$$

We then find

$$\vartheta = a_m \cos (nt + \varphi_m) + s \cos n_\omega t, \quad \ldots \ldots \quad (24)$$

in which a_m and φ_m are the amplitude and the phase (for $t = 0$) of the free swinging and s the amplitude of the forced swinging in the wave-period. s is given by

$$s = \frac{n^2}{n^2 - n_\omega^2} \frac{a}{g} = \frac{1}{1 - \varepsilon^2} \frac{a}{g}, \quad \ldots \ldots \quad (25\,A)$$

where

$$\varepsilon = \frac{n_\omega}{n} = \frac{T}{T_\omega}. \quad \ldots \ldots \ldots \ldots \ldots \quad (25\,B)$$

As we have already mentioned at the end of the previous paragraph, T_ω is seldom smaller than 7 sec and as T is about 1 sec, ε is small and so we may write with an approximation of a few percent

$$s = \frac{a}{g} \quad \ldots \ldots \ldots \ldots \quad (26)$$

The approximate solution of the equation of motion (21) for the fictitious pendulum does not show a second term of the first order in $\cos n_w t$; it is

$$\theta = a_v \cos (nt + \varphi_v), \quad \ldots \ldots \ldots \quad (27)$$

a_v and φ_v being the amplitude and the phase of this pendulum for $t = 0$.

Introducing (23), (24) and (27) in (21) we can derive the more exact solution of (21) and thus we find the correction δg to be applied to the result for g. For the terms of the period of the waves and eventual other terms of periods differing much from T we may simply conclude from (21)

$$\frac{\delta g}{g} = \overline{- \frac{1}{24} \theta^2 - \frac{1}{2} \vartheta^2 + \frac{\ddot{y}}{g} \vartheta}. \quad \ldots \ldots \quad (28)$$

The dash again indicates the mean value taken over the whole duration of the observation. As, however, the main terms of θ and ϑ have the pendulum period, we have to apply a more general method for deriving the complete value of δg.

Referring to the general solution of the pendulum equation on pages 5 and 6 of the publication of 1929, we shall use formula 6 A of that publication for the disturbance δT of T and we derive from it

$$\frac{\delta g}{g} = -2 \frac{\delta T}{T} = \frac{2l}{g} \cdot \overline{\frac{S}{a_v} \cos (nt + \varphi_v)}, \quad \ldots \quad (29\,A)$$

in which S is the disturbance term in the equation of motion, so in our case

$$S = -\frac{g}{l}\,\theta\left[\frac{1}{24}\,\theta^2 + \frac{1}{2}\,\vartheta^2 - \frac{\ddot{y}}{g}\,\vartheta\right]$$

and

$$\frac{\delta g}{g} = -2\,\overline{\cos^2(nt+\varphi_v)\left[\frac{1}{24}\,\theta^2 + \frac{1}{2}\,\vartheta^2 - \frac{\ddot{y}}{g}\,\vartheta\right]} \quad . \quad (29\,B)$$

Introducing $\ddot{y}$, ϑ and θ of (23), (24) and (27) we obtain

$$\frac{\delta g}{g} = -\frac{1}{12}\,a_v^2\,\overline{\cos^4(nt+\varphi_v)} - a_m^2\,\overline{\cos^2(nt+\varphi_v)\cos^2(nt+\varphi_m)} +$$
$$+ s\left(2\,\frac{a}{g} - s\right)\overline{\cos^2(nt+\varphi_v)\cos^2 n_\omega t} -$$
$$-2\,a_m\left(s - \frac{a}{g}\right)\overline{\cos^2(nt+\varphi_v)\cos(nt+\varphi_m)\cos n_w t} \quad . \quad (30)$$

Taking the mean values over a duration that is long with regard to T and T_ω the first two terms reduce to

$$\frac{\delta g}{g} = -\frac{1}{32}\,a_v^2 - \frac{1}{4}\,a_m^2\cos^2(\varphi_m - \varphi_v) - \frac{1}{8}\,a_m^2 \quad . \quad (31\,A)$$

and after some reduction this can be brought to coincidence with formula (21 A) for the reduction to infinitely small amplitude on page 19 of the publication of 1929:

$$\frac{\delta g}{g} = -2\,\frac{\delta T}{T} \doteq -\frac{1}{8}\,a_v^2 - \frac{1}{8}\,a_2^2 - \frac{3}{8}\,a_v\,a_2\cos(\varphi_v - \varphi_2) -$$
$$-\frac{1}{4}\,a_2^2\cos^2(\varphi_v - \varphi_2). \quad . \quad (31\,B)$$

This formula is a function of the amplitude a_v of the fictitious pendulum, the amplitude a_2 of the main term of θ_2, i.e. of the elongation of one of the pendulums of the pair, and of their phase-difference $\varphi_v - \varphi_2$. We can compute the value of this formula by means of the records of the fictitious and of the middle pendulum.

a_2 is the amplitude of the main term of θ_2, i.e. the term corresponding to the free swinging of this pendulum in its own period. In the same way, however, as ϑ, θ_2 has also a second term because of the forced swinging in the period of the waves. It is easy to see that this term is the same as that of ϑ, i.e. $s\cos n_w t$. This term does not appear in the record of the middle pendulum because this record gives the relative position of the pendulum with regard to the damped pendulum swinging in a parallel plane and, according to formulas (16 A) or (16 B), the amplitude of the latter is practically the same as the amplitude s given by formula (25 A); ε^2 as well as ε_1^2 and $\varepsilon_1\,\varepsilon_2$ are small and differ only slightly according to the figures given for T_1 and T_2 on page 12. So the record of the middle pendulum practically only gives the main term of θ_2 which we require for the computation of (31 B).

The last two terms of (30) give, when introducing (25 A) for the value of s

$$\frac{\delta g}{g} = \frac{(1-2\varepsilon^2)}{4(1-\varepsilon^2)^2}\left(\frac{a}{g}\right)^2\left[1+\frac{1}{2}\,\overline{\cos\{2(n-n_\omega)t+2\varphi_v\}}\right]+$$
$$-\frac{\varepsilon^2}{4(1-\varepsilon^2)}\,a_m\,\frac{a}{g}\left[\overline{2\cos\{(n-n_\omega)t+\varphi_m\}}+\right.$$
$$\left.+\overline{\cos\{(3n-n_\omega)t+2\varphi_v+\varphi_m\}}+\overline{\cos\{(n-n_\omega)t+2\varphi_v-\varphi_m\}}\right], \quad (32\,A)$$

which reduces to the first term when n_w is not nearly equal to n or to $3\,n$

$$\frac{\delta g}{g} = \frac{1-2\,\varepsilon^2}{4(1-\varepsilon^2)^2}\left(\frac{a}{g}\right)^2 = \frac{1}{4}\left[1-\frac{\varepsilon^4}{(1-\varepsilon^2)^2}\right]\left(\frac{a}{g}\right)^2. \quad (32\,B)$$

For the computation of this correction we must know the value of a/g and for obtaining this we need the record of the new apparatus for the measuring of $\ddot{y}$ which we shall treat of in Chapter II.

If T_w were about equal to T or to $\frac{1}{3}\,T$ the corresponding cosine of formula $(32\,A)$ would not disappear when taking the mean value over the time of the observation and so it would contribute to the correction δg. As we have already mentioned, this will not occur when T_w is the period of the wave-movement in a submerged submarine and therefore we may neglect these possibilities when we restrict ourselves to the investigation of the effect of the ship's movements. These periods might, however, occur because of other effects and it is worth while to know that such periods may be dangerous because of the possibility of their having systematic disturbing effects of the second order on the pendulums.

If the ship's movement would be composed of more than one periodic term, we have to add corresponding terms to (23) and to the other formulas including the resulting formulas $(32\,A)$ and $(32\,B)$.

If T_w would be 7 sec or more, as is probably the case in a submerged submarine, and if T is about 1 sec, ε is small and so we may in good approximation write $32\,B$ in the simple shape

$$\frac{\delta g}{g} = \frac{1}{4}\left(\frac{a}{g}\right)^2 = \frac{1}{2}\left(\frac{\overline{\ddot{y}}}{g}\right)^2, \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (33)$$

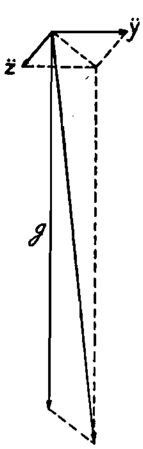

Fig. 8.

which corresponds to the approximate value found in formulas $(8\,A)$ and $(8\,B)$ for the effect of $\ddot{z}$. The formulas $(8\,B)$ and (33) are the corrections as given by BROWNE.

We may point out, as he did too, that these corrections correspond to the difference between the undisturbed value of g and the resultant acceleration when adding to g the components $\ddot{y}$ and $\ddot{z}$. As, however, $\ddot{y}$ and $\ddot{z}$ are variable, their effect on the pendulum is more complicated and so the more exact formulas for δg of this and the previous paragraph slightly differ from this simple shape.

We have still to investigate the effect of the vertical accelerations brought about by the swinging of the gimbals round an axis parallel to the knife-edges. This swinging consists of a free swinging in the period T_a of the gimbals and a forced swinging in the period T_w of the waves. If θ_a is the deviation from the vertical we have

$$\theta_a = p_a \cos n_a t + q_a \cos n_w t, \qquad n_a = 2\pi/T_a, \quad (34A)$$

and from (23) and the equation of motion

$$\ddot{\theta}_a + n_a^2 \theta_a - n_a^2 \frac{\ddot{y}}{g} = 0$$

follows

$$q_a = \frac{1}{1-\varepsilon_a^2} \frac{a}{g}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (34B)$$

where

$$\varepsilon_a = \frac{n_w}{n_a} = \frac{T_a}{T_w}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

Of the two main pendulums Nos 1 and 2 we assume No 2 to hang in the middle of the apparatus and so its knife-edge coincides with the axis of rotation of the gimbals and does not move because of their swinging. If j is the distance of the knife-edge of No 1 from that of No 2, this second knife-edge undergoes a vertical acceleration

$$\ddot{x}_1 = -j\,\ddot{\theta}_a = j\,p_a n_a^2 \cos n_a t + \frac{j\,a}{(1-\varepsilon_a^2)g} n_w^2 \cos n_w t =$$

$$= \frac{j\,T^2}{l\,T_a^2}\, g\, p_a \cos n_a t + \frac{j\,\varepsilon^2}{l\,(1-\varepsilon_a^2)}\, a \cos n_w t. \quad . \; . \quad (36)$$

The knife-edge of the third main pendulum No 3 undergoes the contrary movement.

Both terms of this formula are small. As j/l is about 0,5 and ε seldom larger than 1/7, the second term is of the order of one percent of the horizontal acceleration $\ddot{y}$. So its effect, which is proportional to the square of its value may be neglected. As to the first term, the amplitude p_a of the gimbal-swinging is seldom larger than 1/200 and the square of the ratio T/T_a about 0,1 and so the value of the term is of the order of 1/4000. We may therefore, neglect its effect likewise. Incase, however, a stronger swinging of the gimbals would occur, a further investigation would be required. We shall shortly indicate a way to do this and the result thus obtained.

The acceleration $\ddot{x}_1$ brings about a term $-\ddot{x}_1 \theta_1/l$ in the equation of motion (19A) of the first pendulum. As $\theta_1 = \vartheta + \frac{1}{2}\theta$ the equation of motion of the fictitious pendulum becomes

$$\ddot{\theta} + \frac{g}{l}\,\theta \left[1 - \frac{1}{24}\,\theta^2 - \frac{1}{2}\,\vartheta + \frac{\ddot{y}}{g}\,\vartheta^2 - \frac{\ddot{x}_1}{2g}\right] - \vartheta\,\frac{\ddot{x}_1}{l} = 0. \quad . \quad (37)$$

The effect of the last term between the brackets comes to the same as that of an increase of g by $-\frac{1}{2}\ddot{x}_1$. So we can combine this small acceleration with the vertical acceleration $\ddot{x}$ of the ship, the effect of which we shall derive in the next paragraph. The effect of the last term of (37) can be derived by means of formula (29 A) in which we may substitute this term for S. We find that we may neglect the mean value of this formula unless T_w would be about equal to 2 T or ½ T. For wave-movements this practically does not occur.

§ 3. *The effect of the vertical component of the acceleration.*

In this paragraph we shall investigate the effect on the pendulums of the vertical component $\ddot{x}$ of the acceleration. For doing this we shall follow the solution of the equation of motion of a pendulum given on page 5 of the publication of 1929. In our case this equation is

$$\ddot{\theta} + \frac{g}{l}\,\theta + \frac{\ddot{x}}{l}\,\theta = 0,$$

in which the last term is the disturbance term brought about by the varying vertical acceleration $\ddot{x}$. Introducing the two variables, φ the phase-angle, and a the amplitude of the pendulum, which we define by

$$\theta = a \cos \varphi$$

and

$$\dot{\theta} = - a\, n \sin \varphi,$$

we find the following formula corresponding to formula (4 A) of the publication of 1929

$$\dot{\varphi} - n = \frac{\ddot{x}}{nl} \cos^2 \varphi = \frac{\ddot{x}}{2\,nl} + \frac{\ddot{x}}{2\,nl} \cos 2\,\varphi. \quad . \quad . \quad (38)$$

As we do not want to neglect terms of the order of the second power of the disturbing acceleration $\ddot{x}$, we can not here substitute for the phase-angle φ the value $nt + \varphi_v$ as we have done in formula (29) where S was a quantity of the second order. So we shall have to put

$$\varphi = nt + \varphi_v + \psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

in which ψ is a variable of the same order as $\ddot{x}$. Neglecting the third power of $\ddot{x}$ and of ψ we may write

$$\cos 2\,\varphi = \cos 2\,(nt + \varphi_v) - 2\,\psi \sin 2\,(nt + \varphi_v).$$

Introducing likewise, in agreement with formulas (3) and (23)

$$\ddot{x} = c \cos n_w\, t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

formula (38) becomes

$$\dot{\psi} = \frac{c}{2\,nl} \cos n_w t + \frac{c}{2\,nl} \cos n_w t \cos 2\,(nt + \varphi_v) - \\ - \frac{c\,\psi}{nl} \cos n_w t \sin 2\,(nt + \varphi_v)$$

or

$$\dot{\psi} = \frac{c}{2\,nl} \cos n_w t + \frac{c}{4\,nl} \cos \Big[(2\,n + n_w)\,t + 2\,\varphi_v \Big] + \\ + \frac{c}{4\,nl} \cos \Big[(2\,n - n_w)\,t + 2\,\varphi_v \Big] - \frac{c\,\psi}{2\,nl} \sin \Big[(2\,n + n_w)\,t + 2\,\varphi_v \Big] - \\ - \frac{c\,\psi}{2\,nl} \sin \Big[(2\,n - n_w)\,t + 2\,\varphi_v \Big] \quad . \quad . \quad . \quad (41)$$

By means of the pendulum observations we determine the mean value of the phase-velocity φ during the time of observation and so our problem is to derive the mean value of the disturbance $\dot{\psi}$ of this phase-velocity. As the time of observation is large with regard to the periods T and T_w and as T_w is not equal to $\frac{1}{2}$ T or, in other words, $2n - n_w$ not to zero, we may neglect the mean value of the cosines of the first three terms of formula (41). So the mean value of $\dot{\psi}$ reduces to the mean value of the last two terms where the sines are not multiplied by a constant factor but by a factor containing ψ. For computing their mean value we require an approximate value of ψ accurate up to the first order of the disturbance. This follows from (41) by integrating the first three terms; we may here neglect the last two as being of the second order. We obtain

$$\psi = \frac{c}{2\,nn_w\,l}\sin n_w t + \frac{c}{4\,n\,(2\,n + n_w)\,l}\sin\left[(2\,n + n_w)\,t + 2\,\varphi_v\right] + \\ + \frac{c}{4\,n\,(2\,n - n_w)\,l}\sin\left[(2\,n - n_w)\,t + 2\,\varphi_v\right] \quad . \quad . \quad (42)$$

Introducing this in the last two terms of (41) and taking the mean value by making use of

$$\overline{\sin^2\left[(2\,n + n_w)\,t + 2\,\varphi_v\right]} = \overline{\sin^2\left[(2\,n - n_w)\,t + 2\,\varphi_v\right]} = \tfrac{1}{2},$$

we find the correction of $\overline{\dot{\varphi}}$ to be

$$\overline{\dot{\psi}} = -\frac{c^2}{16\,n^2\,l^2}\left[\frac{1}{2\,n + n_w} + \frac{1}{2\,n - n_w}\right] = - \\ = -\frac{c^2}{16\,n^3\,l^2\left(1 - \dfrac{n_w{}^2}{4\,n^2}\right)} = -\frac{c^2\,n}{16\,g^2\left(1 - \dfrac{T^2}{4\,T_w{}^2}\right)} \quad . \quad (43)$$

For the correction to the result g we derive from $g = n^2\,l$, putting again $T/T_w = \varepsilon$

$$\frac{\delta g}{g} = 2\,\frac{\overline{\dot{\psi}}}{n} = -\frac{1}{8\,(1 - \frac{1}{4}\,\varepsilon^2)}\left(\frac{c}{g}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (44)$$

As we shall see in the next chapter, we can derive the data for c/g needed for the computing of this formula from the fluctuation of the second-marks in the pendulum-records.

In obtaining formula (43) we have neglected the mean value of $\sin n_w\,t \times$ $\times \sin\,(2\,n - n_w)\,t$. This may only be done when T_w is not about equal to the pendulum period T. We have already mentioned that this case does not occur for ship's movements caused by waves.

Formula (44) gives the effect brought about by a vertical acceleration $\ddot{x} = c\,\cos n_w\,t$. If the ship's movement is subject to waves of more than one period we have to add corresponding terms of the same shape. We further have to add the effect of the small vertical acceleration caused by the gimbal-swinging as given by half of the value of formula (36) of the preceding paragraph. We have, however, to take into account that usually the horizontal accelerations given by (23), which are the cause of this term, differ 90° in phase from the vertical accelerations given by (40) and so, assuming this to be the case, we get the complete formula

$$\frac{\delta g}{g} = -\frac{1+\dfrac{j^2\,\varepsilon^4}{4\,l^2\,(1-\varepsilon_a{}^2)^2}\,\dfrac{a^2}{c^2}}{8\,(1-\frac{1}{4}\,\varepsilon^2)}\left(\frac{c}{g}\right)^2 - \frac{1}{32\left(1-\frac{1}{4}\,\dfrac{T^2}{T_a{}^2}\right)}\,\frac{j^2}{l^2}\,\frac{T^4}{T_a{}^4}\,p_a{}^2. \quad . \quad (45)$$

Unless the free gimbal-swinging in its own period is abnormally large, the last term is negligible. If ε is small, as it usually is, we may substitute 1 for the numerator of the first term and approximately we may do the same for the factor of the denominator. We thus obtain

$$\frac{\delta g}{g} = -\frac{1}{8}\left(\frac{c}{g}\right)^2 = -\frac{1}{4}\left(\frac{\overline{\ddot{x}}}{g}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Incase of more than one wave-period we have to add terms of the same shape to the first of the two formulas (46); the second remains the same. This formula was given by BROWNE. In the same way as the corrections (8) and (33) this correction is of the second order, i.e. it is proportional to the square of the acceleration, but it has contrary sign.

As we are studying second order corrections, we have still to investigate whether the disturbance brought about by one of the components of the acceleration does not affect the amount of disturbance caused by another component; in this and the preceding paragraph we have treated of the effect of each component separately. It is easy to see, however, that this is not the case. If we determine the magnitude of the effect δg which we thus neglect, we find $\delta g/g$ to be of the order of

$$\frac{\ddot{x}\,\ddot{y}^2}{g^3} \quad \text{resp.} \quad \frac{\ddot{x}\,\ddot{z}^2}{g^3}$$

and we assume this to be negligible.

§ 4. *The effect of the rotational movements.*

As we have already treated of the effect of the swinging of the gimbals in the first two paragraphs, we have only to investigate the disturbance brought about by the rotation round a vertical axis. We have already taken up this problem in the publication of 1929. According to formula (15) on page 15 of that publication the effect of a rotation with an angular speed $\dot{\gamma}$ on the result g is given by

$$\frac{\delta g}{g} = -2\,\frac{\delta T}{T} = -\frac{T^2}{4\,\pi^2}\,(\dot{\gamma})^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

As the disturbance term S in the pendulum equation is of the order of θ multiplied by the square of the disturbing angular speed $\dot{\gamma}$ (see page 15, publ.

1929) we may apply the same reasoning as at the end of the preceding paragraph and state that we need not take into account a mutual effect of this disturbance and of that of one of the previous paragraphs; such an effect must be negligible.

Strictly speaking, formula (47) solves the problem taken up in this paragraph. We shall, however, proceed one step further here and we shall try to get an idea of the relative size of this correction in comparison with those of the other paragraphs. We suppose of course that the rotation is purely caused by the wave-movement and that no changes of the ship's course do occur. For this purpose we shall have to make assumptions about the ship's and wave movements. As we shall more at large mention in Chapter III, the observations, as far at least as conclusions were possible, have confirmed GERSTNER's or STOKES's theory about the wave-movement, and so we shall take them as a base for our considerations. This theory supposes each water-particle to describe a circular movement in a vertical plane that coincides with the direction of the propagation of the waves. Let us assume this direction to make an angle a with the ship's axis (fig. 4). Let us further suppose that the bow and the stern of the ship are carried along entirely by the components S_1 and S_2 of the water-movement in planes perpendicular to the ship's axis. Lastly we shall assume that the movement there is in opposite phase. These assumptions appear particularly unfavourable for our case; they suppose a rotation of the ship round a vertical axis that is probably greater than it really occurs. The last assumption implies that the projection $A_1 B$ of the ship's axis L on the direction of propagation is equal to half of the wave-length λ, so

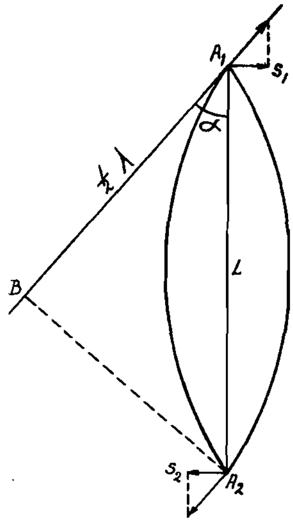

Fig. 4.

$$\lambda = 2\, L \cos a.$$

If d is the centrifugal acceleration of each water-particle and if we suppose the whole ship to move in circles of the same size as those described by the water-particles, d would be equal to the vertical acceleration c of the formulas (45) and (46) and also to the horizontal acceleration of which the quantities a of the formulas (32) and (33) and b of the formulas (7) and (8) are the components. Putting the phase-velocity of the water and ship's movements again at n_w, the horizontal components of the velocities S_1 and S_2 are given by

$$S_1 = - S_2 = \frac{d}{n_w} \sin a \cos n_w t,$$

and so we find

$$\dot{\gamma} = \frac{d}{n_w L} \sin a \cos n_w t$$

and

$$\frac{\delta g}{g} = -\frac{T^2}{4\pi^2}\,\overline{(\dot{\gamma})^2} = -\frac{T^2\,d^2}{8\pi^2\,n_w^2\,L^2}\,sin^2\,\alpha = -\frac{T^2\,T_w^2\,d^2}{32\,\pi^4\,L^2}\,sin^2\,\alpha.$$

In order to get a better idea about the size of this correction we may use the following formula for the wave-length as given by the wave-theory

$$T_w^2 = 2\,\pi\,\frac{\lambda}{g},$$

and introducing the relation of L and λ we find

$$\frac{\delta g}{g} = -\frac{T^2\,d^2}{2\,\pi^2\,T_w^2\,g^2}\,sin^2\,\alpha\,cos^2\,\alpha. \quad \ldots\ldots \quad (48)$$

This becomes a maximum for $\alpha = 45^\circ$ and so we obtain for this maximum value

$$\frac{\delta g}{g} = -\frac{\varepsilon^2}{8\,\pi^2}\left(\frac{d}{g}\right)^2. \quad \ldots\ldots\ldots \quad (49)$$

As ε is seldom larger than 1/7 we see that this correction is small and comparing (49) with (8), (33) and (46) we find it to be about 1/1000th part of the effect of the horizontal components of the acceleration and about 1/500th part of that of the vertical component. As we shall see in Chapter III this leads us to values of δg that never come up to one milligal and as we have obtained this estimate as a maximum value which will probably never be attained, we may conclude that this correction may be neglected. This is an important result because, if it would be otherwise, we should have been obliged to construct a special apparatus for determining $\dot{\gamma}$ in order to be able to compute the correction by means of formula (47). We should not have been able to derive it by means of the above considerations as they are certainly not accurate enough to use them for more than a rough estimate.

§ 5. *Observations in harbours at the surface of the water.*

In the preceding paragraphs we have not yet investigated the disturbances for observations in harbours at the water's surface. In this case the ship's movements differ entirely from those at sea. In harbours there are no long waves and the short waves of relatively small amplitude that occur, appear to have an irregular effect. The writer got the impression that it is mainly the breaking of the waves against the hull of the ship that affects the pendulum observations. Besides we may have the effects of shocks caused by the ship bumping against the quay or by its pulling at anchor-cables. The irregularity of the ship's movements is proved by the irregular amplitudes of the pendulums as shown by the record of the middle pendulum; this record is usually much stronger disturbed for observations in harbours than for those at sea. In harbours, on the contrary, the second-marks never show fluctuations and this confirms the surmise that in this case there are no perceptible movements of longer period. This seems in good agreement with the absence of large waves in harbours.

The irregular character of the ship's movements does not make it probable that second order effects will occur. Using the results of the previous paragraphs we shall consecutively discuss the different disturbances, but we do not dispose of sufficient experimental data for arriving at sure conclusions.

1°. *The horizontal component of the acceleration perpendicular to the swinging-plane of the main pendulums.*

If we assume the accelerations to have a purely irregular character, we may suppose that the positions of the damped pendulums are not affected by them. It depends of course of their damping how much amplitude they will get from time to time, but if the damping-device is functioning satisfactorily we may probably neglect these deviations from the equilibrium. Although we have no direct observational evidence for this supposition we shall here assume it to be true.

So, if OX in fig. 5 represents the position of the true vertical, we suppose it also to indicate the position of the damped pendulum. Let the swinging-plane of the main pendulums be perpendicular to the plane of fig 5 and let OX_1 be its cross-section. If OG is the value of gravity g and GA an arbitrary value of the irregular horizontal acceleration $\ddot{z}$, the projection on the swinging-plane of g and $\ddot{z}$ is

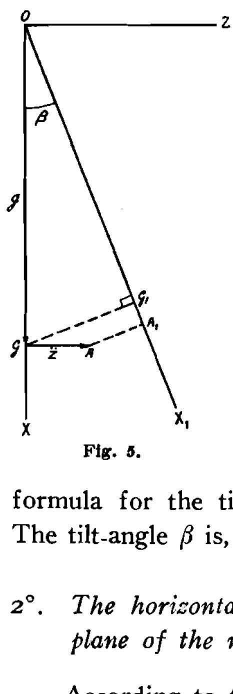

Fig. 5.

$$O\,A_1 = g + \delta g = g\cos\beta + \ddot{z}\sin\beta.$$

Owing to the assumed irregularity of $\ddot{z}$ we may suppose that $\ddot{z}$ is independent of the deviation β of the swinging-plane as the latter is caused by what preceded, and so we see that the mean value of the second term of this formula over the duration of an observation must be negligible. So the correction δg only depends on the angle β and not on the horizontal acceleration $\ddot{z}$; it is given by the normal formula for the tilt-correction, i.e. by the two first terms of formula (7). The tilt-angle β is, as usual, given by the record of the damped pendulum.

2°. *The horizontal component of the acceleration parallel to the swinging-plane of the main pendulums.*

According to the deductions of § 2 the correction for the effect of this component is given by formula $(29\,B)$:

$$\frac{\delta g}{g} = -\,\overline{2\cos^2(nt+\varphi_v)\left[\frac{1}{24}\,\theta^2 + \frac{1}{2}\,\vartheta^2 - \frac{\ddot{y}}{g}\,\vartheta\right]}\,. \quad . \quad (29\,B)$$

Assuming again that $\ddot{y}$ has an irregular character, the elongations θ_1 and θ_2 of the two pendulums that together give the fictitious pendulum, and, therefore, too the mean elongation ϑ show only one period and, besides, the irregularity of amplitude and phase caused by the accelerations $\ddot{y}$. As we suppose the damped pendulums to adopt the position of the true vertical, the record of the middle pendulum that is made with regard to one of them, does not hide terms as it was the case for the wave-period terms discussed in § 2. Owing to this, the two first terms of the last factor of (29 B) only lead to the normal formula for the reduction to infinitely small amplitude as it is given by (31 A) and (31 B); the term $\frac{1}{8}\vartheta^2$ does not in this case give rise to a contribution to a second order correction as it did in that paragraph. The last term of (29 B) that there has led to another term of the second order correction, does not do so now; as $\ddot{y}$ is assumed to be irregular and ϑ brought about by what preceded, the mean value of their product with the purely periodic first factor may be neglected. So here again our assumptions about the irregularity of the acceleration and about the damped pendulum keeping in the true vertical lead to the result that second order effects are negligible.

3°. *The vertical component of the acceleration.*

For this component we come to the same conclusion as for the horizontal components. We derived the effect brought about by the vertical acceleration $\ddot{x}$ in § 3 and we found formula (38) for the value of the mean disturbance of the phase-velocity. Taking its mean value, we have:

$$\frac{\overline{\dot{\varphi} - n}}{n} = \frac{1}{2}\,\frac{\overline{\ddot{x}}}{g} + \frac{1}{2}\,\overline{\frac{\ddot{x}}{g}\,\cos 2\varphi}\,. \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

The mean value over the duration of the observation of the first term is evidently negligible because the mean value of $\ddot{x}$ is the difference in vertical speed at the beginning and at the end divided by the duration. As the phase-angle φ depends on what preceded and, therefore, not on $\ddot{x}$, the mean value of the product of $\ddot{x}$ and $\cos 2\varphi$ may likewise be neglected and so the second term of (38) also disappears.

These conclusions appear to be confirmed by the absence of fluctuations in the second-mark curve. According to formula (54) such fluctuations are proportional to the phase-disturbance ψ, i.e. to the integral with regard to time of $\dot{\varphi} - n$. So the fact that no fluctuations do occur in the curve proves that the disturbances of $\dot{\varphi} - n$ are so irregular that they do not build up in a second to a perceptibly varying value and as a constant change seems out of the question this strongly points to the effect being negligible.

4°. *The rotation round a vertical axis.*

We need hardly mention that in harbours no perceptible rotations round a vertical axis occur and so there is no question there of any disturbing effect of this kind. We have excepted here the possibility to which the observer has to pay attention, that the ship, if moored by only one cable, may turn around during the observation. If this can not be prevented, the rotation has to be observed and the effect to be taken into account by the applying of formula (47).

Resuming we may state that for harbour observations no second order effects save the amplitude and tilt corrections are likely. This conclusion is founded on the suppositions that the ship's movements are entirely irregular and that the damped pendulums practically adopt the position of the true vertical. If the ship's movements would show periods or if the damping of the damped pendulums is not sufficient to keep them in their equilibrium position, the conclusion is no longer valid.

The above conclusion is confirmed by the checks obtained in those harbours where land-observations have been made in the vicinity or where the observations on board ship have been repeated. These checks do not show traces of systematic effects, but their number is not large. They are listed below.

Stations	Land Observations	Observations on board of a submarine
Den Helder	981.338 (1920)	981.344 (1923)
Sevilla	979.968 (Sans Huelin)	979.967 (1925)
Tunis		979.928 (1923)
		979.925 (1925)
Alexandria		979.428 (1923)
		979.436 (1925)
Sabang (East Indies)	978.183 (1923)	978.181 (1923)
		978.188 (1930)
		978.186 (1930, o/b mailboat)
Belawan (East Indies)		978.072 (1930)
		978.069 (1930, o/b mailboat)
Tanjung Priok (Batavia)		978.170 (1927)
		978.163 (1930)
Surabaya		978.137 (1926)
		978.137 (1927)
		978.137 (1935)
Banda Neira (East Indies)		978.274 (1926)
		978.273 (1929)
Amboina (East Indies)		978.182 (1926)
		978.187 (1929)
Punta Delgada (Azores)		980.134 (1932)
		980.129 (1937)
Horta (Azores)		980.162 (1926)
		980.159 (1937)
Funchal (Madeira)		979.778 (1932)
		979.779 (1934)
Curaçao (West Indies)		978.436 (1926)
		978.439 (1937)
San Francisco	980.005 (C. a. G. S.)	979.998 (1926)

CHAPTER II.

The Determination of the Vertical and Horizontal Accelerations.

§ 6. *Introduction.*

If we apply the formulas of the preceding chapter to the case that the ship's movements are brought about by the waves, we can certainly neglect the fourth power of ε with regard to the unity. Assuming that GERSTNER's or STOKES's theory is valid, we find the wave-length of waves of a period of 4 sec to be 25 m. So the amplitude of the movement is halved for every increase in depth of 25/9 m and this leads to the result that the amplitude at a depth of 20 m is reduced to a hundredfiftieth part of its surface value. So waves of that period are no longer percepticle in a submarine submerged at that depth. In the same way we find that waves of a period of 6 sec are reduced at that depth to one ninth of their surface amplitude and those of a period of 7 sec to one fifth. We may conclude that theoretically waves of periods of less than 7 seconds can usually be neglected when the submarine is submerged at a depth of 20 meters and at greater depth this conclusion is still surer. The writer found these conclusions confirmed by the experience obtained during many voyages; this is e.g. shown by a list of wave-periods T_w for all the earlier expeditions which may be found in the table of page 75 e.s.

So values of $T/T_w = \varepsilon$ of $\frac{1}{4}$ are practically out of the question and usually ε will be smaller than 1/6. We may, therefore, certainly neglect ε^4 with regard to the unity and the formulas (7), (32B) and (44) of the first chapter become:

effect of the horizontal component $\ddot{z}$ perpendicular to the swinging-plane:

$$\ddot{z} = \Sigma\, b \cos n_w t \qquad \frac{\delta g}{g} = \Sigma\, \frac{1}{4}\left[1 + 2\,\frac{1+\varepsilon_b^2}{1-\varepsilon_b^2}\,\varepsilon^2\right]\frac{b^2}{g^2}\,, \tag{50 A}$$

effect of the horizontal component $\ddot{y}$ in the sense of the swinging-plane:

$$\ddot{y} = \Sigma\, a \cos n_w t \qquad \frac{\delta g}{g} = \Sigma\, \frac{1}{4}\,\frac{a^2}{g^2}\,, \quad \ldots\ldots\ldots\ldots \tag{50 B}$$

effect of the vertical component $\ddot{x}$:

$$\ddot{x} = \Sigma\, c \cos n_w t \qquad \frac{\delta g}{g} = -\Sigma\, \frac{1}{8}\left(1 + \frac{1}{4}\,\varepsilon^2\right)\frac{c^2}{g^2} \quad \ldots \tag{50 C}$$

The sum has to be taken over all the waves, each with its own period and amplitude, which together form the total ship's movement.

Practically we may not only neglect ε^4 but also ε^2 with regard to the unity; this will seldom bring about errors in δg of one milligal or more. So, unless the observation has been made at shallow depth where small wave-periods may occur or incase the wave movement has been abnormally large, we may write for

the effect of $\ddot{z}$ $$\frac{\delta g}{g} = \Sigma \tfrac{1}{4} \frac{b^2}{g^2} = \overline{\tfrac{1}{2} \left(\frac{\ddot{z}}{g}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51\,A)$$

the effect of $\ddot{y}$ $$\frac{\delta g}{g} = \Sigma \tfrac{1}{4} \frac{a^2}{g^2} = \overline{\tfrac{1}{2} \left(\frac{\ddot{y}}{g}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51\,B)$$

the effect of $\ddot{x}$ $$\frac{\delta g}{g} = -\Sigma \tfrac{1}{8} \frac{c^2}{g^2} = -\overline{\tfrac{1}{4} \left(\frac{\ddot{x}}{g}\right)^2}. \quad . \quad . \quad . \quad . \quad . \qquad (51\,C)$$

The formulas (51 A) and (51 B) have the advantage to be of the same shape and thus to give the same effect for both the components in the horizontal plane. This enables us to combine them in one resulting effect of the total horizontal component of the acceleration. If its amplitude is e we get a formula of the same shape as (51 A) or (51 B) where b resp a has been replaced by e.

Incase GERSTNER's or STOKES's wave-theory were valid this would lead to a simple formula for the total disturbance if at least the pendulum apparatus would in the same way as the water-particles describe a circular orbit with a constant speed. In that case e would be equal to c and we should obtain for the total effect

$$\frac{\delta g}{g} = \Sigma \tfrac{1}{8} \frac{c^2}{g^2} = \overline{\tfrac{1}{4} \left(\frac{\ddot{x}}{g}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

We see that the effect of the vertical acceleration would then take away half of the effect of the horizontal acceleration. For the computing of the disturbance we should in this case only need data about one of the two components, the vertical acceleration or the horizontal acceleration. If, however, we are not sure of the validity of GERSTNER's assumptions we have to apply the formulas (51) and we need data about both. We shall come back to this point in paragraph 8.

The determination of the vertical component of the acceleration does not present serious difficulties. As we shall set forth in the next paragraph, the time-marks made each second by the chronometer in the records of the main pendulums show fluctuations from which we can derive the vertical accelerations. In the pendulum record these time-marks ought to describe a stroboscopic sine-curve but the presence of vertical accelerations makes them fluctuate about this sine-curve with a period equal to that of the accelerations. As, however, the amplitude of the fluctuations is less at the edge of the record, we can only measure them satisfactorily in the middle half part of it. So we shall use the curve of the chronometer-marks from a phase 30° before the passing of this curve through the axis of the record till a phase 30° past it.

It is important to notice that according to this method we can derive the vertical accelerations for all the pendulum observations that have already been made. So, if we could rely on the validity of GERSTNER's or STOKES's theory we should be able to compute their second order corrections. The further consideration of this matter is the subject of the third chapter.

It is not possible to derive the horizontal component of the acceleration from the records of the pendulum apparatus in its old shape. Theoretically we might do it by measuring the amplitude q_b of the wave-period term of the record of the damped pendulum; according to the formula (6) of p. 9 we should be able to derive from it the amplitude b of the horizontal accelerations. In the first place, however, this would only provide us with data for the component perpendicular to the swinging-plane because the second damped pendulum swinging in a plane parallel to it is not recorded separately, but in the second place ε_b is a rather small quantity, certainly for long wave-periods, and so q_b is likewise small; as a consequence of this the fluctuation of this period is difficult to unravel from the other term of this record corresponding to the free swinging of the gimbals. So, practically, this method is not available.

We have, therefore, to develop a special method for determining the two components of the horizontal accelerations. We shall take this problem up in § 8 and § 10 will describe the new apparatus constructed for this purpose. Here we shall only mention the solution, which consists of installing two pendulums in the apparatus of very long period, i.e. one for the determining of each component. The principle of this is, roughly indicated, as follows. The horizontal accelerations bring about an apparent fluctuation of the vertical and the apparatus in its gimbals follows this fluctuation for the greatest part. The long period pendulums, however, are too slow to do this likewise and so the record of their position with regard to the apparatus provides us with the necessary data for finding the fluctuation of the vertical and, therefore, too the horizontal accelerations.

§ 7. *The determination of the vertical accelerations.*

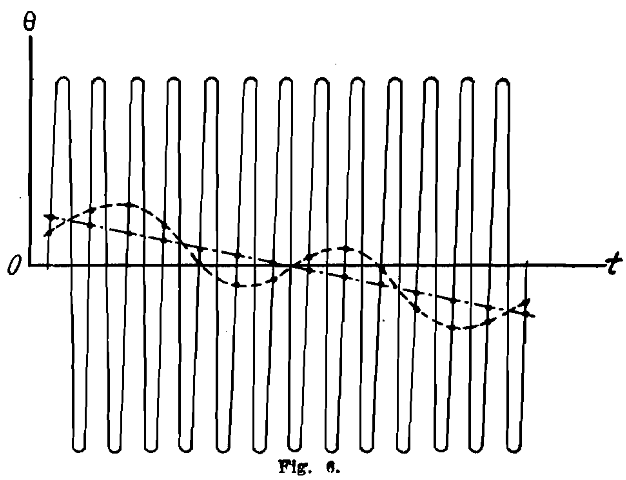

Fig. 6.

Referring to § 3 we found that the ordinate of the record of the fictitious pendulum is given by the formula

$$\theta = a_v \cos \varphi = $$
$$= a_v \cos (nt + \varphi_v + \psi)$$

in which φ_v is the undisturbed phase-angle for $t = 0$ and ψ the disturbance of the phase-angle caused by the vertical acceleration. Other disturbing causes also affect φ but for the fictitious pendulum their effects are of the second order with regard to that of the vertical acceleration.

By means of a shutter operated every second by a chronometer, interruptions are brought about in the record and these second-marks form a stroboscopic curve which is obviously given by

$$\theta' = a_v \cos\left[(n - 2\pi)\ t + \varphi_o + \psi\right] = a_v \cos\left(\varphi_o - \Delta t + \psi\right), \quad (53\,A)$$

in which φ_o is the undisturbed phase-angle for $t = 0$ and

$$\Delta = 2\pi - n = 2\pi \frac{(T-1)}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53\,B)$$

If the vertical acceleration can be neglected, i.e. if $\psi = 0$ the curve is a sine-curve, as indicated in fig. 6 by the point-dot line. As Δ is a small quantity its period is long. Half of the period has often been called the coincidence-interval; it is the time elapsing between an upward and a downward passage of the second-marks through the axis of the record. For the pendulums in use for gravity determinations this period is usually at least 120 seconds and often more; for the pendulums used by the writer for the gravity work at sea it is about 360 seconds. Usually T is greater than one second and so Δ is positive. In that case the phase-angle of the above formula is decreasing with time and the second-marks are receding in the pendulum record.

Incase there are vertical accelerations the angle ψ is no longer negligible. It shows itself in the record as a fluctuation of the second-mark curve of a period equal to that of the accelerations i.e. to that of the waves. So this period, ranging from 6 seconds to 15 seconds, is small with regard to the main period of the curve. A fluctuating curve of this type has been indicated in fig. 6 by the dotted line and clear examples may be found on the Plates III to VII. Neglecting the second power of ψ we find the following formula for the fluctuations with regard to the undisturbed curve

$$F = a_v \cos\left(\varphi_o - \Delta t + \psi\right) - a_v \cos\left(\varphi_o - \Delta t\right) = - a_v \sin\left(\varphi_o - \Delta t\right) \times \psi. \quad (54)$$

By using the formulas for ψ of § 3 we may express F in the vertical accelerations and this gives us the base for deriving these accelerations from the fluctuations F. For this purpose we can only use the middle part of the record where the value of $\sin\left(\varphi_o - \Delta t\right)$ is large enough for getting a sufficient value of the fluctuation F; we shall restrict our measurements of F to those parts where this sine is larger than 0.5 i.e. to values of $\varphi_o - \Delta t$ between 60° and 120° and between 240° and 300°. So we can only use one third part of the total curve but this is quite sufficient for obtaining a result accurate enough for the computation of the second order corrections.

Introducing formula (42) in (54) and substituting $n_w/n = T/T_w = \varepsilon$ and $n^2 l = g$ we obtain

$$F = -\frac{c\, a_v}{2\varepsilon g} \sin\left(\varphi_o - \Delta t\right) \left[\sin n_w t + \frac{\varepsilon}{4\left(1 + \frac{1}{2}\varepsilon\right)} \sin\left(n_w t - 2\Delta t + 2\varphi_o\right) - \right.$$
$$\left. - \frac{\varepsilon}{4\left(1 - \frac{1}{2}\varepsilon\right)} \sin\left(n_w t + 2\Delta t - 2\varphi_o\right)\right].$$

We shall again neglect ε^4 with regard to the unity and find

$$F = -\frac{c\,a_v}{2\,\varepsilon\,g}\sin(\varphi_o - \varDelta\,t)\left[\left\{1 - \tfrac{1}{4}\varepsilon^2\cos 2(\varphi_o - \varDelta\,t)\right\}\sin n_w t + \right.$$
$$\left. + \tfrac{1}{2}\varepsilon\sin 2(\varphi_o - \varDelta\,t)\cos n_w t\right]$$

or

$$F = -\frac{c\,a_v}{2\,\varepsilon\,g}\sin(\varphi_o - \varDelta\,t)\left[1 - \tfrac{1}{8}\varepsilon^2\left\{2\cos 2(\varphi_o - \varDelta\,t) - \right.\right.$$
$$\left.\left. - \sin^2 2(\varphi_o - \varDelta\,t)\right\}\right]\sin(n_w t + o) \quad . \quad . \quad (55\,A)$$

with $\qquad tg\,o = \tfrac{1}{2}\,\varepsilon\sin 2(\varphi_o - \varDelta\,t)$. . . . . (55 B)

So we find the fluctuation F to have a difference in phase of $90° + o$ from the vertical acceleration $\ddot{x}$. According to (55 B) the angle o is small and its value is zero for $\varphi_o - \varDelta\,t$ equal to 0°, 90° or 180°, i.e. for the edges and the axis of the record. Its maximum value is about $\tfrac{1}{2}\,\varepsilon$ or in degrees $90\,\varepsilon°/\pi$.

The amplitude of the fluctuation is given by the whole first part of formula (55 A). For the axis of the record, i.e. for $\varphi_o - \varDelta\,t = 90°$ it is

$$a_{F.o} = \frac{c\,a_v}{2\,\varepsilon\,g}\left(1 + \tfrac{1}{4}\varepsilon^2\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\,A)$$

and so we find the correction for the vertical accelerations as given by formula (50 C) to be

$$\delta\,g = -\tfrac{1}{2}\left(1 - \tfrac{1}{4}\varepsilon^2\right)\varepsilon^2\,\frac{a^2_{F.o}}{a_v^2}\,g. \quad . \quad . \quad . \quad . \quad (56\,B)$$

We can use this formula for the computation of the correction if we measure the amplitude of the fluctuation F in the middle of the record and usually we can omit the factor $1 - \tfrac{1}{4}\varepsilon^2$ because of the smallness of ε. If we determine the mean value of the amplitude of the fluctuations in the whole middle half of the record, i.e. for $\varphi_o - \varDelta\,t$ ranging from 120° to 60°, we have to derive the mean values over this range of the functions $\sin(\varphi_o - \varDelta\,t)$, $\sin(\varphi_o - \varDelta\,t)\cos 2(\varphi_o - \varDelta\,t)$ and $\sin(\varphi_o - \varDelta\,t)\sin^2 2(\varphi_o - \varDelta\,t)$ of formula (55 A). We find successively $\dfrac{3}{\pi}$, $-\dfrac{5}{2\pi}$ and $\dfrac{17}{20\pi}$. Substituting these values in (55 A) we obtain the following formula for the mean amplitude

$$a_{F.m} = \frac{3\,c\,a_v}{2\,\pi\,\varepsilon\,g}\left(1 + \frac{39}{160}\,\varepsilon^2\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (57\,A)$$

and for the correction

$$\delta\,g = -\frac{\pi^2}{18}\left(1 - \frac{19}{80}\,\varepsilon^2\right)\varepsilon^2\,\frac{a^2_{F.m}}{a_v^2}\,g. \quad . \quad . \quad . \quad (57\,B)$$

For normal values of ε we can omit the factor and we have

$$\delta\,g = -\frac{\pi^2}{18}\,\varepsilon^2\,\frac{a^2_{F.m}}{a_v^2}\,g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57\,C)$$

This formula has been applied for the gravity observations at sea of the Netherlands Geodetic Commission. In the middle half of one of the records of the two fictitious pendulums, i.e. the part of the record comprised between two lines on both sides parallel to the axis of the record and halfway between the axis and the edges, the amplitudes of the fluctuations of the second-marks were measured. Their mean value was divided by the amplitude of the pendulum record at that place and the square of that ratio was taken. This was done for ten different parts of the second-mark curve, chosen at random in the record, and the mean of these squares was introduced in formula (57 C). In order to find ε, the value of T_w had to be determined. This was done by counting the number of wave-lengths in one minute of the second-mark curve, estimating also fractions when it was not a whole number. This counting was executed for the same ten parts where also the mean amplitudes of the fluctuations had been determined and from the mean value of these numbers T_w was derived. In this way the data have been obtained for applying formula (57 C) for all those records that show a strong fluctuation of the second-marks.

In case the amplitude of the fluctuation did not exceed 0.6 mm, only four parts of the second-marks curve have been measured for obtaining the mean value. As the correction in this case is only about 5 to 9 milligal this gives a more than sufficient accuracy. For those records where the amplitude was not more than 0.2 mm the correction has been neglected; it amounts in this case to a value of between 0.5 and 1 mgal.

The above method is accurate in case the ship's movements have only one period. In case the waves have more than one period, terms of the same shape must be added to the formulas (56 B), (57 B) and (57 C); the corrections then are the sum of as many terms as there are periods and in each term the amplitude $a_{F.o}$ resp $a_{F.m}$ is the amplitude of the fluctuation of the period corresponding to that term. For applying the formulas in that case we should have to separate the different harmonic terms of the fluctuation of the second-marks in order to get the values of the amplitudes of all the terms. Such a treatment, however, would be complicated and laborious and it would scarcely be justified for the computation of a small correction. So we have not done it for the gravity records of the Netherlands Geodetic Commission and we have kept to the above formulas and method for the single harmonic fluctuation by estimating as well as possible the amplitude and the period of the composite fluctuation as if it were a single periodic one. No investigation has yet been made of the errors thus incurred; probably they are not large.

§ 8. *The determination of the horizontal accelerations.*

The problem to determine the horizontal accelerations is an important one for the gravity work at sea. Not only the right correction for the second order effects of the future observations depends on its solution, but also the question whether we should be entitled to apply the assumptions of GERSTNER'S or STOKES'S theory for correcting the past observations. The solutions given by seismology are not available because of the mobility of the ship in which the measurements have to be made. In his paper in the M.N.R.A.S. of Sept 1937 BROWNE indicated a method for determining the horizontal accelerations by observing the position of the horizon with regard to the apparent direction of the vertical. In this way we should be able to determine the apparent fluctuation of the vertical and thus to find the horizontal accelerations. This, however, is impossible in a submerged submarine and so another method had to be developed.

We mentioned already in the introduction to this chapter that the record of the angle β of the damped pendulum can not well serve for this purpose although the record contains a term depending on the horizontal accelerations; for long waves this term is too small to give sufficient accuracy. So we can not solve our problem satisfactorily by making a record of the two damped pendulums of which the apparatus in its old shape only records the one swinging in a plane perpendicular to the swinging-plane of the main pendulums.

A better solution was obtained by introducing in the apparatus two new pendulums of very long period, one for each component of the horizontal acceleration and each swinging in a vertical plane parallel to this component. By recording the long period pendulum with regard to the apparatus we obtain the difference in movement of this pendulum and of the apparatus in its gimbals. As the period of the first is considerably larger than that of the waves and that of the gimbals smaller, the forced swinging brought about by the horizontal accelerations for the long period pendulum and for the gimbals have different amplitudes and so their differential movement shows a clear term of the wave-period. From the amplitude of this term we may derive the horizontal accelerations.

For obtaining the formulas we have to solve the equations of movement for the slow pendulum and for the gimbals under the effect of the ship's movements. Leaving for a moment aside the effect of the vertical accelerations, both equations have the shape

$$\ddot{\theta}_s + n_s^2\,\theta_s - \frac{n_s^2}{g}\,\ddot{y} = 0. \qquad \ldots\ldots \quad (58\,A)$$

If we may put

$$\ddot{y} = a \cos n_w t,$$

the solution is

$$\theta_s = a_s \cos(n_s t + \varphi_s) - \frac{n_s^2}{n_w^2 - n_s^2}\,\frac{a}{g}\,\cos n_w t, \quad \ldots \quad (58\,B)$$

The first term of this formula corresponds to the normal swinging of the slow pendulum resp. the gimbals and the second to the forced swinging in the period of the waves. As we mentioned, the photographic record of the slow pendulum gives the difference of these terms for the slow pendulum and for the gimbals. So, if we keep to the sub-index s for the slow pendulum and if we use the sub-index a for the gimbals, the record gives an amplitude A for the wave-period term

$$A = \left(\frac{n_s^2}{n_w^2 - n_s^2} - \frac{n_a^2}{n_w^2 - n_a^2}\right)\frac{a}{g} = F\,\frac{a}{g} \quad \ldots \quad (59\,A)$$

We might call F the enlargement factor as it is the ratio of the amplitude of the record and the amplitude a/g of the fluctuation of the vertical. By introducing

$$\delta_s = \frac{n_s}{n_w} = \frac{T_w}{T_s} \quad \text{and} \quad \varepsilon_a = \frac{n_w}{n_a} = \frac{T_a}{T_w}, \quad . \quad . \quad (59\,B)$$

the formula for F becomes

$$F = \frac{\delta_s^2}{1-\delta_s^2} + \frac{1}{1-\varepsilon_a^2} = \frac{1-\delta_s^2\,\varepsilon_a^2}{(1-\delta_s^2)\,(1-\varepsilon_a^2)} = \frac{1-\delta_s^2\,\varepsilon_a^2}{(1-\delta_s\,\varepsilon_a)^2-(\delta_s-\varepsilon_a)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59\,C)$$

The quantities δ_s and ε_a are both small. Their product $\delta_s\,\varepsilon_a = T_a/T_s$ is a constant depending on the periods of the gimbals and the slow pendulum. The enlargement factor F however varies with the period of the waves and so, if more than one wave-period is present, the enlargement differs for the various terms. This is a disadvantage for our purpose of deriving from the record the amplitudes a of the different terms of the horizontal acceleration For a fairly large range of wave-periods, however, we can make F practically constant by making $\delta_s = \varepsilon_a$ for the mean value of T_w of the range. In that case the second term of the denominator of formula (59 C) is small with regard to the first term. Incase we may neglect it, the formula for F becomes

$$F = \frac{1+\delta_s\,\varepsilon_a}{1-\delta_s\,\varepsilon_a} = \frac{T_s+T_a}{T_s-T_a}, \quad . \quad . \quad . \quad . \quad (60\,A)$$

which is independent of the wave-length.

The condition $\delta_s = \varepsilon_a$ gives $T_s = T_w^2/T_a$ and so we can realize it by a suitable adjustment of the period of the slow pendulum. As T_a is between 3 and 4 sec and the mean wave-period about 10 sec, we find a value for T_s of about 25—30 sec.

Incase we are not sure that T_w is inside the tolerated range, we can make use of formula (59 C) in its first shape which, by introducing the periods, becomes

$$F = \frac{1}{1-T_a^2/T_w^2} + \frac{1}{T_s^2/T_w^2-1}, \quad . \quad . \quad . \quad (60\,B)$$

or we may derive from (59 C) in its third shape

$$\frac{1}{F} = \frac{T_s-T_a}{T_s+T_a} - \frac{(T_w/T_m-T_m/T_w)^2}{T_s/T_a-T_a/T_s}, \quad . \quad . \quad . \quad (60\,C)$$

with

$$T_m = \sqrt{T_a\,T_s}$$

in which the first term, corresponding to (60 A), is constant for a certain apparatus and likewise the denominator and the value of T_m in the small second term. In this shape it is useful for the making of a table for F for a certain apparatus, giving the values of F for different values of T_w. We can also apply it for determining the range of values of T_w where the second term is negligible and where we, therefore, may use formula 60 A.

For determining the second order corrections for the horizontal accelerations we can now apply (50 A) and (50 B) by introducing

$$\frac{a}{g} = \frac{A_y}{F_y}, \quad \ldots \ldots \ldots \quad (60\,D)$$

$$\frac{b}{g} = \frac{A_z}{F_z}, \quad \ldots \ldots \ldots \quad (60\,E)$$

where A_y and A_z are the amplitudes A derived from the records of the slow pendulums swinging in the XY and XZ planes.

Before further discussing these results and the methods for making the necessary measurements in the record, we shall first have to consider the matter whether we were justified in leaving aside the effect of the vertical accelerations. If we take them into account, the equation of motion for the slow pendulum and for the gimbals becomes, using for both the index s

$$\ddot{\theta}_s + n_s^2\,\theta_s - \frac{n_s^2}{g}\,\ddot{y} + \frac{n_s^2}{g}\,\theta_s\,\ddot{x} = 0. \quad \ldots \quad (61)$$

As the last term contains the factor θ_s, we may surmise at once that its effect is indeed much smaller than that of the third term, but we shall make sure of its size by developing the solution. Assuming in connection with the theory of the wave-movements that the vertical accelerations differ $\frac{1}{2}\pi$ in phase from the horizontal accelerations, we shall put

$$\left.\begin{aligned} \ddot{y} &= a\,\cos\,n_w\,t, \\ \ddot{x} &= c\,\sin\,n_w\,t. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (62)$$

Introducing

$$\theta_s = \theta_s' - \frac{n_s^2}{n_w^2 - n_s^2}\,\frac{a}{g}\,\cos\,n_w\,t, \quad \ldots \ldots \quad (63\,A)$$

we find

$$\ddot{\theta}_s' + n_s^2\,\theta_s' - \frac{n_s^4}{2\,(n_w^2 - n_s^2)}\,\frac{a}{g}\,\frac{c}{g}\,\sin\,2\,n_w\,t + n_s^2\,\theta_s'\,\frac{c}{g}\,\sin\,n_w\,t = 0.$$
$$\ldots \ldots \ldots \ldots \quad (63\,B)$$

So the third term of (61) has disappeared by this substitution but another term containing ac/g^2 i.e. one order higher, has been introduced. We can make it disappear again by putting

$$\theta_s' = \theta_s'' - \frac{n_s^4}{2\,(4\,n_w^2 - n_s^2)\,(n_w^2 - n_s^2)}\,\frac{a}{g}\,\frac{c}{g}\,\sin\,2\,n_w\,t, \quad . \quad (63\,C)$$

and this introduces in (63 B) a term of the third order. If we restrict our solution of θ_s, however, to terms of the first and second power of θ_s, a/g or c/g, we may neglect this term and we get

$$\ddot{\theta}_s'' + n_s^2\,\theta_s'' + n_s^2\,\theta_s''\,\frac{c}{g}\,\sin\,n_w\,t = 0. \quad \ldots \quad (63\,D)$$

So the equation of θ_s'' contains only the disturbance term caused by the vertical accelerations and for its solution we may apply the method of § 3.

Putting

$$\theta_s'' = a_s \cos \varphi,$$

$$\dot{\theta}_s'' = - a_s n_s \sin \varphi,$$

and applying formulas (4 A) and (4 B) of page 5 of the publication of 1929 we find

$$\dot{\varphi} - n_s = n_s \frac{c}{g} \sin n_w t \cos^2 \varphi, \quad . \quad . \quad . \quad . \quad . \qquad (64\,A)$$

$$\dot{a}_s = n_s a_s \frac{c}{g} \sin n_w t \sin \varphi \cos \varphi. \quad . \quad . \qquad (64\,B)$$

If we indicate the disturbances of the phase-angle and the amplitude by ψ and δa_s, so

$$\varphi = n_s t + \varphi_0 + \psi,$$

$$a_s = a_0 + \delta a_s,$$

φ_0 and a_0 being the values of φ and a_s for $t = 0$, we find ψ and δa_s by integrating (64). In doing this we may, in connection with the degree of accuracy adopted, substitute for φ in the right member of the formulas (64) the value $n_s t + \varphi_0$. We can use the results thus obtained for computing θ'' by means of

$$\theta_s'' = a_s \cos (n_s t + \varphi_0 + \psi) = a_0 \cos (n_s t + \varphi_0) -$$
$$- \psi a_s \sin (n_s t + \varphi_0) + \delta a_s \cos (n_s t + \varphi_0)$$

and combining this with the two other parts of θ given by (63 A) and (63 C) we finally find

$$\theta_s = a_0 \cos (n_s t + \varphi_0) - \frac{n_s^2}{n_w^2 - n_s^2} \frac{a}{g} \cos n_w t -$$
$$- 2 \frac{n_s^3 a_s}{n_w (n_w^2 - 4 n_s^2)} \frac{c}{g} \cos n_w t \sin (n_s t + \varphi_0) +$$
$$+ \frac{n_s^2 a_s}{n_w^2 - 4 n_s^2} \frac{c}{g} \sin n_w t \cos (n_s t + \varphi_0) +$$
$$- \frac{n_s^4}{2 (4 n_w^2 - n_s^2)(n_w^2 - n_s^2)} \frac{a}{g} \frac{c}{g} \sin 2 n_w t.$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (65)$$

For the gimbals we should have to change the index s in a.

The formula shows that our surmise was right and that we may indeed neglect the three last terms brought about by the vertical accelerations. This is valid as well for the slow pendulum for which n_s is smaller than n_w as for the gimbals for which n_a is larger than n_w. The product of a_s resp a_a and c/g is so small that in both cases the terms may be left away and the same is true for the last term because of the smallness of the product of a/g and c/g. Incase we should have assumed another phase-difference of $\ddot{x}$ and $\ddot{y}$ than that assumed in the formulas (62) we should have found a somewhat different formula for θ_s but the conclusion about the last three terms would have been the same.

Returning to the results found in formulas (60) for the enlargement factor

F, we saw that it depends on the period of the slow pendulum and of the gimbals. As we have to determine the components of the horizontal accelerations parallel and perpendicular to the swinging-plane of the main pendulums, we have to apply these formulas for both directions. Although not strictly necessary, we can make the computations easier by making the enlargement factors F_y and F_z the same in both senses. This leads us to making the periods equal for the two slow pendulums and likewise for the gimbals in its two senses. So, besides constructing the new apparatus containing the slow pendulums, we have also renewed the gimbals. In their new state shown by Plate II the periods in the two directions can be adjusted separately; both have been adjusted at about 3.6 sec.

For determining the horizontal accelerations by means of the formulas (59 A) and (60) we have to measure the amplitudes of the wave-period terms of the records of the slow pendulums. This is rendered difficult by the many terms of different period of these records. Besides the wave effect which itself may consist of more than one term, it shows the free swinging in their own periods of the slow pendulum as well as of the gimbals; this corresponds to the first term of formula (58 B) applied to both. As Plate III shows clearly, the long wave of the slow pendulum is especially troublesome for the reading of the other periods. It would, therefore, be of advantage to introduce a damping of this pendulum but the question is whether we can do this without disturbing its main function. Two ways of damping appear feasible, a damping with regard to the apparatus and a damping with regard to space. The last is much more difficult to realize than the first, but if necessary it could be done. So we shall investigate whether a damping with regard to the apparatus may be allowed and only if the result is negative we shall look at the second possibility. We shall see that this will not be necessary as the first system of damping does not present difficulties.

The question to be answered is whether a damping of the slow pendulum with regard to the apparatus does not disturb too much the wave-terms of their relative position from which we derive the horizontal acceleration. For this purpose we have again to solve the equation of motion of the slow pendulum, taking into account this time a damping term. We may assume that this term is proportional to the rate of change of the angle $\theta_s - \theta_a$, i.e. of the relative angle of elongation of the slow pendulum with regard to the apparatus in the gimbals. So the equation (58 A) becomes

$$\ddot{\theta}_s + 2k\dot{\theta}_s + n_s^2\theta_s - 2k\dot{\theta}_a - n_s^2\frac{a}{g}\cos n_w t = 0. \quad . \quad . \quad (66\,A)$$

For the equation of motion of the gimbals we find an equation of the same shape in θ_a but the damping-constant k is inversely proportional to the moments of inertia round the axis of rotation and as this moment is many thousands times greater for the apparatus in the gimbals than for the slow pendulum, we may neglect this damping term in the gimbal equation. So for the gimbals we may continue to use the solution given by (58 B) and so

$$\theta_a = a_a \cos (n_a t + \varphi_a) - \frac{n_a^2}{n_w^2 - n_a^2} \frac{a}{g} \cos n_w t. \quad . \quad . \quad (66\,B)$$

Substituting this value for θ_a in $(66\,A)$ we obtain

$$\ddot{\theta}_s + 2\,k\,\dot{\theta}_s + n_s^2\,\theta_s + 2\,k\,n_a\,a_a \sin (n_a t + \varphi_a) -$$

$$-2\,k\,\frac{n_a^2\,n_w}{n_w^2 - n_a^2}\,\frac{a}{g} \sin n_w t - n_s^2\,\frac{a}{g} \cos n_w t = 0. \quad . \quad (66\,C)$$

The solution does not give difficulties. Subtracting the value of θ_a from the result, we find after taking together the terms of the same period

$$\theta_s - \theta_a = a_s\,e^{-kt} \cos (n_s' t + \varphi_s) - F\,\frac{a}{g} \cos \gamma \cos (n_w t + \gamma) -$$

$$- a_a \cos \mu \cos (n_a t + \varphi_a + \mu), \quad . \quad . \quad . \quad (67\,A)$$

in which F is the enlargement factor if there is no damping, given by formula $(60\,B)$ or eventually $(60\,A)$. Further

$$n_s' = \sqrt{n_s^2 - k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67\,B)$$

$$tg\,\gamma = \frac{2\,k\,n_w}{n_w^2 - n_s^2} = \frac{k\,T_w}{\pi\,(1 - \delta_s^2)}, \quad . \quad . \quad (67\,C)$$

$$tg\,\mu = \frac{2\,k\,n_a}{n_a^2 - n_s^2} = \frac{k\,T_a}{\pi\,(1 - \delta_s^2\,\varepsilon_a^2)} \quad . \quad . \quad (67\,D)$$

The first term of $(67\,A)$ corresponds to the free swinging of the slow pendulum and we see that this term now indeed contains a damping-factor as we intended. The second term is the wave-period term, i.e. the term we want to use for the determination of the horizontal acceleration, and if we compare its amplitude with the amplitude Fa/g of formula $(59\,A)$ for the case there is no damping, we see that the effect of the damping is a factor $\cos \gamma$; the phase of this term is shifted over the angle γ. As for all practical cases γ is a small angle, the cosine is so near to the unity that we may omit this factor in the amplitude formula. For the new apparatus constructed in the Netherlands the damping of the slow pendulums e.g. is such that the amplitude of their free swinging sinks to two thirds of its value during one complete period of 26 sec and this is more than sufficient for a satisfactory damping; as Plate IV shows, the initial free swinging disappears in a few minutes and it practically keeps away during the whole further observation. This damping corresponds to a value of k of 0.0156 and a value of $\cos \gamma$ of 0.9983; γ is about 3°. As the determination of the horizontal accelerations never attains a precision of 0.0017 we may leave away the factor $\cos \gamma$ from the formulas and we can keep to the formulas (60) for the enlargement factor F.

The phase of the wave-period term of the slow pendulum record is that of the horizontal acceleration increased by the small angle γ. Assuming that, according to GERSTNER's or STOKES's wave-theory, the vertical accelerations differ $\frac{1}{2}\,\pi$ in phase from the horizontal accelerations, we may conclude from formulas (40) and $(55\,A)$ that the fluctuations F of the second-marks in the pendulum-records have nearly the same or opposite phase, i.e. the phase is that of the horizontal accelerations increased by the small and fluctuating angle σ as given by (55 B), and eventually by π. So in that case the phase of the fluctuations of the wave-period terms of the slow pendulum records differs only by the small angle $\gamma - \sigma$ from that of the fluctuations of the second-marks in the pendulum-records, eventually from the opposite phase.

The last term of formula (67 A) shows in comparizon with the first term of the formula (66 B) that the effect of the damping on the term corresponding to the free swinging of the gimbals is a factor $cos\,\mu$ in the amplitude and a shift of the phase over the angle μ. As μ also is a small angle, this effect is negligible in the record. This term of the record has no importance for our purpose to determine the horizontal acceleration. If we should wish to decrease it also in order to make the measuring of the amplitude of the wave-period term more easy, we should have to introduce a suitable damping of the gimbal movement. Because of the great moment of inertia of the gimbals and the apparatus and also because of the period being shorter than that of the slow pendulum, such a damping can never be as successful as for this pendulum. It will be very difficult to make this amplitude disappear entirely from the records of the slow pendulums. If we should wish to get a record free from it, we could introduce a new damped pendulum of a period differing enough from that of the gimbals for avoiding interference, and make a record of the position of the slow pendulum with regard to this new pendulum. This would, however, bring along the introduction in the apparatus of at least two new pendulums, one for each direction, and so the apparatus would again become more complicated. This has withheld the writer from doing so. He neither wished to use the damped pendulums that are already present for this purpose, as in this case we should not be able because of their periods being too small, to realize the advantage mentioned on page 33 of making the enlargement factor F practically constant for a wide range of wave-periods. So he has adopted the solution as it has been described. By drawing a pencil-line through the record, we can with sufficient accuracy straighten the curve and eliminate the gimbal-period effect from the record for those parts where it is troublesome.

If the waves have more than one period, we should, strictly speaking, have to separate the different harmonic terms and measure the amplitudes of each term. This would be a complicated procedure and up to now the writer has not done it, but eventually it must be applied if we otherwise can not get sufficient accuracy. This point brings us back to the important central question in which way we can best make the measurements for the computation of the combined second order correction for the vertical and horizontal accelerations. As far as the writer can see, we can follow three ways.

In the first place we can measure both the vertical and the horizontal accelerations according to the methods indicated in this and the preceding paragraphs, and we can then apply the formulas (51) or, if necessary, (50) discussed in the introductory paragraph of this chapter.

In the second place we can use a few simple and clear parts of the slow pendulum records for measuring the horizontal accelerations and compare them to the vertical accelerations measured at the same places of the record. So we have to choose these parts in those places where the vertical accelerations can be determined, i.e. where the second-marks cross the middle part of the record. From these results we can conclude whether the mean of the squares of the vertical and horizontal accelerations are equal according to GERSTNER's wave-theory. If this should be the case we can adopt the relation as valid for the whole record and we can then determine the combined second order effect by means of the simplified formula (52). For doing this we only need the determination of the vertical acceleration for the whole record. This method saves a good deal of work as we now have only to measure the two curves of the slow pendulums for a few small parts.

In the third place we can follow the same line in the beginning as the preceding method but once the equality of the mean values of the squares of the horizontal and vertical accelerations adopted, we can derive the combined correction from the two slow pendulum records and omit the measuring of the vertical accelerations over the whole record. This saves less work than the second method but we have the advantage that the amplitudes to be measured in the records of the slow pendulums are larger than the fluctuations of the second-marks from which we derive the vertical accelerations. On the other hand the last are free from terms of the gimbal-period.

In the actual state of affairs the writer should not wish to give an opinion about the best choice between these three methods as he does not yet dispose of enough material.

We may mention here a last question in connection with the subject of this paragraph, viz that of the accuracy needed in these determinations of the accelerations and the second order corrections. If we adopt a precision of one or two percent in the measurements, this corresponds to two or four percent in the corrections and this accuracy agrees with the use of the simplified formulas (51). It is usually more than sufficient and will seldom give rise to errors in the second order correction of one milligal. Only in exceptional cases of very strong wave-movements, larger errors will occur. If we should wish to try to obtain a higher accuracy in the results for the corrections, we should also have to take into account the smaller terms, i.e. we should have to apply the formulas (50). This would bring along complications in our considerations about the three methods of computation, because the formulas (50 A) and (50 B) for the two components of the horizontal acceleration are then no longer of the same shape. We shall not enlarge on these problems as it does not seem likely that they will ever occur. It certainly appears desirable to avoid them by trying to reduce the ship's movements in unfavourable circumstances by submerging to greater depth.

There can be no doubt that the study of the second order corrections has still better revealed the great advantage of making these gravity determinations in submerged submarines. In this way it is not only possible to reduce the ship's movements but also to eliminate the smaller wave-periods that do not differ sufficiently from the gimbal-period for avoiding too large disturbing effects. The gimbal-period itself could not be diminished too much without getting too near the periods of the main pendulums.

§ 9. *Summary of the formulas for determining the accelerations and the second order corrections.*

For determining the amplitude c of the vertical accelerations $\ddot{x}$ (form 40) and the corresponding term of the second order corrections δg_1, we measure the amplitude a_F of the fluctuations of the second-marks in the records of the main pendulums. Assuming that the period T_w of the waves is more than 5 sec and that the accuracy in c and δg_1 need not be greater than one percent, we may omit the factor beween the square brackets of (55 A) and we obtain

$$\frac{c}{g} = 2\,\frac{T}{T_w}\cdot\frac{a_{Fo}}{a_v}, \quad \ldots \ldots \ldots \quad (68\,A)$$

and

$$\delta g_1 = -\tfrac{1}{2}\,\frac{T^2}{T_w^2}\left(\frac{a_{Fo}}{a_v}\right)^2 g. \quad \ldots \ldots \quad (68\,B)$$

$T =$ period of the main pendulum (double swing) and so $T \approx 1.00$,
$a_v =$ amplitude of the main pendulum,
$a_{Fo} =$ amplitude of the fluctuations measured in the axis of the record.
If we have measured a_F for a phase-angle $\varphi_o - \Delta t$ of the second-marks (for the axis $\varphi_o - \Delta t = 90°$) we find a_{Fo} by means of

$$a_{Fo} = a_F \ cosec\ (\varphi_o - \Delta t) \quad \ldots \ldots \quad (68\,C)$$

and if we have determined a_{Fm} as the mean value of the fluctuations a_F over the middle half part of the record, i.e. for $\varphi_o - \Delta t$ ranging from 60° to 120°, we may derive a_{Fo} from

$$a_{Fo} = \frac{\pi}{3}\,a_{Fm}. \quad \ldots \ldots \ldots \quad (68\,D)$$

For determining the amplitudes a and b of the two horizontal components of the acceleration (form. (3) and (23)) and their combined second order effect δg_2, we measure the amplitudes a_y and a_z of the records of the slow pendulums. As the focal distance of the recording lens k (fig. 7) is 1.00 m and as the arrangement of the prisms g_1 and g_2 is such that the light-rays get only half of the angular movement of the slow pendulums, the amplitudes of these pendulums are 0.02 a_y and 0.02 a_z. So we obtain from formulas (60B), (60D) and (60E)

$$\frac{a}{g} = 0.02\,\frac{a_y}{F_y}, \qquad F_y = \frac{1}{1 - T_a^2/T_w^2} + \frac{1}{T_{sy}^2/T_w^2 - 1} \quad (69\,A)$$

$$\frac{b}{g} = 0.02\,\frac{a_z}{F_z}, \qquad F_z = \frac{1}{1 - T_b^2/T_w^2} + \frac{1}{T_{sz}^2/T_w^2 - 1} \quad (69\,B)$$

or we may use formula (60 C) for computing F_y and F_z

$$\frac{1}{F_y} = \frac{T_{sy} - T_a}{T_{sy} + T_a} - \frac{(T_w/T_{my} - T_{my}/T_w)^2}{T_{sy}/T_a - T_a/T_{sy}} \qquad (T_{my} = \sqrt{T_a T_{sy}}) \ . \quad (69\,A^1)$$

$$\frac{1}{F_z} = \frac{T_{sz} - T_b}{T_{sz} + T_b} - \frac{(T_w/T_{mz} - T_{mz}/T_w)^2}{T_{sz}/T_b - T_b/T_{sz}} \qquad (T_{mz} = \sqrt{T_b T_{sz}}) \ . \quad (69\,B^1)$$

T_{sy} and T_{sz} = periods of the slow pendulums in Y and Z direction,
T_a and T_b = periods of the gimbals in Y and Z direction.
If both periods are the same for both directions, we have $F_y = F_z$.

According to the formulas ($50\,A$) and ($50\,B$) we obtain, if we express δg_2 in mgal and g in gal,

$$\delta g_2 = 0.1\, g \left\{ \left(\frac{a_y}{F_y}\right)^2 + \left[1 + 2\,\frac{T_w^2 + T_b^2}{T_w^2 - T_b^2} \cdot \frac{T^2}{T_w^2}\right] \left(\frac{a_z}{F_z}\right)^2 \right\} \ . \quad (69\,C)$$

or, if we may omit the factor between the square brackets

$$\delta g_2 = 0.1\, g \left\{ \left(\frac{a_y}{F_y}\right)^2 + \left(\frac{a_z}{F_z}\right)^2 \right\} \ . \ . \ . \ . \ . \quad (69\,D)$$

If this is the case and if, according to GERSTNER's and STOKES's wave-theories, the amplitudes of the vertical and horizontal accelerations of the ship are equal, so $c^2/g^2 = a^2/g^2 + b^2/g^2$, the total second order correction $\delta g = \delta g_1 + \delta g_2$ is

$$\delta g = -\delta g_1 = \tfrac{1}{2}\,\delta g_2 \ . \ . \ . \ . \ . \ . \ . \ . \quad (69\,E)$$

which we may derive from ($68\,B$) of from ($69\,D$). Otherwise we have to use both formulas and to add their values for finding the total δg.

§ 10. *The apparatus for the determination of the horizontal accelerations; the gimbals.*

As we have seen in the preceding paragraph, the problem to be solved was to construct an apparatus with two slow pendulums of a period of 25 — 30 sec, swinging in two planes parallel and perpendicular to the swinging-plane of the main pendulums. If possible this apparatus had to be made in such a way that it could be combined with the existing pendulum apparatus and that the slow pendulums could be recorded by means of the same apparatus recording the main pendulums. Considering the arrangements of the old apparatus it was obvious to try to use the free space between the pendulum part and the recording part. As this space has a height of only eight centimeters, this gave a narrow limit for the vertical dimensions of the new instrument. We could of course have decided to modify the pendulum apparatus for increasing the available space, but the writer wished, if possible, to avoid this because of the complications this would have brought along for the introduction of the new device in the different existing pendulum apparatuses. As the attempt has been successful, there is no reason to come back on this decision.

The slow pendulums with accessories.

In connection with these considerations the slow pendulums have been constructed in the shape of horizontal brass rods of a length of 25 cm. The general arrangement of these pendulums is clearly seen in fig. 7 and on Plate I while Plate II shows how the new apparatus fits in the old; one of the pendulums is parallel to the front of the whole apparatus and fills up the back of the free space between the upper and lower parts, while the other is perpendicular to this direction and is located in the right part of this space. They have originally been projected to have as little damping as possible because the writer was not yet sure that such a damping with regard to the old pendulum apparatus would not unfavourably affect their working. As we have seen in the preceding paragraph this is unlikely and so afterwards a damping device was introduced for suppressing the free swinging of the slow pendulums. In connection with the original idea of reducing the damping, the vertical cross-section of the new pendulums was chosen higher than broad, viz. 2.5 cm high and 0.5 cm broad, with pointed upper and lower edges.

In order to obtain the long periods that are wanted, the distance from the centre of gravity of the pendulum to the axis of rotation must be about 20 μ. It would be difficult to make the pendulums so accurately that this claim would be fulfilled straight away and so the pendulums have each been provided with a screw in the middle that can be screwed up or down for adjusting the centre of gravity in the desired position. The lateral position of the centre of gravity in the sense of the horizontal axis of the pendulum is still more difficult to ensure correctly without an adjusting arrangement; a shift of one μ in this sense gives already a deviation of the pendulum from the horizontal position of 3°. So each pendulum has been provided with two small screws at each end of the rod that can be screwed in or out in a horizontal sense and this allows a sufficiently accurate adjustment of the centre of gravity in a horizontal sense.

For a good functioning of the pendulums we must dispose of high quality knife-edges. For the first apparatus that was constructed, the writer had suspended each of the pendulums by two tiny springs, but this construction has not proved satisfactory. The writer has tried it during the voyage with Hr. Ms. Submarine O 12 in the autumn of 1937 from Curaçao to Holland and though it has not quite failed, only a small part of the record could usually be used; for the other parts it was irregular and evidently disturbed or it disappeared entirely. The trouble was that the elasticity of the springs was not sufficiently constant because of their distortion and other too large deformations. They could not be made so thin that their elasticity would not play a part because then they would not be strong enough for carrying the weight of the pendulum. For the second and final edition of the apparatus, constructed after his coming back from this trial trip, the writer went back to knife-edges. For this solution he feels again indebted to Mr. Browne who pointed out to him that the same problem had been coped with for the construction of sensitive balances and that for this purpose knife-edges had proved entirely satisfactory. In the present apparatus the knife-edges are made of steel and the supporting planes on both sides of the pendulums of *vidia,* a modern hard alloy. These planes have been checked for being in the same plane by optical means, i.e. by covering them with an optically plane piece of glass and by observing the interference figures. The knife-edges have been checked by putting them on these planes and lighting them from behind. The records given by the free swinging of the slow pendulums show accurate sine-curves and this proves the parts of the knife-edges that come into contact with the supporting planes to be satisfactory.

The knife-edges must of course be protected from undue wear by lifting them from the planes during the intervals between the observations. The lever-mechanism for this purpose is handled by means of a little wheel with projecting spokes, which can be turned around so lightly that there is no objection to do it when the pendulum apparatus is in action and the gimbals unclasped. The two spoke-wheels are visible on Plate I at the left and the right of the picture; the lifting levers can be noticed in the middle of the left and right glass windows.

Another pair of levers can clasp the slow pendulums for fixing them when out of action. For operating these levers we have first to lift the main pendulums as well as the others from their knife-edges because the fixing can not be done without a risk of shocks. For the left pendulum this lever can be detected in Plate I in its lowest position just on the left of the lifting lever. The handles for operating them are at the back of the pendulum boxes.

For making the pendulums ready for work we begin by unclasping them and then, after preparing the whole pendulum apparatus for action, we lower them on their knife-edges. It is, however, difficult to do this without giving them amplitude and so we need a device for taking this amplitude as well as possible away without disturbing the apparatus. For this purpose each slow pendulum has been provided with a pair of balanced levers that when unfastened, lightly press on the upper and lower sides of the pendulum; this small pressure is exerted by the weight of the levers. In this position they are visible in Plate I in the window to the right; they keep the pendulum in its equilibrium position. The levers can be spread apart by means of a small spring which can be released from the outside of the pendulum box. We see this release mechanism in Plate I above the spoke-wheels of both slow pendulums. For the left pendulum the levers are spread apart and we see the pendulum swinging; the right part is higher than the left in the picture. We may notice the lower lever of this pendulum below it at the left side of the window.

For deriving the second order correction from the record we need the period of the slow pendulums. Because of the short distance between the centre of gravity and the axis of rotation, slight deformations of these pendulums may already cause a considerable change of period and so the writer has been afraid that such changes might occur e.g. by irregularities of temperature or by strong ship's movements during the interval between the observations. This would imply the need for frequently redetermining the period of the slow pendulums, if necessary even for each observation or part of it. These considerations have led him to introduce the following mechanism.

Each pendulum has been provided with two grooves, one on each side of the knife-edge at a distance of 1.08 cm away from it. Through small tubes we are able to drop light balls in these grooves and each weight must cause a deviation of the pendulum from its equilibrium which can be determined by measuring the deviation of the axis in the record. If we choose the smallest type of balls of a ball-bearing, which have a diameter of 1.60 mm and a weight of 16.15 milligram, one ball gives a moment of force of 0.0175 gram cm. If s is the resulting deviation of the axis of the record in cm we shall see below, in treating of the recording system, that this corresponds to an angular deviation of the pendulum of 0.02 s. As the weight of the pendulums is about 330 gram, we find for the distance h from the centre of gravity to the axis of rotation expressed in μ

$$h = \frac{0.0175}{330 \times 0.02\, s} = \frac{26.5}{s}$$

As the moment of inertia of the pendulums around the axis of rotation is about 13800 gram cm^2 we obtain for the mathematical length l of the pendulums

$$l = \frac{13800}{330 \times 0.00265}\, s = 15800\, s \text{ cm}$$

and the resulting figure for the period of the pendulum is

$$T_s = 25.2 \sqrt{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

in which s is expressed in cm.

Because of some uncertainty about the above figure for the moment of inertia it is better to determine the figure of this formula experimentally by swinging the slow pendulums at a land station and measuring their periods as well as the deviations of the axis of the records brought about by the dropping of one ball on each pendulum.

As the balls of a ball-bearing have accurately the same diameter, we may be sure of their weights being sufficiently equal for our purpose. As the balls are small, the dropping of a ball during the pendulum observation does not disturb the main pendulums and so we can check the periods of the slow pendulums as many times during the observation as we wish. The writer has not yet obtained enough experience about the new apparatus for knowing whether such checks are necessary and, if so, how often. Future observations will soon allow an answer to this question.

As each pendulum box has been provided with a scale for roughly reading the positions of the pendulum, we can already get a provisional figure for the pendulum period during the observation,. This scale also allows the checking of the equilibrium position of the pendulum, which likewise might change because of small deformations of the pendulum; if this position should have altered, we can readjust it by means of the small screws at the ends of the

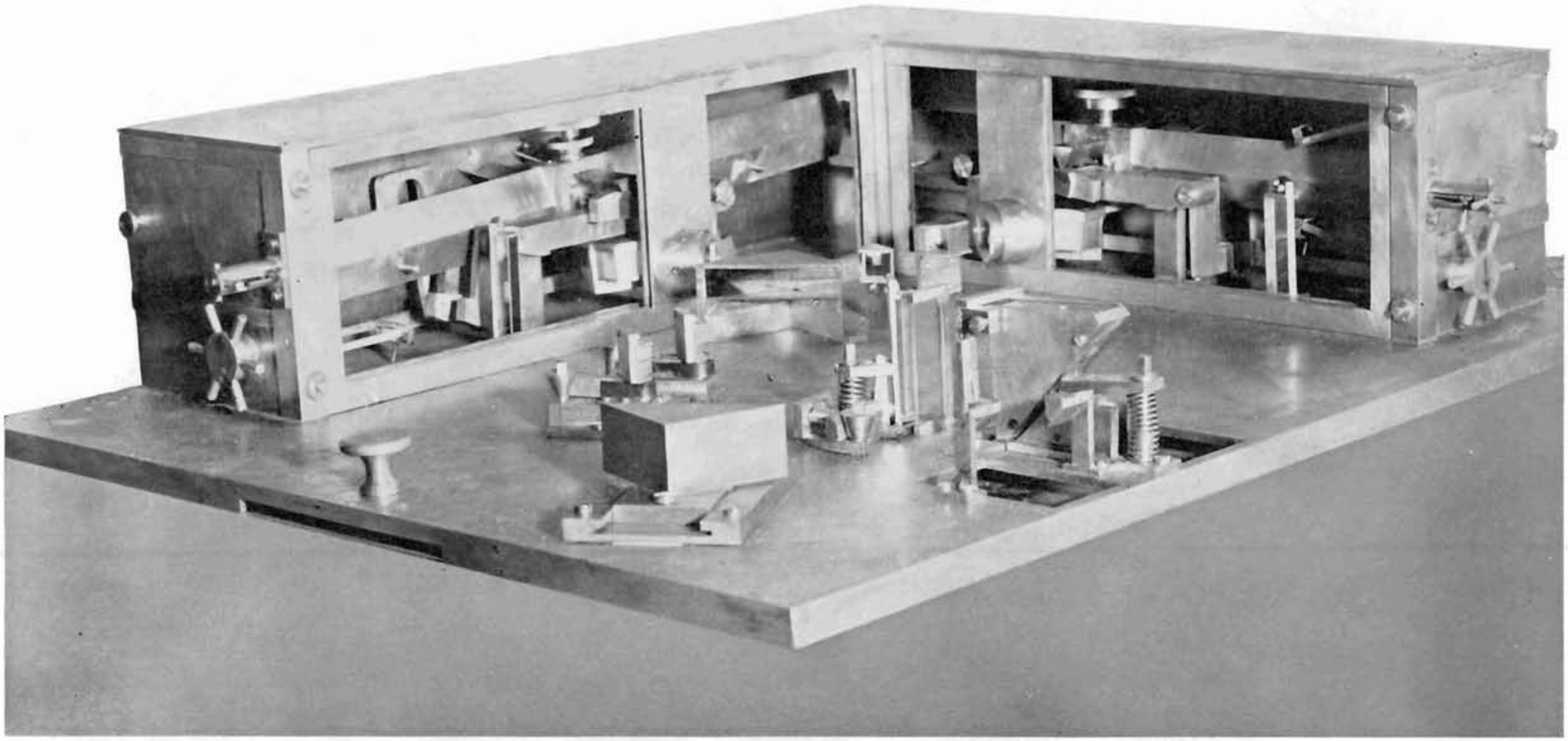

Plate I. Slow pendulum apparatus for measuring the horizontal accelerations.

pendulum. The scale further permits a better observation of the action of the pendulum than would be possible without it and so we can see at once whether there are strong horizontal accelerations during an observation; the pendulum boxes have been provided with large windows for the same purpose. This is important for deciding whether submerging at greater depth would be desirable. The scales have been added after the taking of the pictures of plates I and II and so they are not visible there.

The damping device of the pendulums has also been made after these pictures were taken; Dr. NIEUWENKAMP has taken care of this part of the construction. Each pendulum has been provided with vanes which during the swinging of the pendulum move in a closely fitting case without touching it. The damping, therefore, is an air-damping; as we mentioned already in the preceding paragraph, it diminishes the amplitude of the pendulums to about two thirds in a complete period of 26 sec and this is more than sufficient for the purpose of suppressing the free swinging of these pendulums. For both pendulums the damping is located in the back corner of the apparatus as it is represented by plate I.

The recording part.

The construction of the recording part of the new apparatus presented a few problems. In the first place there was not much room available in the incoming and returning light-beams of the old pendulum apparatus for taking off rays for the two new pendulums. It could, however, just be managed and so we could avoid the construction of an entirely new recording system. For the incoming rays we could put in two small new prisms indicated in fig. 7 on the next page by a_1 and a_2. They take off squares of 5 by 5 mm from the beam of rays going on to the pendulum apparatus and if we pursue their course we find that a_1 casts a shadow in the beam used for the recording of the middle pendulum and a_2 in that recording the thermometer. The first does not interfere with the record of the middle pendulum as long as this recording beam is kept in the left part of the 12 cm control window below the recording apparatus, i.e. there where the writer has always had it. For the recording of the thermometer only a small beam is left but this is not serious because the temperature curve is practically a straight line which requires only a small amount of light for its recording.

The returning beams of the slow pendulums move up and down and so we need a larger prism for reflecting them upwards towards the recording apparatus; for this purpose we have introduced prism l of fig. 7. This prism has been located at the left side of the beam coming back from the pendulum apparatus. At this side a little space was available and further this prism occupies part of the space that was used for the recording rays of the left damped pendulum and of the temperature, i.e. the rays nos 4 and 5 of the figure on page 50 of the publication of 1929. For leaving enough space for this new prism, these beams had to be shifted a little towards the beam coming back from the left fictitious pendulum, i.e. towards beam no 1 of this figure. As fig. 7 shows, the new prism is used for both rays returning from the slow pendulums.

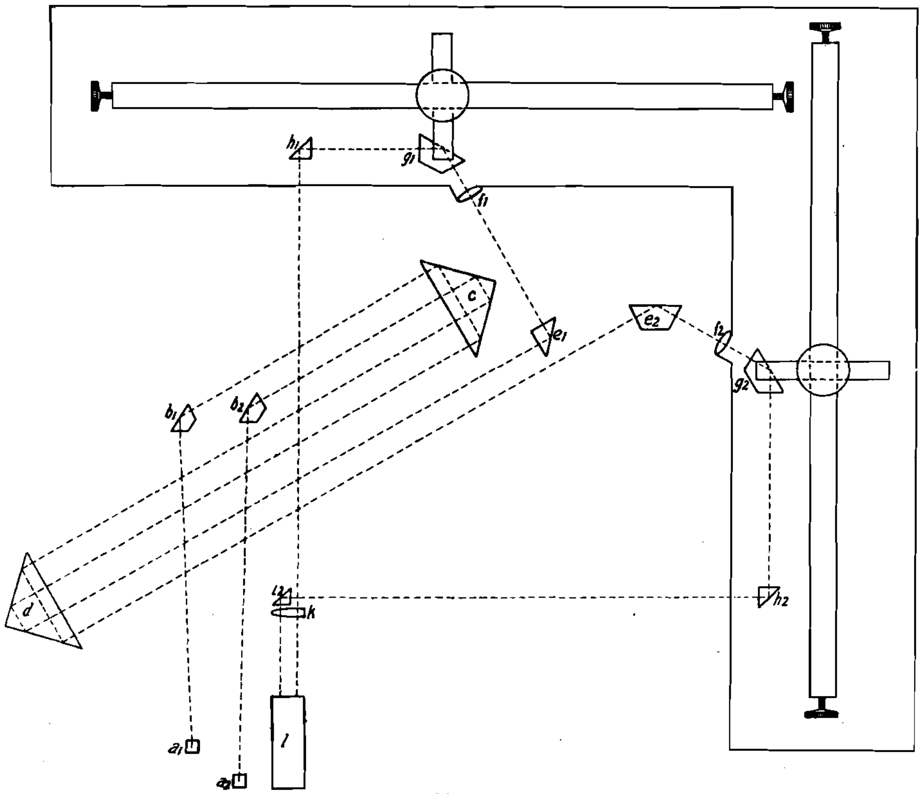

Fig. 7.

SCHEMATIC PLAN OF THE LIGHT-RAYS IN THE SLOW PENDULUM APPARATUS.
The figure indicates the two slow pendulums, each with two small adjustment screws at the ends and a third larger adjustment screw in the middle over the knife-edge. The prisms g_1 and g_2 are attached to the slow pendulums. The two light-rays enter the apparatus by the prisms a_1 and a_2 and leave it by the prism l.

A second problem was to introduce two new curves in the record of the pendulum apparatus without making this record unreadable. For preventing this it was desirable that the new curves should be so thin and sharp as possible. For fulfilling this requirement the image of the new light-rays has to be as small as possible and so the ratio of the distance from the light-source, i.e. the light-slit, to the ingoing lens of this ray (f_1 or f_2 in fig. 7) divided by the distance from the outgoing lens of this ray (k in fig. 7) to the recording paper has to be as large as possible; we assumed here that the construction has been made in such a way that the rays are parallel between the two lenses. As the last distance could not be altered much because the beam has to travel in the old manner through the recording apparatus, the only possible thing to do was to make the first distance as long as possible. This has led the writer to introduce the prisms c and d for lengthening the light-beams. Besides, the lenses f_1 and f_2 have been put as near to the slow pendulums as possible.

In this way we succeeded in making the focal distances of the lenses f_1 and f_2 equal to that of the lens k, i.e. 100 cm, while the focal distances of the in- and outgoing lenses of the pendulum apparatus have a ratio of about 4 : 7.

For the same purpose of reducing the trouble given by the new curves in the record, we have brought back their amplitudes. If we should have used a normal reflection on a mirror attached to the slow pendulums, the curves should spread over nearly the whole breadth of the record; now it has been reduced to one quarter of its value by making the rays reflect on the prisms g_1 and g_2 attached to the slow pendulums in the way as indicated by fig. 7; the rays thus get one half of the angular deviations of the pendulums instead of double their value.

For the lenses we could use parts of spectacle-glasses of a focal distance of one diopter. In connection with the length of glass the rays have to pass through, the real distances do not equal the optical distances and so the distance from the light-slit to f_1 and f_2 has to be 104.2 cm and from k to the photographic paper 104.6 cm. For k we have used a marginal part of a lens chosen in such a way that the focal axis of the lens coincides with the axis of the total beam returning from the pendulum apparatus. In this way we could keep the prism l in a vertical position, while otherwise we should have to give it a tilt which would have disturbed the direction of the movement of the images on the cylinder lens that precedes the photographic paper in the recording apparatus.

This leads us to the third problem that in general each ray has to fulfill three conditions:

1°. It has to follow the prescribed track and at the end it must strike the above mentioned cylinder lens,

2°. The direction of the movement of the image as it is brought about by the movement of the pendulum with regard to the apparatus, must coincide with the axis of this cylinder lens,

3°. The direction of the image of the light-slit must be perpendicular to the axis of the cylinder lens.

These three conditions present many difficulties. For our case we have chosen the solution as represented by the arrangement of fig. 7; this fulfills the requirements. For the accurate adjustment we need at least one adjustable prism in each ray.

A. *The adjustment of the rays in horizontal sense.*

For obtaining this adjustment the prisms b_1 and b_2 are adjustable round a vertical axis, i.e. they can be turned round this axis by means of adjustment screws of one of which the cone can be seen in plate I. As, however, the prisms are small and the reflections numerous, this adjustment is not sufficient for getting the whole ray right. We have, therefore, also provided the prisms i_2 and l with an adjustment in this sense; i_2 can likewise be turned round a vertical axis, the cone of the adjustment screw being visible in plate I. For prism l this adjustment can be obtained in another way, i.e. by rotating it slightly round a horizontal axis parallel to the triangular side of this prism; it is easy to see that the effect on the rays is the same as a rotation round a vertical axis. The rotation is effected by means of the screws fixing the top of this prism laterally; by loosening one screw and fastening the other, a slight rotation in the above sense can be obtained. As prism l reflects both rays, we have first to adjust the ray indicated in fig. 7 by the sub-index 1 by means of this prism and afterwards the second ray by means of prism i_2.

B. *The adjustment of the rays in vertical sense.*

For the adjustment in this sense we only dispose of the prisms i_2 and l; both these prisms can be turned round a horizontal axis parallel to the reflecting plane. The vertical screws for this adjustment and the springs surrounding them are visible in plate I. We may make the same remark as for the adjustment in horizontal sense; we have to begin by adjusting the first ray by means of l and afterwards we adjust the second ray by means of i_2.

For the following reason we have to take care that the rays falling on the prisms attached to the slow pendulums are horizontal. The reflected rays are moving because of the movement of the pendulums and we find that this movement is not plane but that it takes place along a conical surface. So, strictly speaking, this arrangement does not comply with the second condition mentioned above. As, however, the rays do not deviate much from the vertical plane they ought to move in, we need not object to it, but if the ray falling on the pendulum prism is not horizontal, the deviation gets larger and we see the image in the window below the recording apparatus describe a sloping curve which may deviate so much that the ray disappears from the window at one or at both ends of the movement. If such a deviation occurs, we can remedy it by changing the slope of the ray falling on the pendulum prism and this can be effected in the easiest way by putting a metal foil below the prisms e_1 or e_2 for giving them a slight tilt.

For the first adjustment of the rays we have to follow them with a paper and eventually adjust the consecutive prisms by means of introducing metal foils below them incase we can not get the ray right with the above adjustable prisms. The smallness of the prisms makes this initial adjustment more laborious than for the pendulum apparatus where the greater dimensions of the prisms allow more liberty in the course of the rays. Once done, the further adjustment by means of the adjustable prisms is simple.

C. *The focussing of the images on the recording paper.*

We can focus the images on the recording paper by means of the adjustable prism d. This prism can slide in a sense parallel to the rays that are

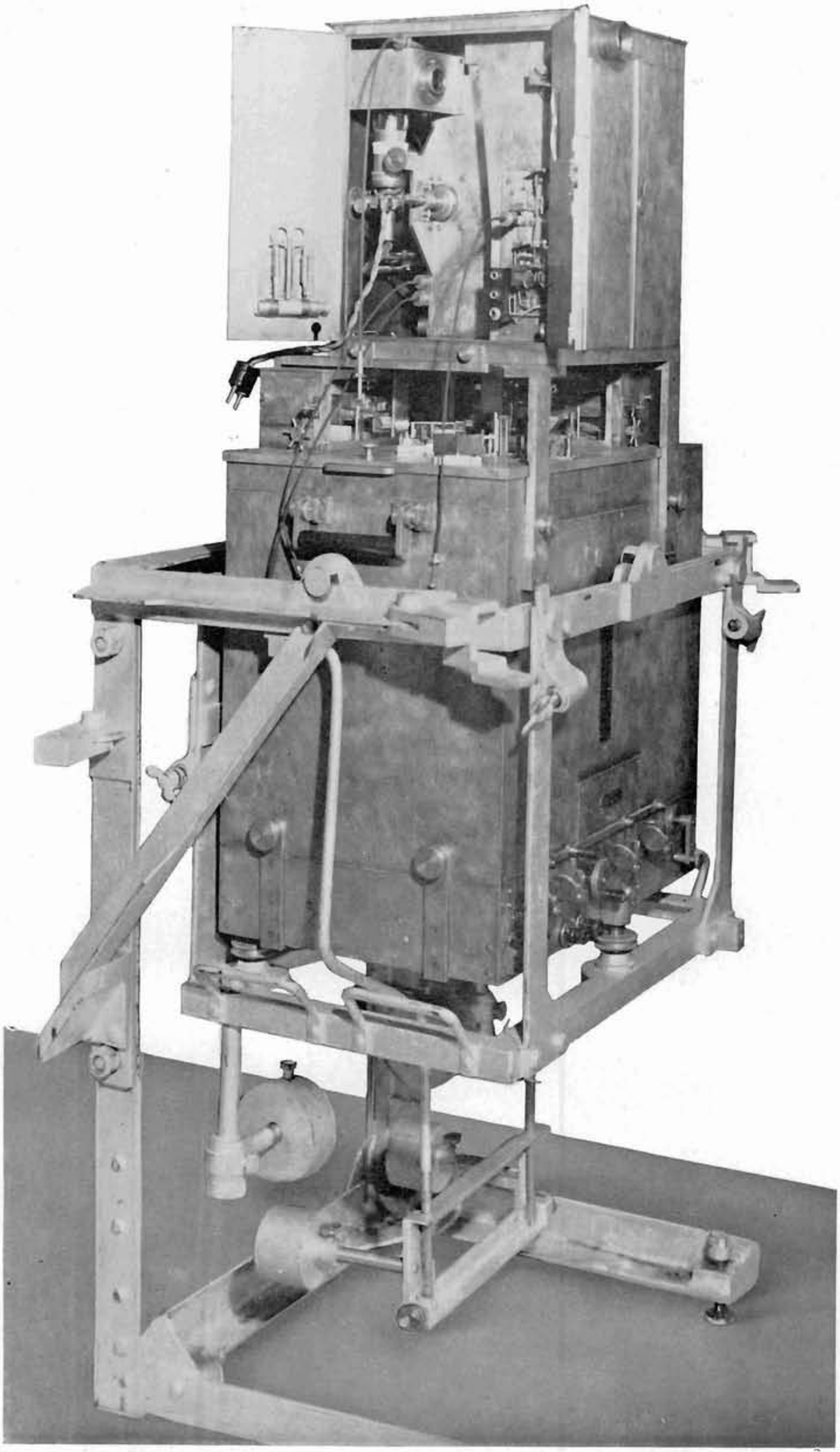

Plate II. Pendulum apparatus for gravity determinations at sea; the slow pendulum apparatus of Plate I is introduced between the recording and the pendulum apparatus. New gimbal system of which the swinging-period for both directions can be independently adjusted.

reflected on it and so, by moving this prism, we can lengthen or shorten the rays till the images in the control-window below the recording apparatus are sharp.

The operation of the new apparatus.

The new apparatus does not give much trouble during the pendulum observations. After releasing the gimbals we unclasp the slow pendulums and we release the light levers that now keep them in position. After having unclasped the main pendulums we lower the slow pendulums on their knife-edges before doing the same for the main pendulums and when we have set these last pendulums swinging, we carefully withdraw the light levers for releasing the slow pendulums with as little amplitude as possible. We can then proceed with the normal pendulum observations without the slow pendulums requiring any further care. At the end of the observation we release the light levers again and we lift the slow pendulums from their knife-edges after having first done the same for the main pendulums. We clasp them after having fixed the main pendulums and we then withdraw the light levers.

The gimbals etc.

There is no need to say much about the modification of the gimbals. We have reconstructed them in such a way that we can adjust the swinging-periods for both directions independently of each other. Plate II shows their new shape. The position of the lower weights determines the period of the swinging round the axis parallel to the swinging-plane of the main pendulums and the upper weights the period in the other sense. The periods have both been adjusted at about 3.6 sec.

In connection with the introduction of the new apparatus, a slight modification was also necessary of the brass supports of the recording apparatus. Except this change and that of the gimbals the new apparatus can be introduced in the old pendulum apparatus without any further alteration. For transportation it is not necessary to take the slow pendulums out of the apparatus.

CHAPTER III.

The Determination of the Second Order Corrections for the Old Observations.

§ 11. *The ship's and wave movements.*

For the old observations we only have the figures of the vertical accelerations; as we have explained in § 7 we can derive them from the fluctuations of the second-marks in the pendulum-records. We have no means of deriving the horizontal accelerations during these observations and so we can only determine the second order corrections if we should be able to derive the horizontal accelerations from the vertical ones. If GERSTNER's wave-theory would be valid, i.e. if the water-particles describe circular orbits in vertical planes and if the ship does the same, the solution would be simple; the mean value of the squares of the horizontal accelerations which we need for the correction, would be equal to the mean of the squares of the vertical acceleration which we can derive. We should then be entitled to apply the simple formula (52) which expresses the entire correction in the last mentioned mean value. Objections have, however, been raised against GERSTNER's theory and other solutions proposed, among others by STOKES and JEFFREYS, and the question is how these affect our problem. For shallow water, moreover, GERSTNER's theory is certainly not valid and we have theories given by LAPLACE and others; the same question has here to be investigated. In this paragraph we shall, therefore, consecutively discuss the following points:

1°. the relation of the movements of the ship and of the water-particles,
2°. GERSTNER's wave-theory,
3°. STOKES's wave-theory,
4°. JEFFREYS's wave-theory,
5°. LAPLACE's wave-theory for shallow water.

For these discussions we have used the representation of these theories given by Dr. H. THORADE in *„Probleme der Wasserwellen"*, HENRI GRAND, Hamburg, 1931.

1°. *The relation of the movements of the ship and the water-particles.*

It is not easy to derive the movements of the ship theoretically, even if the movements of the water-particles are entirely known. If the ship is small with regard to the wave-length, we may probably assert that the ship will follow the movement entirely because the displacement in this case of a group of water-particles by a rigid ship does not materially change the conditions of movement of the water. So the ship must be carried along as the displaced water would have been done. If, however, the ship is large with regard to the waves, the presence of the ship makes a difference for the water-movement and so we get a much more difficult problem to solve. In case of a surface ship it is further complicated by the presence of a free water surface. As this last difficulty does not occur for a submerged submarine we can leave it aside here, but even without this complication our problem is a difficult one; we shall not try to attack it here. We shall adopt as a rough estimate that the movement of the centre of gravity of the submarine has a velocity equal to the mean speed of the water-particles that would have been in the place of the ship if it had been absent and the water-movement undisturbed. Incase the orbits of the water-particles are circular, it is easy to see that this mean speed has the same character as the speeds of the water-particles, i.e. that it must be a vector of constant length revolving with constant speed; if we determine the resultant of the speed-vectors of all the water-particles we obtain a certain vector and as the speed-vectors revolve with constant speed without changing their length, the resultant does the same and the mean speed likewise. We may conclude that if our assumptions are true, the centre of gravity of the submarine describes a circular orbit and as the pendulum-apparatus practically coincides with it, the apparatus does the same.

Our reasoning, however, becomes more uncertain for ships that have larger dimensions with regard to the waves. In that case the speeds of the different water-particles cancel each other partially when determining the mean speed and so this mean does not only become smaller but it is also more uncertain in direction as well as in size. So we can not expect that the above result will be valid for ships larger than 1/2 or 2/3 of a wave-length. A submarine, however, is usually not large; if its length is 60 m we have a good chance of our conclusion still being valid for waves of a length of 90 m, i.e. for periods of 7.6 sec or larger and as we have mentioned on page 12, smaller periods seldom occur when we submerge at a depth of at least 20 m. So we may conclude that if GERSTNER's theory is valid, we may probably adopt the equality of the means of $\ddot{x}^2$ and $\ddot{y}^2 + \ddot{z}^2$, i.e. of the squares of the vertical and horizontal accelerations of the apparatus, which we require for the computation of the second order correction for the old observations.

We may find a confirmation of the surmise that in most cases the submarine fairly accurately follows the movement of the water-particles in the fact that during submergence in rough sea the depth-gauge is usually remarkably steady. As we shall see in the next part of this paragraph, the surfaces of equal pressure for GERSTNER's assumption go up and down with the water-particles and so, as the gauge is worked by the water-pressure, its steadiness points to the submarine following the water-movement. We may find another confirmation in the results mentioned in the next paragraph.

Inside the limits of accuracy of the observations they confirm the supposition of the equality of the maximum values of the horizontal and vertical accelerations and this makes it of course probable that the ship follows the wave-movements because otherwise we should have to assume that the deviation would just be compensated by a deviation from the circular orbits of the water-particles.

2°. GERSTNER'S *wave-theory* (THORADE, P. D. W., p. 20 e.s.).

As we have mentioned already, GERSTNER supposes the water-particles to describe circular orbits. The coordinates of a particle are represented by the formulas

$$x = x_0 - r \cos \varkappa \, (z_0 - q\,t), \quad (71\,A)$$

$$z = z_0 - r \sin \varkappa \, (z_0 - q\,t). \quad (71\,B)$$

The X axis has been chosen vertically downwards and the Z axis in the sense of the propagation of the waves *), x_0 and z_0 are the coordinates of the centre of the orbit. If λ is the wave-length, $\varkappa$ is given by

$$\varkappa = \frac{2\pi}{\lambda} \quad (71\,C)$$

The speed q of the waves is

$$q = \sqrt{\frac{g}{\varkappa}} = \sqrt{\frac{g\lambda}{2\pi}} = 1.25\sqrt{\lambda} \text{ in m/sec.} \quad (72\,A)$$

Because $\lambda = q\,T_w$ we have also

$$T_w = \sqrt{\frac{2\pi\lambda}{g}} = 0.80\sqrt{\lambda} \; (\lambda \text{ in m}) \quad (72\,B)$$

and reversely

$$\lambda = \frac{g}{2\pi} T_w^2 = 1.56\, T_w^2 \text{ in m.} \quad (72\,C)$$

The radius r of the orbits decreases logarithmically with the depth and is given by

$$r = a e^{-\varkappa x_0} \quad (72\,D)$$

As

$$e^{\frac{2\pi}{9}} = 2.010,$$

we can say that the waves decrease to half their amplitude for an increase of the depth of one ninth part of the wave-length; the type of movement remains the same in deeper layers.

These equations give a wave-movement which is dynamically possible and which satisfies the condition div $V = 0$ (V = speed-vector) i.e. the condition for non-compressibility. It has, however, not been proved that this movement really occurs nor has it been proved that the effect of the wind at the surface leads to such a movement.

*) Thorade takes the Z axis vertically upwards and the X axis in the sense of the propagation.

It is hardly necessary to repeat here the obvious fact that the circular orbit and uniform speed of the water-particles imply the equality of the mean values of $\ddot{x}^2$ and $\ddot{z}^2$, the squares of the vertical and horizontal accelerations.

For deriving the pressure p belonging to this type of water-movement (THORADE p. 21), we follow the method of Lord RAYLEIGH for making the movement stationary by adding a speed q contrary to the sense of the propagation. In this way the waves of the water-surface become stationary and the water-movement changes in a flowing between this wavy surface and the plane bottom. The stationary character of the movement also follows from the equations (71) if we add $-\,q\,t$ to z and substitute for $z_0 - q\,t$ the new coordinate z'; we get

$$x = x_0 - r \cos \varkappa z', \quad \ldots \ldots \ldots \quad (73\,A)$$

$$z = z' - r \sin \varkappa z'. \quad \ldots \ldots \ldots \quad (73\,B)$$

These equations no longer contain the time t.

For this stationary movement we can use the theorem of BERNOUILLI in order to derive the pressure p. Making use of (72 A) we find

$$p = \frac{g}{\varrho}\,x_0 - \frac{q^2}{2\,\varrho}\,(1 + \varkappa^2 r^2) + \text{const},$$

in which ϱ is the density. The constant is given by the condition that p is zero at the surface i.e. for $x_0 = 0$. If the amplitude of the wave-movement there is a, i.e. the height of the waves between the highest and lowest points $2\,a$, we obtain

$$p = \frac{g}{\varrho}\,x_0 + \frac{q^2 \varkappa^2}{2\,\varrho}\,(a^2 - r^2). \quad \ldots \ldots \ldots \quad (74)$$

Because particles of the same original depth x_0 describe orbits of the same radius r, we may conclude that the surfaces of equal pressure are given by $x_0 =$ constant. We see likewise that these surfaces go up and down with the particles coinciding with that surface at a given moment; the particles continue to coincide wih it. We used this result in the preceding part of this paragraph.

3°. STOKES's *wave-theory* (THORADE, P. D. W., p. 28 e.s.).

STOKES has objected to GERSTNER's wave-theory on the ground that curl V is not zero for this solution; it is even contrary to the sense of the movement of the water-particles in their circular orbits. He has, therefore, proposed a modification and we shall discuss here whether that modification introduces a difference between the mean squares of the vertical and horizontal accelerations. We shall find this to be the case but the difference is negligible for our purpose of computing the second order corrections. We may conclude that we can not draw an inference from the observations of the accelerations in a submarine whether the water-particles follow the orbit as prescribed by GERSTNER or by STOKES and at the same time that the question does not matter for our problem. In this publication we have, therefore, everywhere mentioned both theories together but, strictly speaking, the circular movement of the water-particles is only true for GERSTNER's supposition although we shall see that the deviation from this orbit according to STOKES's equations is small for the wave-movement of a submerged submarine.

Personally the writer thinks that there are not only theoretical reasons for preferring STOKES's solution, but that the observation of small floating particles favours the idea that there is a slow transfer in the sense of the wave-movement in the way STOKES's equations predict. This transfer likewise seems to find confirmation in the fact that a ship looses less speed in a rough sea when waves come from the stern than when they come from the bow, though the writer admits that other circumstances may also play a part in this question.

STOKES' solution is given by the following formulas for the speed in a fixed point x, z. We have chosen again the X axis vertically downwards and the Z axis in the sense of the propagation of the waves; u is the component of the speed in the sense of the X axis and w that in the sense of the Z axis.

$$u = fq\, e^{-\varkappa x} \sin \varkappa\, (qt - z), \qquad (75\,A)$$

$$w = fq\, e^{-\varkappa x} \cos \varkappa\, (qt - z). \qquad (75\,B)$$

For deriving the orbit of a water-particle we first deduce its accelerations du/dt and dw/dt. We find

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} = \varkappa f q^2 e^{-\varkappa x} \cos \varkappa\, (qt - z) - \varkappa f^2 q^2 e^{-2\varkappa x}, \qquad (76\,A)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z} = - \varkappa f q^2 e^{-\varkappa x} \sin \varkappa\, (qt - z). \qquad (76\,B)$$

The general solution of these equations for finding the orbit is difficult and so we contented ourselves with constructing the curve for a special case. From (75) follows for the speed V

$$V = fq\, e^{-\varkappa x}, \qquad (77)$$

and for finding the radius of curvature of the orbit we deduce the square of the speed V divided by the component of the acceleration perpendicular to the speed as it follows from (76)

$$\varrho = \frac{f}{\varkappa e^{\varkappa x} - \varkappa f \cos \varkappa\, (qt - z)} \qquad (78)$$

By means of these two quantities we can construct the curve with sufficient accuracy. For small but constant intervals of time Δ_t we successively computed the values of ϱ and of the corresponding element of the orbit $\Delta_s = V\,\Delta_t$ and we built up the curve starting from the point $x = z = 0$. Assuming a value of f of $1/4$, of $f/\varkappa$ of 10 cm and of Δ_t of $10\,\pi/180$ sec, we found the curve as given by fig. 8 on a reduced scale. For gaining insight in the meaning of the quantities f and $f/\varkappa$ we may derive from this figure that $f/\varkappa$ is roughly one half of the distance from the highest point A to the lowest B, i.e. of the height $2a$ of the wave. So

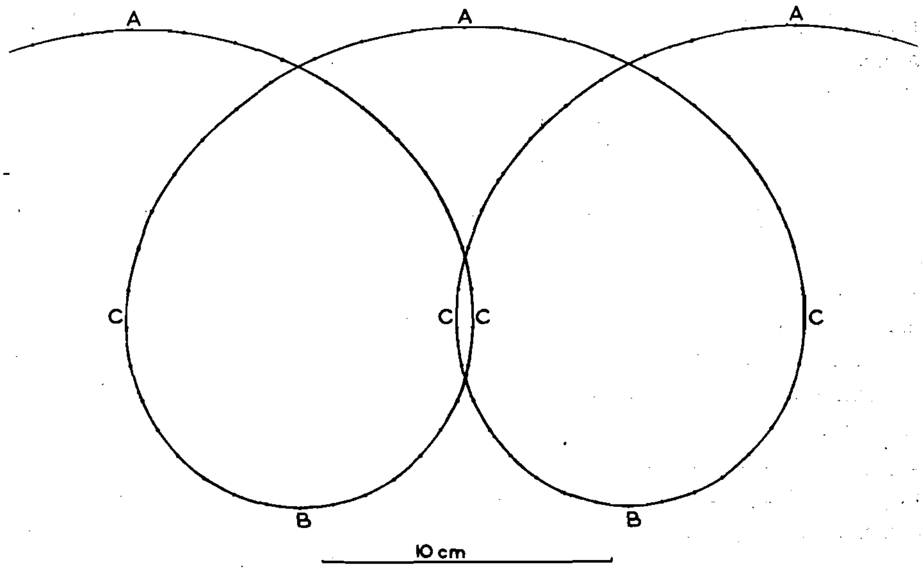

Fig. 8.

THE ORBIT OF A WATER-PARTICLE ACCORDING TO STOKES's WAVE-THEORY.

The curve is constructed for a wave of a height of about 1/15 th of the wave-length. The points indicate positions at equal time-intervals. The value of 10 cm inscribed in the scale-measure gives the scale of the curve as mentioned in the text; for a wave of a length λ this measure represents $f/\varkappa = \lambda/8\pi = 0.040\,\lambda$.

$$f \approx \varkappa a = \pi \times \frac{\text{wave-height}}{\text{wave-length}} \quad \ldots \ldots \quad (79)$$

A value of f of $1/4$ corresponds to a wave of a height of about one fifteenth of the wave-length and this only occurs in very rough seas. In a submerged submarine we may expect values that are considerably smaller. We can obtain a good estimate of the maximum value of f in that case by the following reasoning. As we shall see below, the values of du/dt very nearly follow a sine-curve and so formula (40) for the vertical accelerations well represents it. So the quantity c of (40) approximately equals the maximum value of formula (76 A), which is found by introducing $x = z = t = 0$, and we obtain

$$c = \varkappa f q^2 - \varkappa f^2 q^2,$$

If we neglect f^2 with regard to f and if we apply formula (70) of GERSTNER's theory for the wave-speed q, which is a good approximation as the formula on page 30 of „Probleme der Wasserwellen" by THORADE shows, we obtain the following approximative formulas for c and f

$$c \approx fg,$$

and so

$$f \approx \frac{c}{g}. \quad \ldots \ldots \ldots \quad (80)$$

Introducing this in formula (52) for the total second order correction δg, we get

$$f \approx \sqrt{\frac{8\,\delta g}{g}} \qquad \qquad (81)$$

A value of δg of 32 mgal seldom occurs and may well be considered as a maximum value. According to this formula this corresponds to a value of f of 0.016. So we see that the values of f which we may expect in a submerged submarine are indeed small with regard to the value of 1/4 assumed in figure 8.

Fig. 8 shows us that, as we mentioned in the beginning of this discussion, the water-particles do not move, according to STOKES' supposition, in closed orbits but that in each revolution they advance somewhat in the sense of the propagation of the waves. A further investigation discloses that this advance is approximately proportional to the height of the waves and to f or, in other words, to the square of the height and inversely proportional to the wave-length. For small wave-heights STOKES' theory is practically identical with GERSTNER's. We can draw this same conclusion directly from the equations (75) and (76); for small values of u and w we may put $du/dt = \partial u/\partial t$ and $dw/dt = \partial w/\partial t$ and the equations (75) and (76) tend to being identical with GERSTNER's.

The purpose of our discussion is to derive the ratio of the mean squares of the vertical and the horizontal accelerations and so we computed for our example by means of the formulas (76) the values of du/dt and dw/dt, both divided by $\varkappa^2 q^2$ for practical reasons. For x we introduced the values measured in the curve of fig 8 and we made the computations for the same time-intervals Δ_t used for its construction. The following table gives the results; the first column contains the angles $\varkappa\,(qt - z)$, the last value has been adjusted at 180° by making the last time-interval 1.028 times as large as the normal intervals.

$\varkappa\,(qt - z)$		$\frac{\ddot{x}}{\varkappa^2 q^2}$	$\frac{\ddot{z}}{\varkappa^2 q^2}$	$\varkappa\,(qt - z)$		$\frac{\ddot{x}}{\varkappa^2 q^2}$	$\frac{\ddot{z}}{\varkappa^2 q^2}$
0°	0′	7.500	0.000	82°	5′	−0.495	7.909
7	30	7.401	1.302	91	49	−1.733	7.714
15	1	7.110	2.561	101	52	−2.931	7.310
22	40	6.622	3.760	112	16	−4.062	6.710
30	25	5.963	4.842	122	58	−5.094	5.930
38	20	5.146	5.793	133	56	−5.980	4.971
46	30	4.193	6.598	145	8	−6.697	3.869
54	56	3.123	7.227	156	31	−7.238	2.660
63	40	1.965	7.658	168	3	−7.572	1.371
72	44	0.745	7.889	180	0	−7.691	0.000

If we study these values, both components of the acceleration turn out to follow very nearly a sine-curve if we plot them with regard to a time-coordinate. This points to the conclusion that the movement of the water-particles is approximately a uniform circular movement combined with a uniform transition in the sense of the wave-propagation, i.e. a cycloidal movement. The fact that also according to STOKES's theory the components of the accelerations are sine-functions of a quantity proportional to the time, further confirms the assumptions of the formulas (3), (23) and (40).

By means of the above table we determined the ratio α of the mean values of $\ddot{x}^2$ and $\ddot{z}^2$ and we found

$$\alpha = \frac{\overline{\ddot{z}^2}}{\overline{\ddot{x}^2}} = 1.097.$$

So, according to the formulas (51), we should have to multiply formula (52) for the total second order correction in our case by the factor

$$2\,\alpha - 1 = 1.194.$$

The error of formula (52) thus turns out to be about 20 percent in this case.

The case, however, will never occur for observations on board of submerged submarines; as we saw, the quantity f is at the utmost of the order of one fifteenth of the value of f for our example. So from these considerations we may conclude that it is improbable that the error will exceed one or two percent. We shall, however, make a more accurate estimate of the error for arbitrary small values of f and this will show that it is even entirely negligible.

For this purpose we may remark that we get a fair approximation of the factor α by taking the square of the ratio of the maximum value of $\ddot{z}/\varkappa^2 q^2$ which we may estimate at 7.94 and of $\ddot{x}/\varkappa^2 q^2$ which we may put at the mean of the values 7.500 and 7.691, i.e. at 7.596. We thus find $\alpha = 1.090$. Taking this method for deriving α as a base for our deductions for the general case of an arbitrary small value of f, we shall determine the value of the horizontal acceleration for the point C of the curve and the mean of the vertical accelerations for the points A and B.

For the point A we can derive at once from formula (76 A), putting $t = z = 0$,

$$\ddot{x}_a = \varkappa f q^2 - \varkappa f^2 q^2 = \varkappa f q^2 (1 - f) \quad . \quad . \quad . \quad . \quad . \quad (82)$$

For the points B and C we can not give accurate formulas for the accelerations in the general case because we do not have the equation of the curve. So we have to make estimates of their coordinates for introducing them in (76 A) and (76 B). For x_b we have done this in the following way.

Starting with a first approximation by integrating formula (75 A) while keeping the factor $e^{-\varkappa x}$ constant at the mean of its values in A and B and neglecting the variability of $\varkappa z$, we find

$$x_b \approx \frac{f}{\varkappa} (1 + e^{-\varkappa x_b}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Multiplying by $\varkappa$ we get an equation in $\varkappa x_b$ from which we derive by developing in series

$$\varkappa x_b = 2f - 2f^2 + 4f^3 - 9\frac{1}{3}f^4 + \ldots \quad (84)$$

and also from (83)

$$e^{-\varkappa x_b} = \frac{\varkappa x_b}{f} - 1 = 1 - 2f + 4f^2 - 9\frac{1}{3}f^3 + \ldots \quad (85)$$

Introducing this in (76 A) and putting $qt - z = 180°$, we obtain for the vertical acceleration in B

$$\ddot{x}_b = -\varkappa f q^2 \left(1 - f + \frac{8}{3}f^3 + \ldots\right). \quad (86)$$

The mean of (82) and of (86) with a positive sign gives for the maximum value $\ddot{x}_m$

$$\ddot{x}_m = \varkappa f q^2 \left(1 - f + \frac{4}{3}f^3 + \ldots\right). \quad (87)$$

Basing our estimate on fig. 8 we assume for the coordinate x_c of C the mean value of $f/\varkappa$ and $\frac{1}{2}x_b$ and so we put

$$\varkappa x_c = f - \tfrac{1}{2}f^2 + f^3 + \ldots$$

and developing in series

$$e^{-\varkappa x_c} = 1 - f + f^2 - \frac{5}{3}f^3 + \ldots \quad (88)$$

Introducing this in (76 B) and putting $qt - z = 90°$, we obtain for the maximum value of the horizontal acceleration

$$\ddot{z}_m = \varkappa f q^2 \left(1 - f + f^2 - \frac{5}{3}f^3 + \ldots\right) \quad (89)$$

From (87) and (89) we derive

$$a = \frac{\ddot{z}_m^{\,2}}{\ddot{x}_m^{\,2}} = 1 + 2f^2 - 4f^3 + \ldots \quad (90)$$

According to this value of a we have to add a factor to formula (52) for the total second order correction of

$$2a - 1 = 1 + 4f^2 - 8f^3 + \ldots \quad (91)$$

We may no doubt expect this formula to be valid for the small values of f in submerged submarines. For the adopted maximum value of f of 0.016, the factor becomes

$$2a - 1 = 0.0010.$$

This effect is certainly negligible for the computation of the second order corrections. We thus get to the important conclusion that if STOKES's wave-theory is valid and not GERSTNER's, we may still use formula (52) for that purpose.

4°. JEFFREYS' *wave-theory* (THORADE, P. D. W., p. 34 e.s.).

JEFFREYS has investigated how the wave-movement changes when the waves have finite dimensions in the sense at right angles to the propagation, as we see it often to be the case at sea. The equations thus become three-dimensional. Assuming the cross-section of the water-surface in this sense to be a sine-curve of a wave-length $\lambda' = 2\pi/\varkappa'$, he gives the following formulas; the X direction has again been chosen vertically downwards and the Z direction in the sense of the propagation,

$$u = \varkappa\, q a\, e^{-rx} \sin \varkappa\, (qt - z) \cos \varkappa' y, \quad . \quad . \quad . \quad . \quad . \quad (92\,A)$$

$$v = \frac{\varkappa \varkappa'}{r}\, q a\, e^{-rx} \sin \varkappa\, (qt - z) \sin \varkappa' y, \quad . \quad . \quad . \quad . \quad (92\,B)$$

$$w = \frac{\varkappa^2}{r}\, q a\, e^{-rx} \cos \varkappa\, (qt - z) \cos \varkappa' y, \quad . \quad . \quad . \quad . \quad (92\,C)$$

in which

$$r^2 = \varkappa^2 + \varkappa'^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92\,D)$$

These formulas fulfill again the conditions div $V = 0$ and curl $V = 0$. The movement is dynamically possible when

$$q = \sqrt{\frac{g\lambda}{2\pi}} \sqrt{1 + \frac{\lambda^2}{\lambda'^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

We shall now determine the mean values of the squares of the horizontal and vertical accelerations which we need for the computing of the second order correction. We may assume again that in a submerged submarine the amplitude of the waves is so small that in the same way as we have discussed previously for the theory of STOKES, we may put $du/dt = \partial u/\partial t$, $dv/dt = \partial v/\partial t$ and $dw/dt = \partial w/\partial t$; the advance of the water-particles in the sense of the propagation in this case is practically negligible. We find

$$\ddot{x} = \frac{du}{dt} = \varkappa^2 q^2 a\, e^{-rx} \cos \varkappa\, (qt - z) \cos \varkappa' y, \quad . \quad . \quad (94\,A)$$

$$\ddot{y} = \frac{dv}{dt} = \frac{\varkappa^2 \varkappa'}{r} q^2 a\, e^{-rx} \cos \varkappa\, (qt - z) \sin \varkappa' y, \quad . \quad . \quad (94\,B)$$

$$\ddot{z} = \frac{dw}{dt} = -\frac{\varkappa^3}{r} q^2 a\, e^{-rx} \sin \varkappa\, (qt - z) \cos \varkappa' y. \quad . \quad (94\,C)$$

For the mean values of the squares we obtain

$$\overline{(\ddot{x})^2} = \tfrac{1}{2} \varkappa^4 q^4 a^2 e^{-2rx} \cos^2 \varkappa' y,$$

$$\overline{(\ddot{y})^2} + \overline{(\ddot{z})^2} = \tfrac{1}{2} \varkappa^4 q^4 a^2 e^{-2rx} \left(\frac{\varkappa'^2}{r^2} \sin^2 \varkappa' y + \frac{\varkappa^2}{r^2} \cos^2 \varkappa' y \right),$$

and so, according to formulas (51), the total second order correction is

$$\frac{\delta g}{g} = \frac{1}{8} \frac{\varkappa^4 q^4 a^2}{g^2} e^{-2rx} \left(\frac{2\varkappa^2 - r^2}{r^2} \cos^2 \varkappa' y + 2 \frac{\varkappa'^2}{r^2} \sin^2 \varkappa' y \right)$$

or by applying (92 D)

$$\frac{\delta g}{g} = \frac{1}{8} \frac{\varkappa^4 q^4 a^2}{g^2} e^{-2r z} \left(\cos^2 \varkappa' y - 2 \frac{\varkappa'^2}{r^2} \cos 2 \varkappa' y \right) . \quad (95\,A)$$

We can also write this

$$\frac{\delta g}{g} = \frac{1}{4} \left(\frac{\overline{\ddot{x}}}{g} \right)^2 \left(1 - 2 \frac{\varkappa'^2}{r^2} \frac{\cos 2 \varkappa' y}{\cos^2 \varkappa' y} \right) \quad . \quad . \quad . \quad . \quad (95\,B)$$

So, comparing with formula (52), the factor between the brackets has been added. If the sea would entirely correspond to the formulas of JEFFREYS and if the submarine would exactly follow a course parallel to the Z axis, the angle $\varkappa' y$ would be constant during the observations and we should have to apply this factor. In reality this will never occur and the ship will meet with continuously varying angles $\varkappa' y$. So we may be sure that the mean value of the factor of the second term of (95 A) is zero and the second term of the factor of (95 B) will disappear. We may conclude that if the wave-movement would in great lines follow the formulas of JEFFREYS, we are entitled to continue to use formula (52) for the computation of the second order corrections. For separate waves, however, we shall find differences from the circular orbits of GERSTNER. When we study the components of the acceleration as given by the fluctuations of the second-marks in the pendulum-records and by the records of the slow pendulums, we shall usually not find the values corresponding to circular orbits in vertical planes but the values as given by the formulas (94) which correspond in general to elliptic orbits in sloping planes. Incase $\varkappa' y$ has a value of 90° we must even find a linear and horizontal orbit, perpendicular to the sense of the propagation.

5°. LAPLACE's *theory for wave-movements in shallow water* (Thorade, P. D. W., p. 180 e.s.).

We shall discuss here the wave-movement in shallow water, viz. for such sea-depths that a movement corresponding to one of the preceding types is no longer possible because of the vertical component of the speed assumed by these theories at that depth. We shall not consider depths that are so small that practically only horizontal movements of the water-particles can occur; there is no reason to study that case as it is outside the range of submarine gravity observations.

We shall further restrict ourselves to small amplitudes of the water-movement; if the amplitudes were large, no gravity determinations would be possible in shallow water because the submarine can not submerge deep enough for escaping the strong movements. As we assume small amplitudes we shall only discuss the wave-theory of LAPLACE and we shall leave aside the more complicated formulas of KORTEWEG and DE VRIES and of STRUIK, which have special significance for larger amplitudes. For the same reason we may put $du/dt = \partial u/\partial t$ and $dw/dt = \partial w/\partial t$. We may again neglect the horizontal shift of the water-particles in the sense of the propagation of the waves, discussed for STOKES' wave-theory.

If h is the sea-depth and if we again take the X axis vertically downwards and the Z axis in the sense of the propagation, the equations of LAPLACE are

$$u = fq\,(e^{-\varkappa x} - e^{-\varkappa(2h-x)})\,\sin\varkappa\,(qt-z), \quad . \quad . \quad . \quad (96\,A)$$

$$w = fq\,(e^{-\varkappa x} + e^{-\varkappa(2h-x)})\,\cos\varkappa\,(qt-z)\,. \quad . \quad . \quad . \quad (96\,B)$$

These equations fulfill the conditions div $V = 0$ and curl $V = 0$ and they make the vertical component of the speed u zero for $x = h$. The movement is dynamically possible for

$$q = \sqrt{\frac{g\,\lambda\,(e^{\varkappa h} - e^{-\varkappa h})}{2\pi\,(e^{\varkappa h} + e^{-\varkappa h})}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

According to these equations the water-particles describe elliptic orbits of which the focal distances are everywhere the same. The vertical axis diminishes with the depth and becomes zero for a depth h; the particles there move only horizontally.

We shall again determine the mean values of the squares of the vertical and horizontal accelerations, which we require for the deriving of the second order corrections. We obtain

$$\ddot{x} = \frac{du}{dt} = \varkappa f q^2\,(e^{-\varkappa x} - e^{-\varkappa(2h-x)})\,\cos\varkappa\,(qt-z), \quad (98\,A)$$

$$\ddot{z} = \frac{dw}{dt} = -\varkappa f q^2\,(e^{-\varkappa x} + e^{-\varkappa(2h-x)})\,\sin\varkappa\,(qt-z), \quad (98\,B)$$

and the mean values of the squares are

$$\overline{(\ddot{x})^2} = \tfrac{1}{2}\,\varkappa^2 f^2 q^4\,(e^{-2\varkappa x} + e^{-2\varkappa(2h-x)} - 2\,e^{-2\varkappa h}),$$

$$\overline{(\ddot{z})^2} = \tfrac{1}{2}\,\varkappa^2 f^2 q^4\,(e^{-2\varkappa x} + e^{-2\varkappa(2h-x)} + 2\,e^{-2\varkappa h}).$$

Applying the formulas (51) we find for the total second order correction

$$\frac{\delta g}{g} = \frac{\varkappa^2 f^2 q^4}{8\,g^2}\,(e^{-2\varkappa x} + e^{-2\varkappa(2h-x)} + 6\,e^{-2\varkappa h}) \quad . \quad . \quad (99\,A)$$

or

$$\frac{\delta g}{g} = \tfrac{1}{4}\,\overline{\left(\frac{\ddot{x}}{g}\right)^2}\left[1 + \frac{8}{(e^{\varkappa(h-x)} - e^{-\varkappa(h-x)})^2}\right] \quad . \quad . \quad . \quad (99\,B)$$

We see that a factor has been added to formula (52) and we can not omit this factor as we could in the previous cases; the second term between the brackets can not always be neglected. This term depends only on the quantity $\varkappa\,(h-x)$ or, according to (69 C) on $(h-x)/\lambda$, i.e. on the ratio of the wave-length to the distance from the ship to the bottom of the sea.

For applying this formula we, therefore, need both quantities. This does not bring about difficulties for the old observations; as the sea-depth and the depth of submergence is known for all of them, we can determine the distance from the submarine to the bottom, and we can derive the wave-length or $\varkappa$ from the period of the waves T_w as it follows from the fluctuations of the second-marks in the pendulum-records. For this purpose we can use the following formula deduced from (97) in combination with $\lambda = q\,T_w$

$$\varkappa = \frac{4\pi^2}{g\,T_w^2}\left(1 + \frac{2}{e^{2\varkappa h} - 1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (100)$$

In the right member of this formula we can practically substitute the value of $\varkappa$ as it is given by the factor before the brackets.

By means of (99 B) and (100) we derived the following tables for the determination of the factor between the brackets of (99 B). We start by finding P by means of Table I and then we use Table II for determining the factor F. In Table II we added a second factor N representing the ratio $\ddot{z}/\ddot{x}$ of the amplitudes of the horizontal and vertical accelerations (form. 98 B and 98 A), which at the same time gives the ratio of the horizontal and vertical axis of the elliptic orbits of the water-particles.

We have used these tables for the computation of the second-order corrections for the few old observations in shallow water for which these corrections could not be neglected. We do not yet dispose, it is true, of much experimental evidence about the validity of LAPLACE's theory for the wave-movement in shallow water, but as far as it goes, it is in good agreement with the formulas. We may hope to obtain more data about it in the future by means of further pendulum observations combined with observations of the slow pendulum instrument.

Table I for the factor P of formula (100).

$$P = 1 + \frac{2}{e^{2\varkappa h} - 1}\ ; \ \alpha = \frac{h}{T_w^2}\ , \ h = \text{sea-depth in meters.}$$

α	P	α	P	α	P
0.20	1.50	0.30	1.20	0.40	1.083
0.22	1.41	0.32	1.16	0.45	1.055
0.24	1.34	0.34	1.14	0.50	1.036
0.26	1.28	0.36	1.12	0.60	1.016
0.28	1.23	0.38	1.10	0.70	1.007

For $\alpha > 0.70$ we may put $P = 1.00$.

Table II for the factor F of formula (99 B) *and the factor* $N = \ddot{z}/\ddot{x}$.

$$F = 1 + \frac{8}{[e^{\varkappa(h-x)} - e^{-\varkappa(h-x)}]^2}\ ; \ \beta = P \times \frac{(h-x)}{T_w^2}\,,$$

h = sea-depth in meters. x = depth of submergence in meters.

β	F	N	β	F	N	β	F	N	β	F	N
0.150	5.85	1.85	0.20	3.49	1.50	0.30	1.86	1.19	0.50	1.148	1.037
0.155	5.51	1.80	0.21	3.21	1.45	0.32	1.71	1.16	0.55	1.098	1.024
0.160	5.20	1.76	0.22	2.97	1.41	0.34	1.59	1.14	0.60	1.065	1.016
0.165	4.92	1.72	0.23	2.76	1.37	0.36	1.49	1.12	0.65	1.043	1.011
0.170	4.66	1.68	0.24	2.58	1.34	0.38	1.41	1.10	0.70	1.029	1.007
0.175	4.42	1.65	0.25	2.42	1.31	0.40	1.35	1.08	0.75	1.019	1.005
0.180	4.20	1.61	0.26	2.28	1.28	0.42	1.29	1.07	0.80	1.013	1.003
0.185	4.00	1.58	0.27	2.16	1.26	0.44	1.25	1.06	0.90	1.006	1.001
0.190	3.82	1.55	0.28	2.05	1.23	0.46	1.21	1.05	1.00	1.002	1.001
0.195	3.65	1.52	0.29	1.95	1.21	0.48	1.17	1.04			

From the discussions in this paragraph we may conclude that, as far as the existing wave-theories go, we may adopt formula (52) for the second order corrections of the old observations, except for the observations in shallow water where (99 B) must be used. In the next paragraph we shall see that the experimental evidence which we obtained, also confirms this conclusion.

§ 12. *Observations of the ship's movements.*

In this paragraph we shall discuss the results of the experiments made with the slow pendulum apparatus and especially their bearing on the question whether the accelerations correspond to GERSTNER's and STOKES's wave-theories. For this purpose we dispose of the results of the voyages of Hr. Ms. Submarine O 12 from Curaçao to Holland in the autumn of 1937, of Hr. Ms. O 13 to the end of the Channel in May 1938 and of Hr. Ms. O 19 in the North Sea in July 1939.

During the first voyage the writer made experiments with the new apparatus in its first shape, consisting of only one slow pendulum which could be attached to the pendulum apparatus in two positions, one parallel to the swinging-plane of the main pendulums and one at right angles to it. During the observations the ship was following different courses with regard to the direction of the waves, in most cases by slowly describing a circle or part of it. In two cases observations have been made at the same station at two different depths, 20 m and 40 m, and both accompanied by pendulum observations in order to see whether we should be able to draw conclusions about the second order corrections from the difference of the results; this has not been possible because of the irregularities in the chronometer rate. Although the results of the observations with the slow pendulum gave a first confirmation of the equality of the amplitudes of the vertical and horizontal accelerations as GERSTNER's and STOKES's theories predict it, they gave less conclusive evidence than the results of the following expeditions. In the first place only parts of the records could be used because the spring suspension of the slow pendulum did not work quite satisfactorily and so the record disappeared from time to time. In the second place the wave-movement was in many cases not regular enough to determine a unique direction for a certain wave and so the observation of only one component of the horizontal acceleration at the same time did not suffice for deriving the horizontal acceleration in the plane of the wave-movement. As a consequence of this we shall not mention here the results of this expedition save a few remarks which will follow below. We shall confine ourselves for our purpose of investigating the wave and ship's movements to the results of the two last voyages.

One observation of the first voyage is wordt mentioning here; its record shows evidence of the ship describing an orbit in a tilted plane. The horizontal acceleration during a special course of the ship had a small amplitude and it had the same phase as the vertical acceleration and so this would correspond to a movement in an inclined plane perpendicular to the swinging-plane of the slow pendulum. The record was not complete enough to make sure of the orbit of the ship in this plane. This experience might point to a wave of the type as investigated by JEFFREYS and mentioned under 4° in the preceding paragraph. The result has not been repeated; no other observation of this or the following expeditions has shown a similar effect. As far as they have been analysed the further records all indicate movements in vertical planes.

In the second place we may mention an experience which was confirmed by the later voyages, viz that often during submergence other and longer waves predominated than had been noticed at the surface. This is in good agreement with formula ($72\,D$) according to which a long wave diminishes less in amplitude than a shorter one. It also proves to be difficult to estimate the direction of a surface wave with some accuracy and it is still more difficult to make a satisfactory estimate of its length; in some cases differences could only be attributed to errors in the surface estimates. The estimate of the length is usually too small.

During the voyages of Hr. Ms. O 13 and O 19 the observations have been made with the new apparatus in its second shape as described in this publication. So both components of the horizontal acceleration could be determined at the same time and we could, herefore, derive for each moment the three components together. We thus could better deduce the plane of the movement as well as the orbit of the ship. During the observations the ship has again slowly described parts of a circle for observing the ship's movements for different courses with regard to the direction of the waves. In many cases we also have made observations accompanied by pendulum observations at two different depths at the same station. We shall afterwards come back to the results of these observations; they have been indicated by A and B in the following tables.

We shall begin by discussing the results for the determination of the accelerations by means of the two slow pendulums and the measuring of the fluctuations of the second-marks in the records of the main pendulums. For deriving the amplitudes a and b of the two components of the horizontal accelerations we have used the formulas ($69\,A$) and ($69\,B$) and for computing the amplitude c of the vertical accelerations the formula ($68\,A$). For the voyage of the O 13 we have adopted the following values for computing F_y and F_z

$$T_a = 3.1 \text{ sec}, \qquad T_{sy} = 22.5 \text{ sec},$$
$$T_b = 4.3 \text{ sec}, \qquad T_{sz} = 24.0 \text{ sec},$$

After this voyage the gimbals have been modified according to what has been mentioned on page 36; up to that moment they had been left in their old condition. They have then be adjusted in such a way that the periods in both directions were equal and the same has been done for the slow pendulums. We thus had for the second voyage

$$T_a = T_b = 3.63 \text{ sec}, \qquad T_{sy} = T_{sz} = 26.0 \text{ sec}.$$

The computations of the values of F and of the other quantities mentioned in the following table have been extended at least one decimal further than would correspond to the accuracy of the measured figures. This has been done for not losing accuracy by the computations. We may further mention that between the two voyages the slow pendulums have been provided with the damping device described in § 10 and that, therefore, the measurements of the amplitudes of the slow pendulums could be better made in the records of the last voyage than in those of the previous ones because of the absence of the fluctuations in their own period.

The computations have only been made for a small part or a few small parts of each record, viz for those parts where the records of the slow pendulums showed a fairly regular curve. Three of these parts are represented by the Plates V, VI and VII and so the reader is able to check the corresponding measurements with the figures in the table below. With these results the material obtained during these voyages is far from being exhausted; there are a number of other parts sufficiently regular for the same treatment and by a harmonic analysis of the curves we could even make available the entire records of the slow pendulums as we thus could separate them in their different harmonic terms. The figures that have been obtained seem, however, to be sufficient for a first attempt at drawing a conclusion.

The figures are given by table I. It gives the numbers of the stations of both voyages, the depth of submergence of the ship's centre, i.e. $2\frac{1}{2}$ m less than the depth of the keel as given by the depth-gauge, the double amplitude $2\,a_v$ of the main pendulum of which the record was used for measuring the fluctuations of the second-marks, the double amplitude $2\,a_{F0}$ of these fluctuations reduced by means of $(68\,C)$ to the value that would have been found if they had been measured in the axis of the record, the double amplitudes $2\,a_y$ and $2\,a_z$ measured in the records of the slow pendulums parallel to the swinging-plane of the main pendulums and perpendicular to this plane, the period T_w of the waves as derived from these records, the values of the constants F_y and F_z computed by means of the formulas $(69\,A)$ and $(69\,B)$, the values of a/g and b/g derived by means of the first parts of these same formulas, the resultant e/g of the components a/g and b/g, the value of c/g determined by means of $(68\,A)$, the ratio e/c of the amplitudes of the vertcial and horizontal accelerations, this same ratio e/c (f) as predicted by LAPLACE's, GERSTNER's and STOKES's wave-theories and derived by means of the tables on page 62, and lastly the differences of the values of the last two columns. The mean value of this difference for all the computed parts is given below.

In table II are given the station-numbers in the list of sea-gravity stations of the Netherlands Geodetic Commission, the positions of the stations and the sea-depth, the depth of submergence of the ship's centre, the wave-lengths λ as derived from T_w by means of formula $(72\,C)$ eventually divided by the factor P taken from table I of page 62, the azimuths α of the waves as found by means of

TABLE I.

No	depth of subm.	$2\,a_v$	$2\,a_{Fo}$	$2\,a_y$	$2\,a_z$	T_w	F_y	F_z	$a/g \times 1000$	$b/g \times 1000$	$e/g \times 1000$	$c/g \times 1000$	e/c	$e/c\,(f)$	$e\,c - e/c(f)$
O 13	m	cm	cm	cm	cm	sec									
1	17.5	3.12	0.13	1.38	1.55	6.6	1.378	1.822	10.02	8.51	13.15	12.62	1.04	1.02	+ 0.02
2 *A*	17.5	2.36	0.107	0.880	2.562	6.3	1.405	1.946	6.26	13.17	14.58	14.40	1.01	1.00	+ 0.01
3 *A*	17.5	4.16	0.52	4.20	1.84	8.0	1.322	1.532	31.8	12.0	34.0	31.2	1.09	1.00	+ 0.09
3 *B*	37.5	3.55	0.125	1.113	0.273	8.0	1.322	1.532	8.42	1.78	8.61	9.04	0.95	1.00	— 0.05
4 *A*	17.5	3.60	0.195	1.883	0.978	7.5	1.331	1.598	14.15	6.12	15.41	14.46	1.06	1.00	+ 0.06
4 *B*	37.5	3.50	0.204	1.153	0.947	14.0	1.685	1.619	6.84	5.85	9.00	8.33	1.08	1.00	+ 0.08
5	37.5	3.37	0.115	0.855	0.08	11.0	1.401	1.446	6.10	0.55	6.13	6.21	0.99	1.00	— 0.01
5	37.5	3.03	0.111	0.89	0.38	12.6	1.521	1.512	5.85	2.51	6.36	5.81	1.09	1.00	+ 0.09
6 *A*	17.5	3.31	0.123	1.053	0.810	7.2	1.341	1.654	7.86	4.90	9.26	10.32	0.90	1.00	— 0.10
7 *A*	9.5	3.45	0.112	0.27	1.10	9.5	1.337	1.445	2.02	7.62	7.88	6.87	1.15	1.12	+ 0.03
7 *B*	17.5	3.90	0.055	0.31	0.43	10.0	1.350	1.437	2.30	2.99	3.77	2.82	1.34	1.35	— 0.01
O 19															
1 *A*	17.5	4.43	0.140	0.00	0.92	10.0	1.326	1.326	0.00	6.94	6.94	6.32	1.10	1.01	+ 0.09
1 *A*	17.5	3.85	0.111	0.00	0.755	10.0	1.326	1.326	0.00	5.69	5.69	5.76	0.99	1.01	— 0.02
1 *A*	17.5	3.75	0.117	0.10	0.855	10.0	1.326	1.326	0.75	6.37	6.42	6.24	1.03	1.01	+ 0.02

mean value $e/c - e/c\,(f) = +\,0.02$

$$tg\,(a - a_s) = \pm\,\frac{a}{b} \quad (a_s = \text{azimuth ship's course}) \quad (101)$$

and lastly the wave-lengths λ_e and azimuths a_e as estimated at the sea-surface. In applying formula (101) we find the quadrant of the angle $a - a_s$ by means of the rule that, when counting the azimuth of the incoming wave as is usual from north clockwise, the angle is in the first quadrant when the fluctuations of the records of both the slow pendulums are in the same phase with the fluctuations of those parts of the second-mark curve that correspond to the light-interruptions moving to the right in the control-window below the recording apparatus; if the fluctuation of the record of the slow pendulum swinging in a plane at right angles to the ship's axis is in contrary phase, the angle is in the 3rd or 4th quadrant, if that of the slow pendulum swinging parallel to the ship's axis is in contrary phase, it is in the 2nd or 3rd quadrant.

TABLE II.

No Stat.	No voy.	Latitude *N*	Longitude *W*	Sea-depth	depth of subm.	λ	$a-a_s$	a_s	a	λ_e	a_e
	O 13			m	m	m				m	
837	1	52° 06′	— 02° 46′	42	17.5	68	130°	283°	53°		70°
838 *A*	2 *A*	49 48	+ 03 31	75	17.5	62	25	39	64	40	70
839 *A*	3 *A*	48 56	+ 07 36	135	17.5	100	111	344	95	110	70
839 *B*	3 *B*	48 56	+ 07 36	135	37.5	100	78	350	68	110	70
840 *A*	4 *A*	47 09	+ 10 02	4250	17.5	88	67	1	68	60	40
840 *B*	4 *B*	47 09	+ 10 02	4250	37.5	306	229	50	279	60	40
841	5	47 15	+ 09 09	4400	37.5	189	275	0	355	90	35
841	5	47 15	+ 09 09	4400	37.5	248	293	0	293	90	35
842 *A*	6 *A*	48 36	+ 06 30	125	17.5	81	58	267	325	110	0
843 *A*	7 *A*	52 22	— 03 01	40	9.5	133	195	170	5		15
843 *B*	7 *B*	52 22	— 03 01	40	17.5	144	218	162	20		15
	O 19										
844 *A*	1 *A*	56 18	— 01 39	90	17.5	156	180	180	0	40	0
844 *A*	1 *A*	56 18	— 01 39	90	17.5	156	180	180	0	40	0
844 *A*	1 *A*	56 18	— 01 39	90	17.5	156	187	180	7	40	0

From these tables we can draw several conclusions. In the first place the small values in the last column of table I give us a confirmation of the main point of our investigation. This column lists the differences of the observed ratio of the horizontal and vertical accelerations and the ratio computed according to the wave-theories of GERSTNER, STOKES and LAPLACE and so we see these theories confirmed. We need not stress here again the importance of this result for the reduction of the old pendulum observations at sea for the second order effects; it makes it possible to apply them and the uncertainty of the corrections is even small. This is a remarkable piece of good luck; when Mr. BROWNE discovered these effects in 1937, it was of course likely that it would be impossible to correct the old results for them as no measurements had ever been made to meet the case. The fact that we can derive the vertical acceleration from the fluctuation of the second-marks in the records of the pendulums is the first piece of good luck. The second one is that the ship's movement seems to be so regular that we can derive the necessary data about the horizontal accelerations from those of the vertical accelerations. In the next paragraph we shall make use of this result for deriving the corrections for the old gravity observations at sea.

The last column of table I gives in fact reason to consider it likely that the result will prove generally valid. The values are not larger than we might expect as a result of measuring errors; this is clear when we notice that the values of $2\,a_{F_0}$ are usually of the order of one or two millimeter and the uncertainty in the measuring of the amplitudes of the slow pendulum would explain errors of the same magnitude. The maximum value of the column is only — 0.10 and the mean value, i.e. the root of the mean of the squares, is not more than 0.057.

According to the formulas (51) the total second order correction δg is

$$\delta g = - \left[2 \left(\frac{e}{c} \right)^2 - 1 \right] \delta g_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

where δg_1 is the effect for the vertical accelerations, which we can compute for the old observations. So the mean deviation of e/c of 0.057 gives a mean deviation in the factor of (102) from the value 1 of $2 \times 1.057^2 - 1 = 0.23$ and the result for the total second order correction when computed by assuming this factor to have the value 1, i.e. by applying formula (52), has a mean error of 23 %. As the total correction is only seldom larger than 10 mgal and usually smaller, this does not amount to a serious increase of the total mean error of the old gravity determinations at sea. This estimate of the mean error is less than the first rough estimate of 50 % the writer gave in his paper in the Proceedings of the Royal Academy of Amsterdam of 1938 when he had not yet at his disposal the above figures. He may add here that he should not wish to give already an estimate of the mean error of the total second order corrections for future observations when the records of the slow pendulums will be available for their determination; only one station has been occupied since the slow pendulums have been provided with a damping, which makes their records more regular and more easy to use for this purpose, and this is not sufficient for giving already an estimate of the mean error of these measurements.

The mean value of 0.02 below the first column of table I is hardly larger than the mean error of this mean value, which has a value of $0.057 : \sqrt{14} = 0.015$. So we can not yet decide whether it is a real deviation of the ratio as predicted by GERSTNER and STOKES of the vertical and the horizontal accelerations or if it is brought about by the measuring errors. In the future we may expect more material for giving an answer to this question.

Twelve of the fourteen values of the last column of table I give a confirmation of GERSTNER's or STOKES's equations. In four cases, viz for Nos

1, 7 *A*, 7 *B* and the last, the water was so shallow that we had to apply LAPLACE's equations but only for the Nos 7 *A* and 7 *B* the difference exceeded the mean error; as the column preceding the last shows, the ratio according to LAPLACE for these two cases ought to be 1.12 and 1.35. The observations gave 1.15 and 1.34 and so this gives a good confirmation of LAPLACE's theory. Although two observations are certainly not sufficient for drawing a final conclusion, we may no doubt tentatively adopt it and use the corresponding tables of page 62 for observations in shallow water. In the future more data will no doubt be available for a decision.

We may also conclude from table I that the ratio e/c does niet appear to show a systematic deviation when the angle between the direction of the waves and the ship's axis is small, i.e. when $\alpha - \alpha_s$ is small or near to 180°, as table II shows e.g. to have been the case for the observations 2 *A*, 7 *A* and 7 *B*. So we may infer that also in this case the orbit of the ship's centre is about circular and this is especially remarkable for the observation 2 *A* because in this case the wave-length was only 62 meters; as the ship's length is about the same, we must suppose that during this observation the ship's centre can not entirely have followed the orbit of the water-particles. So this appears to confirm our tentative supposition made on page 51 that even in case the ship is not small with regard to the wave-length the orbit of the ship's centre will be circular, although it may have less amplitude than the orbits of the water-particles. We require, however, more observations to make sure about this point.

For several stations where observations have been made at two depths, the water-movement at the greater depth was not sufficient for giving measurable fluctuations in the second-mark curve of the pendulum-records, but in three cases, viz nos 3, 4 and 7, we, could determine the accelerations for both observations. We shall here compare these data and derive the amplitudes of the wave-movement as they follow from the maximum vertical acceleration *c*. From *c* we derive the radius *r* of the circular orbit of the water-particles, or in case of an elliptic orbit the vertical half axis of the ellipse, by means of

$$r = \frac{T_w^2}{4\pi^2}\, c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (103)$$

and the height of the wave h has double this value. We shall indicate the height of the wave at the smaller depth by h_1 and that at the larger by h_2. By means of formula (72 *D*) we may also derive the radius r at the surface and from this the height h_s of the waves at the surface.

For stations nos 3 and 7 we derived from table II and from the above computations

	depth of submerg.	wave-length	azimuth waves	height waves	height at surface
no 3 *A*	17.5 m	$\lambda = 100$ m	$a = 95°$	$h_1 = 99$ cm	$h_s = 297$ cm
no 3 *B*	37.5 m	$\lambda = 100$ m	$a = 68°$	$h_2 = 28.8$ cm	$h_s = 304$ cm
surface estimate		$\lambda = 110$ m	$a = 70°$		
no 7 *A*	9.5 m	$\lambda = 133$ m	$a = 5°$	$h_1 = 30.8$ cm	$h_s = 48$ cm
no 7 *B*	17.5 m	$\lambda = 144$ m	$a = 20°$	$h_2 = 14.0$ cm	$h_s = 30$ cm
surface estimate		$\lambda =$?	$a = 15°$		

For both stations it is clear that we have measured the same wave at the smaller and at the larger depth and that it is also the same as the estimated wave at the sea-surface; the values for the wave-length and for the azimuth agree inside the limits we may expect. The heights h_1 and h_2 of the waves at the surface as derived by formula (72 *D*) from the height during submergence also agree well. As the height of the individual waves of a system differ much, this agreement must be more or less fortuitous.

For station no 4 we have the following results

	depth of submerg.	wave-length	azimuth waves	height waves	height at surface
no 4 *A*	17.5 m	$\lambda = 88$ m	$a = 68°$	$h_1 = 40.5$ cm	$h_s = 141$ cm
no 4 *B*	37.5 m	$\lambda = 306$ m	$a = 279°$	$h_2 = 81.1$ cm	$h_s = 175$ cm
surface estimate		$\lambda = 60$ m	$a = 40°$		

We see that in this case we have found two different waves, a wave of a length of 88 m coming from *ENE* and a long swell of a length of 306 m coming from a nearly opposite direction. The first has obviously been noticed at the surface but it is a curious fact that the swell has not been observed although its computed amplitude at the surface is even larger than that of the shorter wave. This is no doubt caused by its enormous length.

Knowing it is there, we may find its effect in the record of both slow pendulums made at the depth of 17.5 m; we can e.g. discern it in the part reproduced in Plate V, where the same part has been represented as has been measured. The relative azimuth of the swell with regard to the ship's axis for this part has been computed at 229°; the upper record of this figure is that of the slow pendulum swinging parallel to the swinging-plane of the main pendulums, the lower that of the other. We may expect the swell to be discernible in this record because according to formula (72 *D*) its height at that depth must have been 122 cm. If we derive the height of the shorter wave at the deph of 37.5 m we find 9.7 cm while the swell had an amplitude there of 81.1 cm. So we may understand that the shorter wave can only hardly be detected in this record as we may notice in Plate VI where part of it is reproduced.

Table II gives several other instances of large divergences of the measured wave during submergence from the estimated one at the surface; we may for some of them at least suppose that here also a shorter surface wave of which the effect had practically disappeared at the depth of the observation, has at the surface hidden the longer wave that has been measured.

We may broach here a point that might give misgivings in reading the records. The free swinging of the gimbals in the Z direction, i.e. at right angles to the swinging-plane of the main pendulums, must show in the record of the damped pendulum as well as in that of the slow pendulum swinging in this same sense. The amplitudes of these fluctuations in both directions must have a ratio of 2.26 because the factors of multiplication brought about by the reflection of the recording lightrays on the prisms attached to the damped pendulum and to the slow pendulum are 1.00 and 0.50 while the focal distances of the lenses converging the lightrays towards the photographic paper are 1.13 m and 1.00 m. For a swinging of the gimbals brought about by the ship's movements because of the waves, the ratio of the recorded fluctuations is quite different. In this case the damped as well as the slow pendulum get amplitude and the result for the fluctuations in the records can be derived from formula (6) combined with (16 A) or (16 B) and from formulas (59 A) and (60 B) or (60 C). Many of the fluctuations recorded by the damped pendulum belong to the second class and so the ratio of 2.26 which we might expect, only seldom occurs.

The attempts to check the formulas for the effect of the wave-movements by making pendulum observations at the same station at different depths, for which the wave-movement must be different, have not been successful. For the voyages of Hr. Ms. O 12 and O 13 we have used the old time-keepers and their rate did not prove regular enough to enable us to compare the results of the successive observations at the same station as accurately as was needed for this purpose. Since then the Netherlands Geodetic Commission was fortunate enough to obtain the loan of the crystal chronometer of the Bell Telephone Laboratories, which was generously put at our disposal by Prof. Dr. RICHARD M. FIELD of Princeton. This time-keeper has an extremely regular rate and thus would entirely discard our difficulty. Since it arrived, however, the political situation has prevented scientific expeditions except one opportunity the Netherlands Navy has been able to give, the short trial trip of Hr. Ms. O 19. During this trip in the North Sea Dr. NIEUWENKAMP could make one double observation at 17.5 m and 37.5 m. We shall here mention the results but for three reasons the attempt has not been as precise as was hoped for. In the first place the wave-movement was small and so the second order effect was only 3.7 mgal at 17.5 m depth and nothing at 37.5 m. So we should have needed great accuracy in the observations for checking the formulas in an effective way. In the second place there was some uncertainty about the steadiness of the current during the two observations and in the third place the two observations have not been made at exactly the same place and the gradient of gravity in the sense of the displacement of about two miles was of course not known. The effect of the two latter circumstances can not amount to much but they can doubtless bring about a change of one mgal in the difference of the results for the two observations. So this means a considerable part of the difference in the second order corrections which we wished to determine experimentally.

For increasing the data and for trying to neutralize the two latter causes of uncertainty — the two observations have been made one after the other while keeping the course of the ship the same and the effects of a varying current and of a gradient of gravity may perhaps be assumed to change linearly with the displacement — both observations have been divided in two parts and each part has been separately computed. The results after applying the second order correction as it had been determined by means of (68 B) and (69 D) were

1 A (depth 17.5 m)	part I	$g = 981.6714$,	second order correction	$= 0.0040$,
	part II	$g = .6716$,	,, ,, ,,	$= 0.0034$,
1 B (depth 37.5 m)	part I	$g = 981.6704$,	,, ,, ,,	$= 0.000$,
	part II	$g = .6704$,	,, ,, ,,	$= 0.000$.

We may conclude that the corroboration is as good as we may expect it to be, but that the check does not go further than affirming that there are no large effects of the wave-movements that have been overlooked. We may hope that this line of investigation will allow a more precise check in the future.

§ 13. *The second order corrections* (BROWNE *terms*) *for the old observations of the Netherlands Geodetic Commission.*

Assuming in accordance with GERSTNER's and STOKES's wave-theories that the mean squares of the vertical and horizontal accelerations are equal, we have been able to compute the second order corrections for the old gravity observations at sea of the Netherlands Geodetic Commission. For doing this we had likewise to assume that we may omit the factor between the square brackets of formula (69 C). It is easy to derive the error which thus ensues; only in case of smaller values of T_w than 7 sec and if the wave-direction makes a small angle with the ship's axis, it may attain to values larger than 14 % of the second order correction.

Basing our computations on these assumptions, we have applied the first part of formula (69 E) and so we have adopted the total correction δg to be equal to the correction for the vertical accelerations δg_1 with reversed sign. For determining this correction δg_1 we have followed the way described on page 31 at the end of § 7. For each record showing fluctuations of the second-mark curve we thus found the number of fluctuations per minute N_w and the mean q of the squares of the ratio of a_{Fm} in mm divided by a_v in cm, a_{Fm} being the mean amplitude of the fluctuations measured in the middle half of the pendulum record and a_v the amplitude of the record itself. For deriving δg we introduced these values in the formula (57 C) (= 68 B combined with 68 D) which by substituting the values 980 and 1.0024 for g and T becomes

$$\delta g = 1.500\, q\, N_w^2 \quad \ldots\ldots\ldots \quad (104)$$

with

$$q = 100 \frac{a_{Fm}^2}{a_v^2} \quad \ldots\ldots\ldots \quad (104\,A)$$

and

$$N_w = \frac{60}{T_w} \quad \ldots\ldots\ldots\ldots \quad (104\,B)$$

For shallow water we have applied formula ($99\,B$) corresponding to LAPLACE's wave-theory, i.e. we have used formula (104) multiplied with the factor F of table II on page 62. This gave a difference for only two stations, viz nos 193 and 373. In the provisional list of second order corrections, published in the paper in the Proceedings of the Royal Academy of Amsterdam, 1938, no. 6, we have neglected this effect and have given the figures as computed by means of (104), i.e. — 2.3 mgal and — 3.8 mgal; because of the factors N of 3.76 and 2.67 they change in — 8.6 mgal and — 10.1 mgal.[1])

The second order corrections could not be derived for the stations nos 1 — 32 of the first voyage, because these observations have been made with the old apparatus which did not directly record the fictitious pendulums and so it was practically impossible to find the fluctuations of the second-marks for these records. As it happens the second order effects for these stations will probably not have been large as the observations with the old apparatus could only be made for small ship's movements.

In the following table only those stations have been mentioned for which the corrections are larger than 1 mgal. For station no 183 the record was missing and so the correction could not be determined; for this station we adopted the mean correction of nos 182 and 184, which were observed at time-intervals of about six hours before and after no 183. As they have all three been occupied in the Indian Ocean south of the island of Java where the waves are long and therefore slowly variable, there is a good chance that this estimate is not far wrong; the wave-period T_w for no 182 was 6.5 sec and for no 184 6.6 sec and so this also points to a stable condition of the sea.

The table contains the number of the stations as given in the lists of „Gravity Expeditions at Sea", vols I and II, the number of the observation of the voyages, the latitude, the longitude, the sea-depth, the value of q, the value of N_w, the wave-period T_w derived from N_w and the second order correction δg. Incase two observations have been made at the same station, distinguished by A and B, we have given the correction for each of them and also the mean that has been applied to the mean result of both observations as given in the station-list of Gravity Expeditions at Sea, Vol. II, p. 89 e.s. From the values of T_w in this table we may conclude that our surmise has been right that during submergence the wave-period is usually larger than 7 sec. We see also that generally the period in the open ocean is larger than

1) In studying the matter a printer's error was found in the list of stations of „Gravity Expeditions at Sea", vol II, p. 92; the sea-depth of station no 246 must be 98 m instead of 10 m as given in the list.

in more or less enclosed seas; this concurs likewise with what we may expect.

The results δg show that in general the second order corrections are small; they seldom attain to values of 10 mgal or more. For the majority of the observations they were less than 2 mgal and so they do not appear in the table. The large value for no 57 corresponds to an observation made at the surface of the sea.

TABLE OF THE SECOND ORDER CORRECTIONS (TERMS OF BROWNE) FOR THE OLD GRAVITY OBSERVATIONS AT SEA OF THE NETHERLANDS GEODETIC COMMISSION.

All the corrections have negative sign.

No station	No voyage	Latitude	Longitude	Depth meters	q	N_w	T_w sec.	δg mgal.
	K XI	° ′	° ′					(—)
33*A*	1*A*	49 04 N	05 55 W	109	0.202	6.7	9.0	13.6
33*B*	1*B*	49 04 N	05 55 W	109	0.318	6.8	8.8	22.1
33	mean	49 04 N	05 55 W	109				17.8
34	2	44 59 N	08 39 W	4720	0.268	6.7	9.0	18.1
36*A*	4*A*	36 10 N	03 25 W	620	0.055	8.5	7.1	6.0
36*B*	4*B*	36 10 N	03 25 W	620	0.023	8.3	7.2	2.4
36	mean	36 10 N	03 25 W	620				4.2
37	5	36 52 N	00 01 E	2540	0.172	10.1	5.9	26.3
38*A*	6*A*	37 08 N	04 17 E	2300	0.085	9.3	6.5	11.0
38*B*	6*B*	37 08 N	04 17 E	2300	0.070	9.8	6.1	10.1
38	mean	37 08 N	04 17 E	2300				10.6
39	7	37 05 N	04 28 E	1780	0.060	9.7	6.2	8.5
40*A*	9*A*	36 20 N	15 31 E	1130	0.305	7.3	8.2	24.4
40*B*	9*B*	36 20 N	15 31 E	1130	0.298	7.1	8.5	22.6
40	mean	36 20 N	15 31 E	1130				23.5
41	10	35 13 N	18 31 E	3630	0.067	7.6	7.9	5.8
42*A*	11*A*	33 56 N	22 12 E	2180	0.042	7.5	8.0	3.6
42*B*	11*B*	33 56 N	22 12 E	2180	0.017	7.1	8.5	1.3
42	mean	33 56 N	22 12 E	2180				2.1
	K XIII							
43	1	44 18 N	15 27 W	5200	0.128	6.5	9.2	8.2
44	2	43 11 N	18 44 W	4570	0.038	6.5	9.2	2.4
45	3	41 22 N	21 39 W	4040	0.472	5.6	10.8	22.4
46	4	39 48 N	24 57 W	3580	0.192	5.7	10.5	9.5
48	6	36 23 N	26 43 W	3610	0.073	5.9	10.2	3.7
49	7	33 42 N	24 19 W	5480	0.070	5.8	10.3	3.5
50	8	30 48 N	22 30 W	5170	0.075	6.1	9.8	4.2
51	9	29 20 N	19 15 W	4610	0.108	6.0	10.0	5.8
52	10	28 40 N	15 53 W	3630	0.135	7.1	8.5	10.2
54	12	27 13 N	17 48 W	3730	0.072	7.5	8.0	6.4
55	13	26 33 N	21 37 W	4780	0.120	6.8	8.8	8.3
56	14	25 44 N	25 19 W	4990	0.072	6.1	9.8	4.1
57	15	25 07 N	28 50 W	5310	0.900	5.7	10.5	44.0
58	16	25 00 N	32 30 W	5920	0.171	6.0	10.0	9.1
59	17	24 34 N	35 57 W	5950	0.173	6.0	10.0	9.4
60	18	24 03 N	39 34 W	5490	0.150	5.9	10.2	7.9

No station	No voyage	Latitude		Longitude		Depth meters	q	N_w	T_w sec.	δg mgal.
		°	′	°	′					(—)
61	19	23	44 N	43	00 W	3970	0.172	6.6	9.1	11.3
62	20	23	21 N	47	05 W	3550	0.080	6.2	9.7	4.6
63	21	23	03 N	50	45 W	4870	0.070	6.2	9.7	4.1
64	22	22	45 N	54	34 W	5920	0.065	7.1	8.5	4.9
65	23	22	17 N	58	24 W	5880	0.103	7.2	8.3	8.0
66	24	21	35 N	63	22 W	5690	0.065	6.1	9.8	3.6
67	25	20	44 N	65	37 W	5510	0.052	6.0	10.0	2.9
68	26	19	32 N	66	46 W	8040	0.045	7.0	8.6	3.3
69	27	18	24 N	67	42 W	290	0.030	7.8	7.7	2.8
70	28	16	55 N	67	42 W	4930	0.052	8.0	7.5	5.0
73*A*	31*A*	12	55 N	71	54 W	1570	0.060	6.1	9.8	3.4
73*B*	31*B*	12	55 N	71	54 W	1570	0.060	7.4	8.1	4.9
73	mean	12	55 N	71	54 W	1570				4.2
74	32	10	22 N	77	13 W	3320	0.060	7.0	8.6	4.4
75	33	09	51 N	78	01 W	1480	0.167	6.8	8.8	11.7
78	38	07	01 N	82	37 W	2990	0.125	5.1	11.8	4.8
79	39	10	21 N	88	30 W	3470	0.151	6.1	9.8	8.4
80	40	13	35 N	95	27 W	3870	0.127	5.0	12.0	4.8
81	41	15	03 N	98	20 W	3660	0.181	4.7	12.8	5.9
82	42	15	17 N	98	23 W	3970	0.245	4.7	12.8	8.1
83	43	15	35 N	98	20 W	4730	0.126	5.0	12.0	4.7
84	44	15	55 N	98	16 W	890	0.101	5.4	11.1	4.4
85	45	17	35 N	103	26 W	5030	0.142	5.3	11.3	6.0
86	46	18	01 N	103	25 W	3050	0.122	5.5	10.9	5.6
87	47	18	15 N	103	26 W	710	0.143	5.5	10.9	6.5
88*A*	48*A*	23	06.0N	106	24.8W	Mazatlan	0.045	7.5	8.0	3.8
88*B*	48*B*	23	06.0N	106	24.8W	Mazatlan	0.052	6.1	9.8	3.0
88	mean	23	06.0N	106	24.8W	Mazatlan				3.4
89	49	24	24 N	113	23 W	3610	0.136	5.6	10.8	6.4
90	50	26	32 N	115	40 W	3140	0.213	6.9	8.7	15.1
91	51	27	02 N	115	21 W	3340	0.045	5.9	10.2	2.4
92	52	27	20 N	115	10 W	3080	0.127	6.9	8.7	9.0
93	53	29	13 N	117	06 W	3320	0.058	7.8	7.7	5.3
94	54	31	01 N	119	19 W	3760	0.109	7.8	7.7	10.0
95	55	33	12 N	121	30 W	4010	0.060	7.1	8.5	4.6
96	56	35	50 N	122	43 W	3490	0.092	6.0	10.0	5.1
98	58	37	29 N	123	09 W	680	0.142	5.6	10.8	6.6
99	59	37	06 N	123	54 W	3760	0.143	5.2	11.5	5.8
100	60	36	22 N	125	22 W	4520	0.182	6.2	9.7	10.5

No station	No voyage	Latitude		Longitude		Depth meters	q	N_w	T_w sec.	δg mgal.
		°	′	°	′					(—)
101	61	34	46 N	128	34 W	4920	0.100	6.0	10.0	5.4
102	62	33	15 N	131	52 W	5120	0.070	6.4	9.4	4.3
103	63	32	12 N	134	01 W	4930	0.075	5.0	12.0	2.9
104	64	30	15 N	138	25 W	4820	0.060	5.5	10.9	2.8
105	65	28	57 N	141	14 W	4970	0.112	6.9	8.7	8.0
106	66	27	27 N	144	21 W	4900	0.075	5.6	10.8	3.5
107	67	25	45 N	147	43 W	5210	0.085	6.8	8.8	5.9
108*A*	68*A*	24	19 N	150	48 W	5270	0.065	6.2	9.7	3.7
108*B*	68*B*	24	19 N	150	48 W	5270	0.058	6.2	9.7	3.3
108	mean	24	19 N	150	48 W	5270				3.5
109*A*	69*A*	22	57 N	153	40 W	4590	0.112	6.6	9.1	7.4
109*B*	69*B*	22	57 N	153	40 W	4590	0.088	7.6	7.9	7.7
109	mean	22	57 N	153	40 W	4590				7.6
110	70	22	13 N	155	24 W	4510	0.078	5.8	10.3	4.0
112	72	21	09.0N	157	28.0W	510	0.058	7.1	8.5	4.4
114	74	20	48 N	158	36 W	4290	0.060	7.0	8.6	4.4
115	75	20	29 N	160	30 W	4590	0.065	5.9	10.2	3.4
116	76	19	58 N	164	56 W	4960	0.042	6.5	9.2	2.6
117	77	19	31 N	168	27 W	3520	0.060	6.6	9.1	4.0
119	79	18	37 N	175	00 W	1790	0.088	5.4	11.1	3.9
120	80	18	06 N	178	14 W	3830	0.085	6.0	10.0	4.6
121	81	17	47 N	179	33 W	4900	0.065	6.0	10.0	3.5
122	82	17	02 N	176	24 E	3190	0.052	5.8	10.3	2.6
123	83	16	12 N	171	53 E	5600	0.085	5.9	10.2	4.5
124	84	15	32 N	168	27 E	5600	0.075	5.6	10.8	3.5
125	85	15	07 N	164	56 E	5330	0.050	6.1	9.8	2.8
126	86	14	43 N	161	30 E	5490	0.055	5.9	10.2	2.9
128	88	13	38 N	155	56 E	5910	0.058	5.9	10.2	3.1
129	89	12	42 N	150	58 E	5770	0.058	5.4	11.1	2.5
130	90	12	02 N	147	38 E	5620	0.040	6.1	9.8	2.2
135	95	13	42 N	142	53 E	3610	0.108	5.6	10.8	5.1
137	97	09	57 N	140	05 E	2350	0.068	6.2	9.7	4.0
138	98	09	25 N	138	34 E	7690	0.072	7.0	8.6	5.3
140	100	09	21 N	138	47 E	4510	0.125	5.4	11.1	5.5
141	101	09	30 N	136	36 E	4780	0.098	5.6	10.8	4.6
142	102	09	52 N	132	46 E	6050	0.080	6.6	9.1	5.3
145	105	10	17 N	126	41 E	8740	0.040	6.0	10.0	2.2
170	131	08	51 S	114	38 E	270	0.105	6.8	8.8	7.4
171	132	09	41 S	114	15 E	3950	0.112	5.0	12.0	4.2

No station	No voyage	Latitude		Longitude		Depth meters	q	N_w	T_w sec.	δg mgal.
		°	′	°	′					(—)
172	133	10	23 S	113	55 E	2680	0.104	6.4	9.4	6.4
173	134	10	59 S	113	40 E	6680	0.092	6.2	9.7	5.3
174	135	11	40 S	113	15 E	4280	0.163	6.2	9.7	9.5
176	137	12	24 S	110	00 E	4390	0.080	5.0	10.2	3.0
177	138	11	05 S	110	04 E	5040	0.090	4.9	12.2	3.3
178	139	10	18 S	110	03 E	7080	0.062	5.1	11.8	2.4
179	140	09	31 S	110	18 E	1320	0.095	5.5	10.9	4.3
180	141	09	00 S	110	33 E	3020	0.062	5.2	11.5	2.5
181	142	08	14 S	110	32 E	130	0.058	5.8	10.3	3.0
182	143	07	35 S	106	55 E	230	0.060	6.5	9.2	3.8
183	144	08	26 S	106	36 E	2210				4.8
184	145	09	21 S	106	21 E	6610	0.090	6.6	9.1	5.9
185	146	10	13 S	105	55 E	4780	0.040	5.5	10.9	1.9
186	147	11	42 S	105	24 E	5280	0.080	5.6	10.8	3.7
187	148	10	38 S	102	29 E	5120	0.122	5.1	11.8	4.8
188	149	09	00 S	102	47 E	5380	0.118	4.9	12.2	4.3
189	150	07	46 S	103	09 E	6260	0.100	5.6	10.8	4.7
190	151	07	04 S	102	29 E	4410	0.060	5.1	11.8	2.3
191	152	06	42 S	102	55 E	2410	0.052	5.9	10.2	2.8
192	153	06	58 S	104	07 E	2090	0.068	4.8	12.5	2.3
193	154	06	17 S	104	41 E	50	0.052	5.4	11.1	8.6
	N.E.I.									
197	2	08	54 S	118	13 E	500	0.188	5.6	10.8	8.9
199	4	09	32 S	118	10 E	2298	0.227	5.8	10.3	11.4
200	5	10	23 S	117	56 E	4234	0.122	6.4	9.4	7.5
201	6	11	01 S	117	47 E	4600	0.238	6.2	9.7	13.9
202*A*	7*A*	12	00 S	117	25 E	5400	0.137	6.2	9.7	7.9
202*B*	7*B*	12	00 S	117	25 E	5400	0.194	5.7	10.5	9.5
202	mean	12	00 S	117	25 E	5400				8.7
203*A*	8*A*	12	45 S	117	12 E	5500	0.068	6.0	10.0	3.6
203*B*	8*B*	12	45 S	117	12 E	5500	0.125	5.5	10.9	5.7
203	mean	12	45 S	117	12 E	5500				4.7
204	9	11	36.5 S	118	34.5 E	4900	0.058	5.8	10.3	3.0
205	10	11	13 S	119	20 E	6370	0.125	4.4	13.6	3.6
206	11	10	20.7 S	120	25.6 E	280	0.136	4.6	13.0	4.3
207	12	10	22 S	121	05 E	1582	0.150	4.8	12.5	5.2
208	13	09	29.5 S	121	21 E	2512	0.045	4.6	13.0	1.4
209	14	08	54.5 S	121	36 E	Endeh 960	0.055	5.1	11.8	2.2

No station	No voyage	Latitude		Longitude		Depth meters	q	N_w	T_w sec.	δg mgal.
		°	′	°	′					(—)
210	15	09	35 S	122	32 E	3100	0.052	5.0	12.0	2.0
212	17	10	27.2 S	123	48.2 E	205	0.062	4.5	13.3	1.9
213	18	10	48 S	124	10 E	2240	0.080	6.5	9.2	5.0
214*A*	19*A*	11	10 S	124	21 E	840	0.060	7.5	8.0	5.0
214*B*	19*B*	11	10 S	124	21 E	840	0.045	7.8	7.7	4.1
214	mean	11	10 S	124	21 E	840				4.6
215	20	10	15 S	126	43 E	462	0.062	9.0	6.7	7.6
216	21	09	13 S	126	37 E	1543	0.119	6.9	8.7	8.6
217	22	08	54 S	126	40 E	490	0.048	8.6	7.0	5.3
218	23	09	19 S	127	09 E	2970	0.062	8.6	7.0	6.9
219	24	09	35 S	131	04 E	230	0.072	8.9	6.7	8.6
220*A*	25*A*	08	30 S	131	00 E	1393	0.095	8.5	7.1	10.3
220*B*	25*B*	08	30 S	131	00 E	1393	0.087	8.2	7.3	8.8
220	mean	08	30 S	131	00 E	1393				9.6
225	30	05	40 S	130	12 E	2995	0.102	9.1	5.9	12.7
226	31	05	42 S	130	33 E	2721	0.078	9.5	6.3	10.7
227	32	05	36 S	131	08 E	7330	0.082	9.2	6.5	10.4
228	33	05	39 S	132	05 E	525	0.062	8.1	7.4	6.2
239	44	04	52 S	131	13 E	5080	0.048	9.8	6.1	6.9
242	48	04	13 S	130	21.5 E	5100	0.058	7.9	7.6	5.5
245	51	02	17 S	131	00 E	90	0.039	9.0	6.7	4.3
246	52	02	12.8 S	130	21.6 E	10	0.052	9.4	6.4	6.9
257	64	00	20 N	132	39 E	4500	0.045	9.8	6.1	6.5
258	65	01	03.2 N	131	16.5 E	1297	0.020	7.6	7.9	1.8
267	74	04	54 N	128	13 E	7780	0.040	9.9	6.1	5.9
268	75	04	33 N	127	03 E	980	0.038	8.9	6.7	4.5
269	76	04	23 N	126	25.5 E	3320	0.050	9.1	5.9	6.3
279	86	01	56.6 S	125	19.0 E	895	0.045	9.9	6.1	6.6
280	87	02	52 S	125	08 E	4800	0.048	10.0	6.0	7.3
281	88	03	32 S	124	58 E	5010	0.045	9.1	5.9	5.6
290	97	07	41.7 S	125	42.8 E	3850	0.085	8.5	7.1	9.2
293*A*	100*A*	08	08.2 S	122	43.5 E	1527	0.126	7.0	8.6	9.4
293*B*	100*B*	08	08.2 S	122	43.5 E	1527	0.070	7.4	8.1	5.7
293	mean	08	08.2 S	122	43.5 E	1527				7.6
294	101	08	14.8 S	121	37.0 E	1840	0.182	8.0	7.5	17.6
369	177	05	35.5 S	103	58.0 E	1050	0.055	6.0	10.0	3.0
370	178	04	51.1 S	103	22.0 E	220	0.093	5.9	10.2	4.8
371	179	03	47.0 S	102	14.7 E	Ben-koelen	0.048	6.6	9.1	3.2
372	180	04	09 S	101	45 E	1105	0.040	6.4	9.4	2.4

No station	No voyage	Latitude	Longitude	Depth meters	q	N_w	T_w sec.	δg mgal.
		° ′	° ′					(—)
373	181	04 29 S	101 17 E	50	0.068	6.1	9.8	10.1
374	182	05 01 S	100 30 E	5890	0.109	5.6	10.8	5.1
376	184	06 24 S	98 36 E	5520	0.085	5.9	10.2	4.5
377*A*	185*A*	05 34 S	98 02 E	5520	0.088	6.1	9.8	5.0
377*B*	185*B*	05 34 S	98 02 E	5520	0.110	6.3	9.5	5.8
377	mean	05 34 S	98 02 E	5520				5.4
378	186	04 43 S	97 16 E	4950	0.124	6.1	9.8	6.3
379	187	03 54 S	96 36 E	5070	0.038	6.5	9.2	2.2
380	188	03 11 S	97 39 E	4220	0.068	6.2	9.7	3.6
381*A*	189*A*	02 34 S	98 29 E	5311	0.100	6.0	10.0	4.9
381*B*	189*B*	02 34 S	98 29 E	5311	0.082	5.8	10.3	3.8
381	mean	02 34 S	98 29 E	5311				4.4
388	196	01 21.5 N	97 03.0 E	488	0.042	6.2	9.7	2.4
389	197	00 58.5 N	96 31 E	5280	0.035	6.1	9.8	2.0
390	198	00 20 N	95 36 E	4092	0.065	5.5	10.9	3.0
391	199	00 26 S	94 30 E	4590	0.070	5.8	10.3	3.5
392	200	00 18 N	93 14 E	4530	0.068	5.2	11.5	2.9
397	205	04 33 N	93 32 E	1380	0.102	5.0	12.0	3.8
398	206	05 12 N	94 12 E	2555	0.127	4.0	15.0	6.5
	O 13							
426	1	48 20 N	07 52 W	—	0.098	7.8	7.7	9.0
427	2	47 44 N	09 21 W	3650	0.042	7.1	8.5	3.2
428	3	46 21 N	12 49 W	4040	0.052	6.6	9.1	3.4
430	5	43 58 N	17 48 W	4550	0.032	7.0	8.6	2.3
431	6	43 14 N	19 36 W	4100	0.030	7.9	7.6	2.9
441	16	35 27 N	29 07 W	3600	0.035	7.2	8.3	2.7
442	17	35 51 N	30 19 W	3220	0.042	6.9	8.7	3.0
443	18	36 12 N	31 18 W	2910	0.022	6.8	8.8	1.5
444	19	36 37 N	32 23 W	2520	0.022	6.8	8.8	1.5
458	33	41 08 N	25 18 W	3640	0.042	6.0	10.0	2.3
459	34	40 30 N	24 09 W	4050	0.078	5.9	10.2	4.1
460	35	39 03 N	22 00 W	4410	0.050	5.5	10.9	2.3
470	45	32 50 N	15 54 W	4070	0.035	8.2	7.3	3.5
471	46	33 04 N	14 51 W	4220	0.020	7.6	7.9	1.8
472	47	33 20 N	13 50 W	4520	0.030	7.2	8.3	2.3
482	57	43 47 N	11 34 W	5060	0.035	6.5	9.2	2.2

CHAPTER IV.

The adjustment of the pendulum apparatus and other subjects.

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§ 14. *The adjustment of the pendulum apparatus and other preparatory measures before an expedition.*

Since the publication of 1929 a few new view-points about the adjustment of the apparatus and other preparatory measures have arisen and so we shall deal again with this subject in this paragraph. For a complete adjustment we have to attend to the following program:

I. We put the pendulum apparatus without its cover on a fixed support.

1. We level the apparatus first in the sense of the swinging-plane by putting the long level on the agate planes of the left and the middle pendulum and then in the other sense by putting the pendulum-level successively on the planes of all three pendulums and making the mean of the three positions zero. If we should find that the position of a plane deviates more than two minutes of arc, corresponding to a bubble deviation of more than two parts of the scale, it is better to readjust it by unscrewing the metal block in which it is fixed and putting metal foil below it.

After levelling we note the position of the two outside levels at the bottom of the apparatus which we require for the observations at the sea-stations; if they should deviate too much from the level position, it is advisable to readjust them by putting metal foil below the block in which they are fixed.

2. We adjust the slow lifting levers of the main pendulums by means of their adjustment screws,

a. in vertical sense in such a way that the knife-edges of the three pendulums touch the planes at about the same moment, e.g. after turning the wheel about two turns and a half; by putting a light behind the knife-edges we make sure that each knife-edge touches at the same moment over its full length;

b. in horizontal sense in such a way that the knife-edges, after lowering them on the planes, are parallel. This point, not mentioned in the publication of 1929, has some importance because, if the error would be too large, the basic condition for the functioning of the apparatus would not be fulfilled; we could no longer assume that the horizontal accelerations in the sense of the swinging-plane would be the same for the two pendulums of each pair;

an acceleration e.g. perpendicular to the swinging-plane of one of them would have a component in the swinging-plane of the other and none in that of the first.

For obtaining the parallellism of the knife-edges we put one of the pendulums successively on the three planes, leaving the other planes unoccupied, and we put a telescope in the swinging-plane at a distance of a few meters and at about the same height as the pendulum-mirror. In each position of the pendulum we adjust the lowering levers in such a way that looking in the telescope towards the pendulum-mirror we see the reflected image of the telescope or of a mark vertically below or above its axis. For putting the telescope in the desired position in the swinging-plane, we may put pointed marks, e.g. of cardboard, on two of the planes in such a way that the marks are exactly in the centres of the planes and we so arrange the telescope that we see the two marks in one line. Instead of using a telescope we can make the adjustment, if necessary, also by taking sights from a vertical slit, put up in the same way as the telescope. After making this adjustment we have again to check the adjustment in vertical sense and eventually to readjust the levers in this sense.

3. We make sure that the inner pendulums of the damped pendulums do not touch, i.e. that they can swing freely, and that the oil-filling is such that the damping functions satisfactorily.

II. We put on the cover of the apparatus without its top-plate and we put the recording-apparatus in its place. After switching on the lamp we adjust the light-rays:

4. We adjust the position of the lamp till the beam covers the slit-diaphragm in the light-box. The slit may be shortened if we desire to do so for making the images on the photographic paper sharper.

5. We follow the light-rays on the top-plate of the pendulum-apparatus and we make sure that the last prism in each ray is well lighted. If this would not be the case, we readjust the ray by means of its adjustable prism.

6. Bij means of the adjustable prism of the top-plate of the recording apparatus and those of the different rays in the pendulum apparatus we bring the light-rays in the control-window below the recording apparatus, that of the middle pendulum 0.6 mm higher, that of the right pendulum-pair 0.2 mm higher, that of the left pendulum-pair 0.2 mm lower and that of the damped pendulum for measuring the tilt 0.6 mm lower than would correspond to the horizontal axis of the window; these small shifts are required for obtaining that the images of the slit fall on the central strip of the cylinder-lens, which is about 4 cm nearer to the out-going lens of the pendulum apparatus than the control-window. We then arrange the images in the horizontal sense in the desired positions on the scale of the control-window; for doing this we dispose of the adjustable prism in the front-plate of the pendulum apparatus and the adjustable prism in each ray of this apparatus. For these positions and for those of the light-rays of the two slow pendulums, which we shall

treat of below, we may e.g. choose the following distances from the right edge of the photographic paper:

first fictitious pendulum (right pair of pendulums) . . .	25 mm,
second fictitious pendulum (left pair of pendulums). . .	70 mm.
middle pendulum	100 mm,
first slow pendulum	40 mm,
second slow pendulum	55 mm,
damped pendulum for measuring the tilt.	85 mm,
temperature	95—120 mm.

7. We have to ensure that each of the recording rays of the fictitious pendulums strikes the two pendulum mirrors of that ray under the same angle, because otherwise the ray would not make a record of $\theta_1 - \theta_2$ but of the difference of θ_1 and θ_2 multiplied by different factors. For checking this we lower the three pendulums on their knife-edges while the amplitude-levers are turned back and we do this so carefully that the images of the two fictitious pendulums in the control-window below the recording apparatus get no amplitude. We then give amplitude to the three pendulums by rythmically pushing the pendulum apparatus, taking care while doing so not to disturb the position of the apparatus; for making safe about this last point we may beforehand fix the foot-plates for the foot-screws of the pendulum apparatus. By our manipulation the three pendulums get exactly the same amplitude in the same phase. If our condition about the equality of the angles of incidence of the rays on the mirrors of the pendulums is fulfilled, the images of the two fictitious pendulums get no amplitude in horizontal sense although they may move slightly up and down. If one of them gets amplitude we have to readjust this ray. For the ray of the right pendulum-pair we have to do this by means of prism f of the figure on page 50 of the publication of 1929; we then bring back this ray in the control-window by means of the prism of the top plate of the recording apparatus and, as a result of this, we have to bring back all the other rays by means of the adjustable prism in each ray. For the ray of the left pendulum-pair we may do it by means of the adjustable prism e of the above-mentioned figure and we can bring it back in the control- window by means of prism g. From this program ensues that we have to begin by the adjustment of the ray of the righ pendulum-pair and that we then can adjust the ray of the left pair.

8. We adjust the amplitudes to be given to the outer pendulums by applying the amplitude-levers for these pendulums and adjusting their screws in such a way that the light-images of the two fictitious pendulums in the control-window are shifted over the desired amount, e.g. both over 21 mm. We note the position of the screws of the amplitude-levers. We do the same for the amplitudes needed for the special observations for determining the period of the middle pendulum at the base-station.

9. Having adjusted the light-rays and having shut the pendulum-apparatus, we now can proceed to make a record for determining the position of

the axis of the damped pendulum record with regard to the axis of the records of the two fictitious pendulums for the levelled position of the apparatus; we require this position for the measuring of the tilt during the observations of the expedition.

III. We put the apparatus in the gimbals, taking care not to bring the outside levels into contact with the gimbal-cradle; we fix the gimbals.

10. We so arrange the foot-screws of the apparatus that the distance between the bottom of the apparatus and the bars of the cradle is the same next to each foot-screw; this distance must be so adjusted that the axis of the cradle are within one or two mm in the same horizontal plane as the knife-edges of the pendulums.

11. We arrange the chronometer-currents and we adjust the springs of the two shutter levers in such a way that the light is not interrupted a second time by the rebound of the lever; we may find examples of this in Plates IV and VII.

12. We put the slow pendulum apparatus in its place on top of the pendulum apparatus and we adjust the two light-rays. By means of the prisms b_1 and b_2 we ensure that the last prisms i_2 and l are well lighted and then we adjust in the first place the prism l and then the prism i_2 in such a way that the rays appear in the control-window below the recording apparatus and that they are located in the desired position on the scale e.g. as indicated in the little table on page 83.

13. We put the gimbals into action and we so adjust the counter-weights in both directions that they have the same swinging-period. We check that the gimbals can swing freely without undue friction.

14. We set the gimbals free in the sense of the swinging-plane of the pendulums and we lower the pendulums on their knife-edges. We then carefully tilt the apparatus over e.g. one degree and we check the right adjustment described in 7, by observing whether the light-images of the two fictitious pendulums in the control-window do not move in lateral sense; they may move slightly up and down. At the same time we check whether the damped pendulum for the recording of the middle pendulum is not carried along by the tilting of the apparatus; in that case the light-image of the middle pendulum would show a slight shift during the tilting. We do afterwards the same for the second damped pendulum by setting the gimbals free in the other sense and tilting the apparatus in this direction. If the shifting of the light-image of one of the damped pendulums would indicate a carrying along of this pendulum, this would be a proof of its supporting needles being used and we should have to renew them.

15. We may finally make a record for checking some of the last adjustments. This may show whether the light-interruptions are followed by rebound interruptions, whether the axis of the fictitious pendulums are not shifted by a slight tilting of the apparatus and whether the axis of the records of the middle pendulum and of the damped pendulum for the measuring of the tilt

do not move by tilting the apparatus in the two senses. It likewise gives us the opportunity to check the periods and the damping of the slow pendulums which we should have to readjust if they would differ from the intended values. We also can use it for determining the shift of the axis of the records of the slow pendulums brought about by the dropping of little weights in the grooves with which these pendulums have been provided for this purpose.

This finishes the adjustment program of the complete pendulum apparatus. As further preparatory measures for an expedition we may mention here the checking of the hair-hygrometer e.g. by means of an aspiration psychrometer and eventually the checking of the thermometer of the apparatus.

§ 15. *The determination of the correction for sway and of the differences of the pendulum-periods.*

In this paragraph we shall discuss a few questions not dealt with in earlier papers. We shall keep to the system followed in this publication of indicating the time in which the phase-angle of a pendulum increases by 360° by T, i.e. T is the time of a double swing; this differs from the publication of 1929 where T indicated the time of a single swing. In agreement with the meaning of T adopted here, every correction of the pendulum-period has double the value of that meant in the former publication.

A. *The determination of the sway correction when the apparatus is mounted in the gimbals.*

In this case the sway is large; the correction u to T according to the new meaning of u and T is of the order of 2100 10^{-7} sec. For its determination the simplest way is to use the period T_p shown by the amplitudes and the phase-differences of the three pendulums when they are swinging all three together. This case has been dealt with in the appendix IV of chapter I of the publication of 1929; we found there on page 44 that the period T_p is given by

$$T_p = \frac{T^2}{3u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

For the above value of u the period T_p is about 26 minutes. Besides this period there is a long period of the order of several days which we can neglect here; for the time of an observation it practically gives a linear change in the quantities under consideration.

It is easy to determine T_p; if we make an observation with the three pendulums swinging together, the amplitude of the record of the middle pendulum will show this period and so we can measure it in this record. The amplitude will show greater fluctuations and the period will, therefore, show clearer if the sway of the apparatus is larger, and so we have to choose the amplitudes and phases of the three pendulums in such a way that their effects on the sway of the apparatus do not compensate each other. We may e.g. start with about equal amplitudes in the same phase for the outer pendulums and no amplitude for the middle pendulum. As we have only to measure amplitudes, we can use the small paper-speed for the recording. For making it more easy to count the number of minutes, we put the light-interruption system worked by the chronometer at half second interruptions. For the duration of the observation we require at least one interval T_p in order to guarantee the presence in the record of one maximum and one minimum value of the amplitude at an interval of $\frac{1}{2}$ T_p; we can of course improve the determination by extending it over a longer time, e.g. $1\frac{1}{2}$ T_p, which ensures the obtaining of one more maximum or minimum value. This would imply an observation of about fourty minutes for the present apparatus.

If we should prefer to do so, we may also derive u from the period of the phenomenon when two instead of three pendulums swing together on the apparatus. According to formulas published in an earlier publication [1]) the period in this case is

$$T'_p = \frac{T^2}{2u} \quad \ldots \ldots \ldots \ldots \quad (106)$$

This period is one and a half times as large as in the case of three pendulums and so we require an observation over a time that is in the same proportion longer.

The formulas (105) and (106) remain the same when we use the old system for which T, T_p and u have half the value.

B. *The determination of the sway correction for the apparatus mounted on a fixed support.*

In this case the sway is much smaller and so the above mentioned method would take too much time. We can then appropriately apply the old method for determining the sway correction. We give an amplitude to one of the outer pendulums, e.g. the left pendulum nr 1, while the middle pendulum nr 2 begins by hanging still; the third pendulum is immobilized by means of the amplitude-lever. Pendulum nr 2 gradually gets amplitude and from this amplitude we derive the sway correction u. For making the record we use the small paper-speed and we apply the arrangement for half second interruptions of the light-rays.

If we substitute in the normal formula for this method the amplitude a_v of the fictitious pendulum of the nrs 1 and 2 instead of the amplitude of the pendulum nr 1, and if t is the time since the beginning of the observation, we have

$$u = \frac{T^2}{2\pi} \times \frac{a_2}{t\, a_v}. \quad \ldots \ldots \ldots \quad (107\,A)$$

Because the phase of the movement of the second pendulum is lagging over an angle $\frac{1}{2}\pi$ behind that of the first pendulum, this substitution does not bring about an appreciable error for the time of our observation as long as the second pendulum starts really with zero amplitude. Introducing the value

1) Observations de pendule dans les Pays Bas, publ. Neth. Geod. Comm., WALTMAN, Delft 1923, p. 28 and 29.

1.0024 for T, expressing t in minutes and taking into account that a_2 is recorded on a 1.137 times larger scale than a_v, we obtain

$$u = \frac{23440}{t_m} \times \frac{a_2}{a_v}\, 10^{-7}\, sec, \quad . \quad . \quad . \quad . \quad . \qquad (107\,B)$$

where a_2 and a_v now have the meaning of amplitudes measured in the record. Because it is difficult in this apparatus to lower the middle pendulum so carefully that it does not get a slight amplitude, we shall not always be sure that a_2 is zero at the beginning. On the other hand we have the advantage here of making a record and this enables us to take a small initial amplitude in a simple way into account. For deriving the formulas for this case we shall use the general formula (45) of page 36 of the publication of 1929, which for our case of two pendulums swinging together becomes

$$\dot{q}_2 = -\,i\,d\,q_1 - i\,d\,q_2 + i\,r_2\,q_2, \quad . \quad . \quad . \quad . \quad . \qquad (108\,A)$$

or in connection with (44) on the same page, and dividing by $-\frac{2\pi i}{T^2}\,(q_1 + q_2)$ after bringing the last term to the left side

$$u = i\,\frac{T^2}{2\pi}\;\frac{(\dot{q}_2 - i\,r_2\,q_2)}{(q_1 + q_2)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (108\,B)$$

In the formulas (108) q_1 and q_2 are complex quantities which in the usual way can be represented (fig. 9) by vectors having moduli equalling the

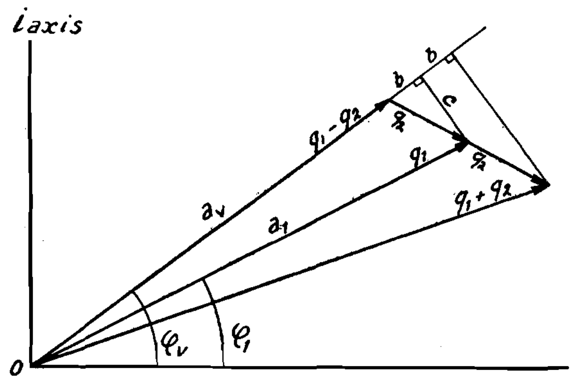

Fig. 9.

amplitudes and arguments equalling the phase-angles of the pendulums nrs 1 and 2. r_2 is $n_2 + ik$, n_2 being the phase-velocity and k the damping of pendulum nr 2; we shall assume the damping of both pendulums to be the same. From these definitions it follows that the vector representing $q_1 - q_2$ has a modulus equalling the amplitude a_v of the fictitious pendulum and an argument equalling its phase-angle φ_v.

If we develop the real part of the right member of (108 B) we find

$$u = \frac{T^2}{2\pi} \times \frac{\left[c - c_0 - \frac{2\pi t}{T^2}\,(T_2 - T_1)\,b_m\right]}{t\,(a_v + 2\,b_m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (109\,A)$$

in which t is the duration of the observation, b_m the mean value during the observation of the component of q_2 in the sense of $q_1 - q_2$, c_0 the component of q_2 perpendicular to $q_1 - q_2$ at the beginning of the observation and c this same component at the end after an interval t. We assume c to be positive when being in phase $\frac{1}{2}\pi$ behind $q_1 - q_2$.

We may find $2b$ by measuring the grey part of the record of the middle pendulum in the same vertical as the maximum amplitude of the second-mark curve in the record of the fictitious pendulum; b is in fact the same quantity as that treated of on page 81 of the publication of 1929 where the method of measuring it is enlarged on. It has positive sign when the grey parts of the records of the fictitious pendulum and of the middle pendulum where the measuring is done, have the black triangle at the same side.

In an analogous way we may measure c in the grey part of the record of the middle pendulum in the same vertical as the passage through the axis of the second-marks in the record of the fictitious pendulum. It is positive when the passage through the axis of the second-marks in the record of the middle pendulum that comes after the place where we do the measuring, is in the same sense as the passage in the record of the fictitious pendulum. The signs of b and c have to be reversed when we should have used the right outer pendulum for the observation instead of the left as we assumed here.

Introducing in formula (109 A) the value of T of 1.0024, taking into account the difference of scale of a_v and of b_m, c_0 and c, and indicating by t_m the interval t expressed in minutes, the formula becomes

$$u = \frac{23440}{t_m} \times \frac{[c - c_0 - 0.375\, t_m\, (T_2 - T_1)\, b_m \times 10^{-4}]}{(a_v + 1.76\, b_m)}\, 10^{-7}\ \text{sec}, \qquad (109\,B)$$

$T_2 - T_1$ is expressed in 10^{-7} sec and a_v, b_m, c_0 and c are lengths measured in the record. No more than for the old method the duration of the observation may be extended too much; it depends on the value of $T_2 - T_1$ how large t_m may be without loosing too much accuracy in applying the formulas.

The formulas (107 B) and (109 B) remain valid for the old system in which T_1, T_2 and u have half the value if we introduce for t_m the double duration of the observation in minutes.

C. *The determination of the difference of the pendulum-periods.*

For an accurate determination of the difference of the pendulum-periods the writer may refer to what has been said about it in the publication of 1929. For the difference of the outer pendulums we can use the normal program for the observation, giving equal amplitude in opposite phase to these two pendulums and no amplitude to the middle one. For the difference of the middle pendulum from the others the schedules mentioned on page 28 of the publication of 1929 may serve the purpose. The writer wishes here only to indicate a quick and simple method for a rough determination of differences of period which may be useful when synchronizing pendulums.

The method has the advantage of requiring no time-keepers and no recording; it works by means of reading amplitudes in the control-window below the recording apparatus. We have to put the pendulum apparatus on a fixed support because we can not apply the method when the sway is too large. We put the pendulums which we wish to compare in one of the outer places and in the middle place and we begin by lowering them so carefully that the fictitious pendulum corresponding to this pair does not get amplitude. If there is a third pendulum in the apparatus, we must have fixed it beforehand by means of the amplitude lever. We then give equal amplitudes in the same phase to the pendulums by rythmically pushing the apparatus, taking care not to push it so hard that it would move. If the pendulums would be exactly synchronous, the image of the fictitious pendulum would remain quiet, but if this is not the case it will gradually get amplitude and this amplitude is a measure for the difference of the periods. So by reading the amplitude after some time we can determine the difference.

It is easy to derive the formula for this method by the following reasoning. In the beginning the two pendulum-vectors q_1 and q_2 are equal and so their difference $q_1 - q_2$, which is the vector corresponding to the fictitious pendulum, is zero. If the pendulums, however, have a difference of period $T_2 - T_1$ their phase-velocities differ over $2\pi (T_2 - T_1)/T_1 T_2$ and so the amplitude of the fictitious pendulum after a time t will be (fig. 10)

$$a_v = \frac{2\pi}{T^2} a_2 t (T_2 - T_1),$$

and we obtain

$$T_2 - T_1 = \frac{T^2}{2\pi} \times \frac{a_v}{t a_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (110\,A)$$

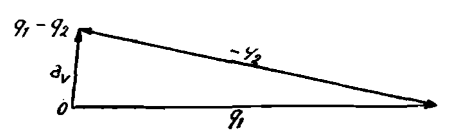

Fig. 10.

Introducing $T = 1.0024$, taking into account that the scale of recording of a_2 is 1.137 times larger than that of a_v and expressing the duration t in minutes, we get

$$T_2 - T_1 = \frac{30310}{t_m} \times \frac{a_v}{a_2} \, 10^{-7} \text{ sec.} \quad . \quad . \quad . \quad . \quad . \quad (110\,B)$$

By reading the amplitudes a_v and a_2 a few times we get a quick determination of the difference of the periods T_1 and T_2. The method does not give great accuracy as it is difficult to realize with high precision the initial condition of zero amplitude for the fictitious pendulum.

The formula (110 B) is valid for the old system for which T_1 and T_2 have half the value, i.e. the value of the time of one swing, if we introduce for t_m the double duration of the observation in minutes.

The Plates III—VII represent records, true to scale, of the complete pendulum apparatus; in all the plates the left and middle pendulum records are the records of the two fictitious pendulums, the right one that of the middle pendulum. The records read from top to bottom. There are four more curves; beginning from the left they are for the Plates III, V and VI the temperature-curve, the slow pendulum in the sense of the ship's axis (Z axis), the slow pendulum in the other sense (Y axis) and the curve of the damped pendulum (tilt-curve); for the Plates IV and VII: the temperature-curve, the slow pendulum in the ship's axis, the damped pendulum and the other slow pendulum.

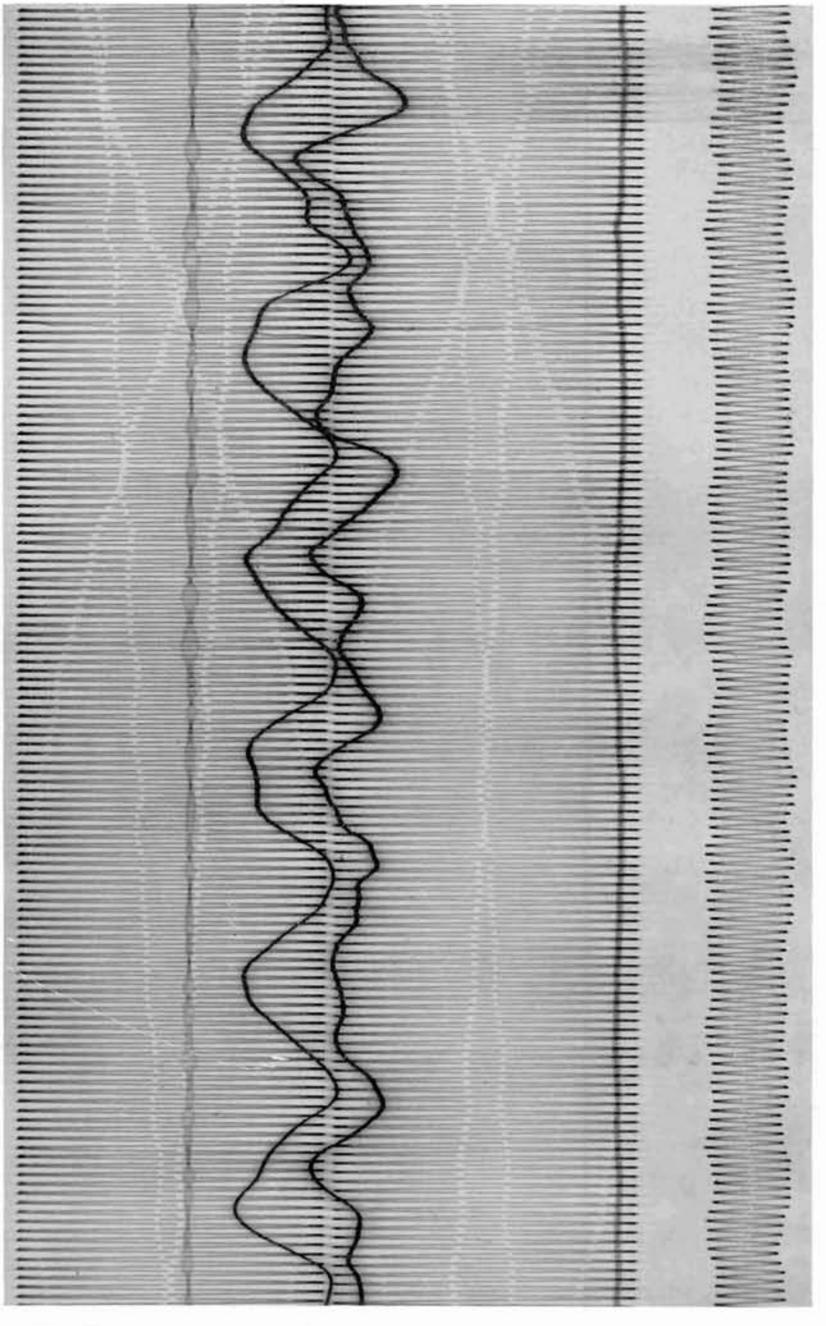

Plate III. No 4 *B*, voyage O 13, depth 37.5 m, course slowly changing from 190° (top) to 170° (bottom); same record as Plate VI but the undamped free swinging of the slow pendulums makes it difficult to measure the wave-period of 14 sec present in the fluctuations of the second-mark curves and clearly visible in Plate VI.

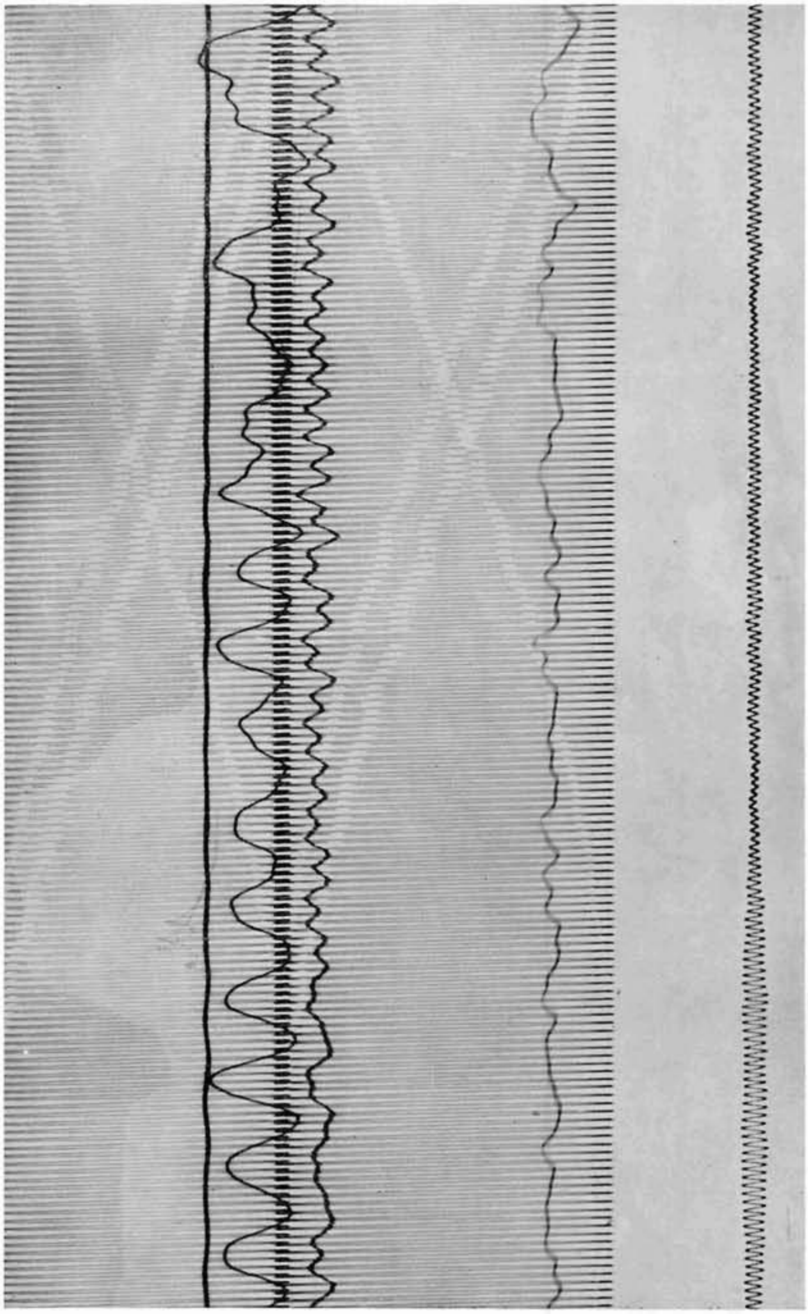

Plate IV. No 1 A, voyage O 19, depth 17.5 m, course 180°; same record as Plate VII. The record shows a strong damping of both slow pendulums, which at the top have amplitudes disappearing quickly towards the bottom. At the bottom the wave-period of 10 sec is clearly visible in the left slow pendulum record.

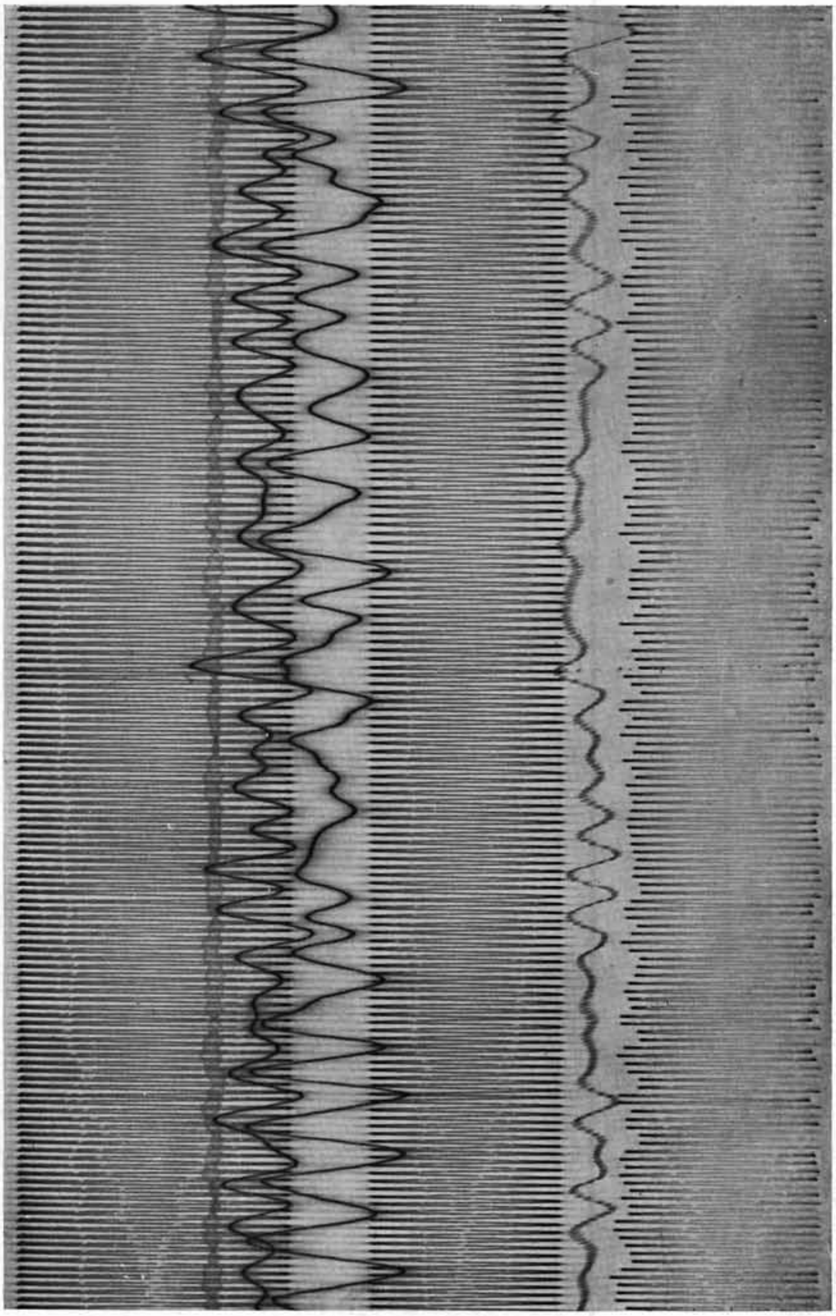

Plate V. No 4 *A*, voyage O 13, depth 17.5 m, same station as the record of Plates III and VI, course slowly changing from 20° (top) to 0° (bottom), lower part measured for obtaining the data of tables I and II of pages 66 and 67.

Principal wave-period 7.5 sec, wave-length 88 m (estimated at surface 60 m), azimuth wave 68° (estimated at surface 40°); height wave 40.5 cm, at surface 141 cm.

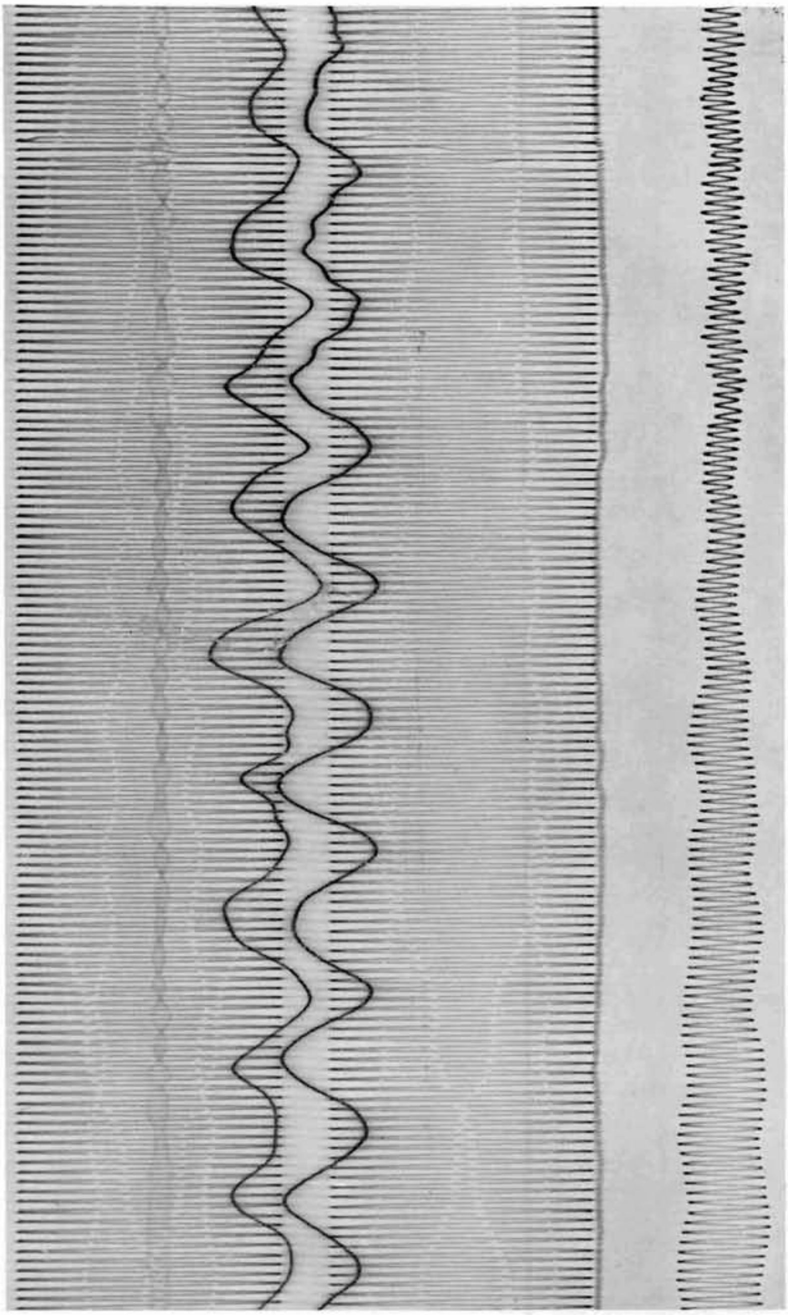

Plate VI. No 4 *B*, voyage O 13, depth 37.5 m, same record as Plate III, same station as Plate V, course slowly changing from 65° (top) to 45° (bottom), part measured for obtaining the data of tables I and II of pages 66 and 67.

Principal wave-period 14.0 sec, wave-length 306 m, azimuth wave 279°, height wave 81.1 cm, at surface 175 cm. Swell not noticed at the surface although higher than the wave of Plate V which was observed at the surface.

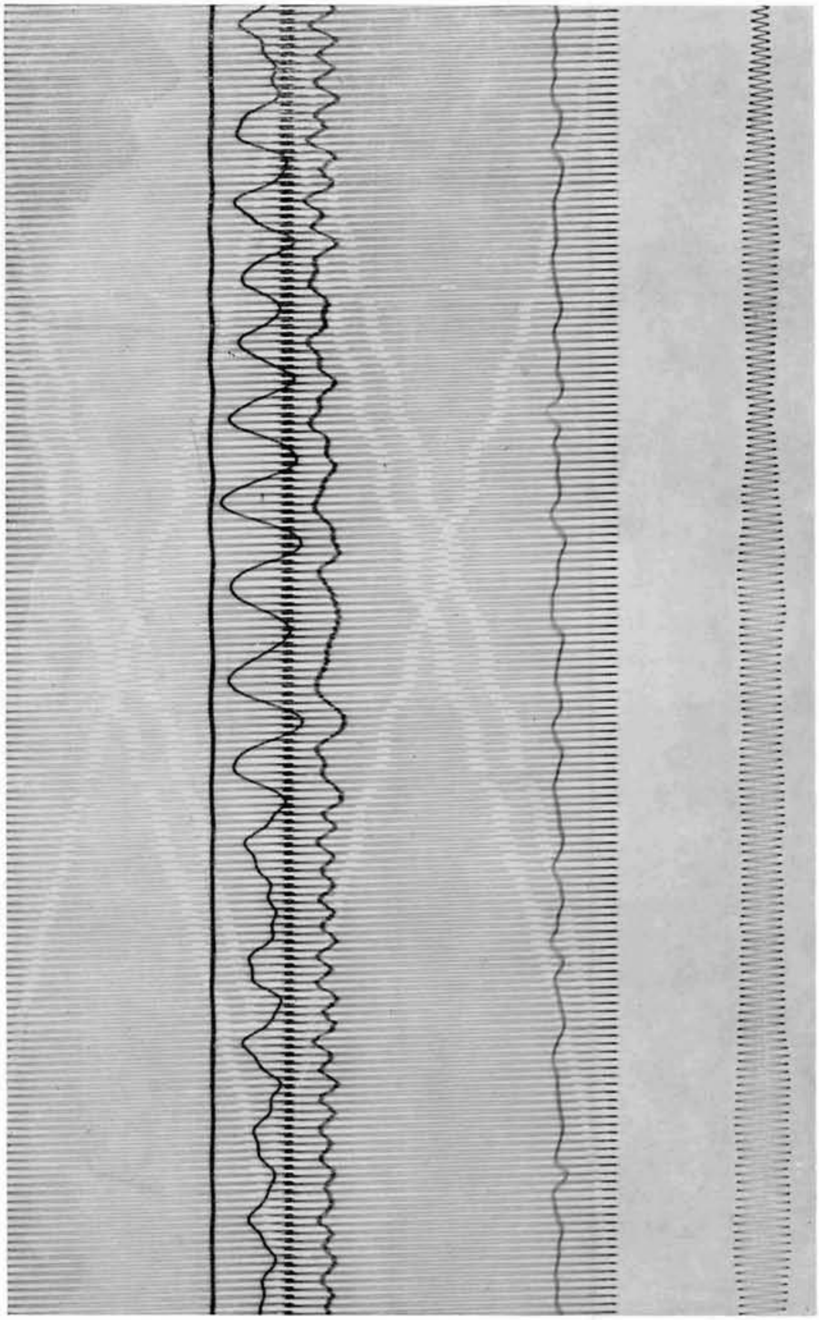

Plate VII. No 1 *A*, voyage O 19, depth 17.5 m, course 180°, same record as Plate IV, upper part measured for obtaining the data for tables I and II of pages 66 and 67 (1st line of 1 *A*).
Principal wave-period 10.0 sec, wave-length 156 m (estimated at surface 40 m), azimuth wave 0° (estimated at surface 0°).